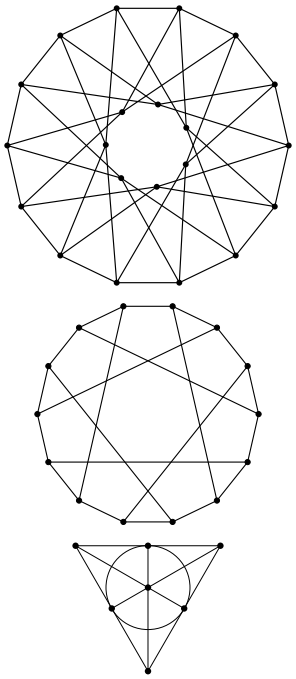
Sira Busch

Groups of Automorphisms of Spherical Buildings



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Fach Mathematik

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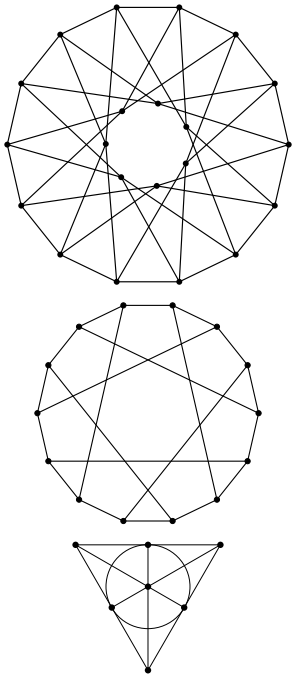

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vorgelegt von Sira Busch,
geb. Detmold, whft. Münster.

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Dekan:in: Prof. Dr. Arthur Bartels

Erstgutachter:in: Prof. Dr. Linus Kramer

Zweitgutachter:in: Prof. Dr. Bernhard Mühlherr

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Introduction

0.1. Abstract

We investigate various groups of automorphisms of thick irreducible spherical buildings of rank at least three, using primarily the tools of incidence geometry, while also offering some algebraic considerations and translations. We give an alternative geometric proof of the fact that buildings of types A_n, B_n, C_n, and D_n are Moufang. This geometric approach yields new insight into the root groups associated with these buildings, namely that every root elation in each point residue can be expressed as an even self-projectivity. Furthermore, it enables us to prove that polar spaces of infinite rank are Moufang; after establishing an equivalent definition for the Moufang property without using the notion of apartments. For buildings of types H_3 and H_4, we present a novel and elementary way of showing why thick spherical buildings of these types cannot exist, avoiding the use of the extension theorem. Following the observation that, in the aforementioned cases, root elations can be described as even self-projectivities, we show more generally, for Moufang buildings of all types, that given an element θ of the little projective group which fixes a certain simplex F pointwise, we can describe θ as an even self-projectivity of F, and conversely, that every even self-projectivity of F is induced by a product of elations that fix F pointwise. This yields a correspondence between the pointwise stabilizers of simplices in the little projective group and the special projectivity groups. We then determine all special and general projectivity groups of simplices with irreducible residues in thick spherical buildings of rank at least three. As we describe, this determines the precise action of the Levi subgroup of a parabolic subgroup on the corresponding residue in the cases where the building is attached to an adjoint Chevalley group, which then coincides with the little projective group of the building. For buildings of simply laced type, we provide both geometric and algebraic proofs for the special projectivity groups, a general diagrammatic rule to decide exactly when the special and general projectivity groups of a simplex coincide, and a new combinatorial characterization of the polar type. To achieve these results, we develop several new tools, including a very slight generalization of Tits's [**Tit74**, Proposition 3.30]: in a building of thickness at least $t+1$, given t simplices of the same type, there always exists a simplex opposite to all of them. We then investigate the case where the building has thickness $t+1$ and we are given $t+1$ vertices of the same type. For finite classical buildings, we prove that a common opposite still exists except when, in the corresponding Grassmannian geometry, the set T of the $t+1$ given vertices forms either a line, a hyperbolic line in a symplectic residue,

or an ovoid in a large symplectic subquadrangle of a rank-2 residue in characteristic 2. This extends a fundamental result of Bose and Burton from the 1960s and determines the minimal blocking sets of finite classical buildings of thickness $t + 1$. For finite buildings of types F_4, E_6, E_7, E_8, and G_2, we prove analogous results for many such associated Grassmannian geometries. This allows us to classify geometric lines of many such geometries in a new way: as point sets of the size of a line, admitting no object opposite to each of its members. As a further corollary, we characterize automorphisms of exceptional spherical buildings as opposition-preserving maps.

0.2. Research Chronology

When Jacques Tits first described buildings, he found a unified approach to study a rich variety of different mathematical objects, uncovered their connections, and shed a bright light on the deep harmony that underlies all of mathematics. Even to this day, mathematicians keep finding new value in the theory of buildings. Recently, buildings have been popularly considered in the study of totally disconnected locally compact groups, expander graphs, strongly regular graphs, braid groups, and RGD-systems. Building theory is neither purely algebra, nor purely geometry, nor purely combinatorics, but lives in the intersection of all three areas, and therefore one can find an entrance from many different directions. No matter from which of these directions one starts, one will be able to profit from seeing the interplay between all the different mathematical concepts that come together in the theory of buildings.

Historically, the theory of buildings evolved during the search for analogues of exceptional simple Lie groups over arbitrary fields. During the late 1950s, it became a widespread desire among mathematicians to classify all finite simple groups. Simple groups are the groups used to assemble all other groups. Traditionally, in the context of Lie groups, people worked over the fields $\mathbb{C}$ and $\mathbb{R}$. Finding analogues of Lie groups over arbitrary fields was of interest since it would allow the field to be finite and, with that, one could hope to find new families of finite simple groups. In 1955, Chevalley managed to construct these analogues, and the groups he found are now known as *Chevalley groups* (cf. [**Che55**]). After Chevalley published his work, Jacques Tits developed the theory of buildings by attaching geometric structures to these groups. After interweaving the groups and the geometric structures, Jacques Tits was able to see even more new groups, such as the *twisted Chevalley group* of type ${}^3\mathsf{D}_4$, for which he gave a construction in an article in 1959 (cf. [**Tit59**]). Ultimately, Chevalley groups defined over finite fields became part of a new class of groups that are now known as *finite groups of Lie type*.

Thus, from the start, the theory of buildings arose from the strong bond between group theory and geometry. Group theory itself can largely be seen as the mathematical study of symmetries, and it has many applications outside of mathematics, from physics to cryptography, chemistry (e.g., in the study of crystal structures or the structure of viruses), and even architecture and mechanical engineering (e.g., in the study of lightweight designs).

In this thesis, we will focus on *spherical buildings*, which, very roughly speaking, can be viewed as complexes made by gluing together triangulated spheres according to certain rules. Our buildings will also be *thick*, meaning that each face of every highest-dimensional analogue of a tetrahedron is contained in at least three of them, giving us richer and more interesting structures. Famously, every building is attached to a *Coxeter group*. Coxeter groups are generalizations of symmetry groups of regular n-gons and include the symmetry groups of the Platonic solids. Given a Coxeter group W and a generating set S for it — and we will also refer to such a tuple (W, S) as a *Coxeter system* — we can construct a *Coxeter complex* and encode its structure in a *Coxeter diagram*. For finite Coxeter groups, these Coxeter complexes are the triangulated spheres that we use as the building blocks for our buildings, and we will call them *apartments*; it will turn out, however, that not all of these Coxeter complexes are suitable for constructing thick spherical buildings. Given a building, every apartment is a Coxeter complex for the same Coxeter group with respect to the same generating set. In particular, all apartments are isomorphic. Thus, we can speak of different *types of buildings* depending on the attached Coxeter system. If the associated Coxeter diagram of a building is connected, we call the building *irreducible*. Thick irreducible spherical buildings will be our main protagonist. Since irreducible finite Coxeter groups have been classified, we can already note that there are not too many types of buildings we will have to consider. The highest-dimensional analogues of tetrahedra inside a building will be called *chambers*, and their faces *panels*. Our protagonist will usually appear with chambers that are at least three-dimensional, which, as we will see, corresponds to the Coxeter diagram having at least three vertices. We will call the number of vertices of the associated Coxeter diagram the *rank of the building*. In case the chambers are only two-dimensional, i.e., the rank is two, then our buildings are so-called *generalized polygons*, whose apartments are just the *ordinary polygons*, and they consist only of vertices and edges. If we are given a non-maximal simplex A in a building Δ, and we denote the set of all chambers in Δ by $\mathscr{C}$, then the set of all simplices in $\bigcup_{\{C\in\mathscr{C}|A\subset C\}} C \setminus A$ is sometimes called the *residue of* A, denoted $\mathsf{Res}_\Delta(A)$, and it is called *irreducible* if it is itself an irreducible building. In that case, its Coxeter diagram is a connected subdiagram of the Coxeter diagram of Δ.

It is said to have been one of Tits's greatest achievements (cf. [**AB08**, §0.5]), or one of the great accomplishments of 20th-century mathematics (cf. [**Wei04**, Preface]), to classify all thick irreducible spherical buildings of rank at least three. Roughly speaking, he showed that almost every such building corresponds to a *simple algebraic group* (cf. [**Tit74**, Theorem 11.4], [**AB08**, §9.1]). To be a bit more exact: every such building corresponds to either an algebraic group, a *classical group*, or a *mixed group*. Many classical groups are algebraic groups, but not all, and mixed groups are only similar to algebraic groups. That means, given a thick irreducible spherical building of rank at least three, we actually always have two groups attached to it: a Coxeter group and a group that is either classical, algebraic, or mixed. Confusingly, sometimes, when the type of a building is mentioned, it does not refer to the attached Coxeter system or Coxeter diagram, but to one of the adjectives classical, algebraic, or mixed. We will have the following convention: if we just mention the type of a building, it will always refer to the attached Coxeter system or Coxeter diagram. Whenever we want to talk about whether the building corresponds to a classical,

algebraic, or mixed group, we will talk about the *type of the group attached to the building*. This still requires the reader to pay good attention, but in this thesis, we will only very rarely talk about the type of the group attached to the building, and when we do, it will be clear in the context that we are not referring to the Coxeter group.

To continue with the story: shortly after Tits classified all such buildings (in [**Tit74**]), he showed in [**Tit77**], see also [**Tit74**, Addenda], that all of them satisfy the so-called *Moufang condition*, or, in other words, have the *Moufang property*, which tells us that our building has a rich automorphism group. Thick irreducible spherical buildings of rank at least 2 satisfying the Moufang condition will be called *Moufang*.[1] In general, it is unclear whether a given building possesses many 'interesting' automorphisms, but for Moufang buildings this is guaranteed. We can roughly visualize the Moufang condition in the following way: we divide every apartment Σ of our building Δ into two hemispheres. These will also be called *half-apartments* or *roots*. An automorphism of Δ that, roughly speaking, fixes a half-apartment pointwise and also fixes neighboring vertices of the half-apartment will be called a *root elation* of Δ. For a given half-apartment α of Δ, we call the group of root elations fixing α pointwise, again roughly speaking, the *root group* associated to α, and usually denote it by U_α or X_α. If a given root group U_α acts transitively on the set of apartments of Δ containing α, then we say that the root α is *Moufang*. If a building satisfies the Moufang condition, then every root of Δ is Moufang. The group generated by all root elations of Δ is also called the *little projective group* of Δ. The little projective group of a Moufang building Δ of rank at least three is always a simple group and acts on Δ.

It is not a coincidence that Tits discovered that all thick irreducible spherical buildings of rank at least three satisfy the Moufang condition when he classified all such buildings. It turns out that, for every Moufang building, one can construct an *RGD-system* and a *BN-pair* by considering its little projective group, and, given an RGD-system of spherical type, one can construct a BN-pair and an associated Moufang building. We will explain the algebraic systems known as RGD-systems and BN-pairs in the course of this thesis. In our context, RGD-systems can be seen as refinements of BN-pairs: in general, every group with a BN-pair gives rise to a building, but spherical RGD-systems give rise to Moufang buildings (cf. [**AB08**, §7.8]). More importantly, a group determined by an RGD-system of spherical type, and hence associated to a Moufang building, can be either classical, algebraic, or mixed (cf. [**Tit92**], [**Tit74**]). For certain Moufang buildings attached to simple algebraic groups, these groups are Chevalley groups and actually coincide with the little projective groups. In fact, all finite groups of Lie type of rank at least 2 possess BN-pairs, are therefore associated with buildings, and can be described as groups of automorphisms of spherical buildings (cf. [**AB08**, pp. 507–508]). The rank-1 groups of Lie type play a special role. For them, one has to consider so-called *Moufang sets*, but these will not appear in this thesis.

Even though it will also not be important for us here, we will mention that all Moufang buildings of rank two are classified. It was conjectured by Tits that this

[1]This might be a more narrow terminology than some readers might be used to. We choose to do it like this, since we will only be interested in spherical buildings in this thesis. The way we defined Moufang buildings is in accordance with [**Wei04**].

could be possible in [**Tit74**, Addenda], an intricate case was worked out in [**MV99**], and the entire classification was done in [**TW02**]. There is also the notion of *split BN-pairs* (cf. Definition L.5.0.4), which is in principle weaker than the Moufang condition. In the finite case, Tits noted that these two notions are equivalent. Paul Fong and Gary M. Seitz classified split BN-pairs in the finite setting (cf. [**FS73**], [**FS74**]), and thus classified finite Moufang polygons. Later, Katrin Tent and Hendrik Van Maldeghem showed that in the infinite case every split BN-pair of rank two likewise defines a Moufang polygon (cf. [**Ten04**], [**TV01**], [**TV03**]), thereby establishing the equivalence also in the infinite setting.

The above description should already give us an idea about how buildings could be helpful to study certain groups, and why groups of automorphisms of Moufang buildings could be particularly interesting. However, buildings are still much more versatile. First of all, we will quickly mention that it is also possible to define buildings as edge-colored graphs with certain properties. This is sometimes referred to as the *combinatorial* or *chamber system view*, whereas our previous approach is referred to as the *simplicial* or *chamber complex view*. In the combinatorial view, the apartments of a building are copies of the *Cayley graph* of the attached Coxeter group, and the chambers are vertices. We will also briefly describe that in Chapter 2. Moreover, with both approaches, one can define a *Weyl-distance function* on the building Δ: roughly speaking, we can measure the distance between two chambers by walking from one chamber to the other, crossing panels, and counting how many panels we have crossed. These legal paths are also called *galleries*. In the combinatorial view, galleries are just the usual paths in the graph. We call two chambers *adjacent* if they intersect in a panel, and express galleries as sequences of adjacent chambers. As we will see, each chamber in a building corresponds to an element of the associated Coxeter group, and chambers are adjacent if, for the corresponding group elements — say w and w' — there exists a generator s such that $ws = w'$. Now, if we walk along a gallery, switching from one chamber to the next with each step, we can imagine that every step corresponds to a generator of the Coxeter group. If we do not care about where we begin or end, but only about the journey, then it might be enough to describe our gallery through those generators, and, if we do that, we call that the *type of the gallery*. Writing all these generators behind each other, we obtain an element of the Coxeter group again, and, if we did not take any detours on our journey, we refer to that element as the *Weyl-distance* between the two chambers. Counting our steps, we obtain a numerical distance, and in this way, we can view buildings as metric spaces. Since our buildings are nicely designed, we can associate reduced words of the attached Coxeter group to galleries of Δ of minimal length.

However, to really get to the heart of what we will discuss in this thesis, we still have to discover yet another side of Moufang spherical buildings, which, again, had been made visible by Jacques Tits through his classification. Indeed, every Moufang building is a *flag complex* of some *Lie incidence geometry*. For example, buildings of type A_n are flag complexes of *projective spaces*, and buildings of type C_n are flag complexes of *polar spaces*. This is something that we will discuss in detail in Chapter 4. Hence, in a certain way, and by abusing language a little bit, buildings can be seen as generalizations of projective spaces. The name for the Moufang condition, which was given by Jacques Tits, actually comes from the person Ruth

Moufang, the first woman ever to receive the title of full professor of mathematics in Germany, who did a lot of research on projective planes.[2]

Incidence geometry is very combinatorial in nature and studies systems of abstract objects, some of which are nested, that relate to each other by either being incident or not. For example, we could consider a non-empty set $\mathscr{P}$, whose elements we will call *points*, and a non-empty set $\mathscr{L}$ of subsets of $\mathscr{P}$ of cardinality at least two, whose elements we will call *lines*, and say that a point $p \in \mathscr{P}$ and a line $L \in \mathscr{L}$ are incident if $p \in L$. In that case, we also talk about *point-line geometries*. Projective spaces can be defined axiomatically as point-line geometries, and we will see that in Chapter 3. More generally, every Lie incidence geometry is a point-line geometry; the reverse is not true. We will study not only projective spaces and polar spaces, but also generalized polygons and parapolar spaces as point-line geometries in the course of this thesis, starting in Chapter 3. That will already give us some first information about the Lie incidence geometries of buildings of each type. More specific properties of the various Lie incidence geometries are discussed in Chapter 4.

Unfortunately, much literature about projective spaces is quite old, and literature about polar spaces and parapolar spaces is sparse. Even though incidence geometry was at the core of Tits's considerations, when he classified all thick irreducible spherical buildings of rank at least three, and even named the famous Moufang condition after a researcher in projective geometry, this side of buildings is not well-known by many. In this thesis, we hope to demonstrate how insightful it can be, how it can lead to more comprehensive and less technical proofs, and how it can sometimes even lead to results for which it is not clear how one could obtain them otherwise, to consider the Lie incidence geometries corresponding to buildings. Because of the state of the literature, and because knowledge of incidence geometry is not widespread, we will recall many known properties about Lie incidence geometries to give the reader unfamiliar with them a chance to follow. To name some very good references, we would like to mention *Buildings of Spherical Type and Finite BN-Pairs* by Jacques Tits [**Tit74**], *Diagram Geometry* by Francis Buekenhout and Arjeh M. Cohen [**BC13**], *Distance Regular Graphs* by Andries E. Brouwer, Arjeh M. Cohen, and Arnold Neumaier [**BCN89**], *Generalized Polygons* by Hendrik Van Maldeghem [**Van98**], *Handbook of Incidence Geometry* by Francis Buekenhout [**Bue95**], *Moufang Polygons* by Jacques Tits and Richard Weiss [**TW02**], *Points and Lines* by Ernest Shult [**Shu10**], *Polar Spaces* by Hendrik Van Maldeghem [**Van24**], and *Projective Geometry over Arbitrary Fields* by Bart De Bruyn, Koen Thas, and Hendrik Van Maldeghem (soon to be published) [**DTV25**].

The initial idea for this thesis, which was given by Linus Kramer, has been to find an alternative geometric proof for Tits's result that all thick irreducible spherical buildings of rank at least three satisfy the Moufang condition, by considering the corresponding Lie incidence geometries and by only using elementary tools. Showing

[2]In the early 1930s, Ruth Moufang studied different projective planes and their properties. One goal was to determine under which circumstances *Desargues's theorem* (cf. Theorem 3.2.0.13) holds for projective planes. In 1933, Ruth Moufang discovered a class of projective planes, now known as the *Cayley planes* (cf. Definition 4.8.0.3), in which Desargues's theorem does not hold. However, she showed that a slightly weaker theorem holds for them as well (cf. [**TW02**, p. 176]). This theorem is known as the *little Desargues theorem*. Projective planes in which the little Desargues theorem holds are called *Moufang planes*.

the Moufang property was done by Tits by proving a theorem about the extensions of isometries (cf. [**Tit74**, Theorem 4.1.2 and Proposition 4.16]), finding an isomorphism φ from an apartment Σ to an apartment Σ', such that Σ and Σ' intersect in a root α and φ acts as the identity on α, and then extending it to an automorphism of the whole building Δ. This proof is general and does not ask for the type of building. However, Jacques Tits himself noted that his extension theorem is rather technical (cf. [**Tit74**, p. 7]). Hence, considering the type of the building and using its specific geometry to find direct, geometric constructions for all root elations, can help illuminate why Tits's famous result holds. The author is only aware of the proof introduced by Tits, which relies on the extension theorem, and is unaware of any other proof. It is worth mentioning that a simplified version of Tits's extension theorem exists for polar spaces (cf. [**BC13**, Theorem 8.5.5]). However, this does not give us a direct construction for root elations and it assumes that we already know that a certain isomorphism exists (cf. [**BC13**, p. 400, l. 2]).

We give the desired constructions for root elations for Moufang buildings of types A_n, B_n, and C_n in Section 6.2 and Section 6.4, but note that the constructions for buildings of type A_n were known before.[3] We also give an equivalent way to define the Moufang condition for polar spaces without using the notion of apartments. This enables us to describe what the Moufang property looks like for polar spaces of infinite rank, and we proceed to prove it for them as well, which has not been done before. Moreover, we use this equivalent definition for Lie incidence geometries corresponding to buildings of type D_n. Buildings of type D_n are *oriflamme complexes* for *top-thin* polar spaces, instead of standard flag complexes, and thus arguing via apartments is not practical. We will talk about that in Section 3.3.1, Note 6.4.0.10, and Note 6.5.0.6. For Lie incidence geometries corresponding to buildings of types E_6, E_7, E_8, or F_4, things are much more complicated, which should become clear in Section 5.6, at the latest. However, we will give some constructions for *generalized homologies* of Lie incidence geometries corresponding to these buildings in Section 10.6.3 and Section 10.9 — every elation can be generated by homologies, but the converse is not true. Furthermore, we will present a general result connecting root elations to 'certain products of projections' in the building in Theorem 2.10.0.3, and determine certain stabilizers in the little projective groups associated to thick irreducible spherical buildings of rank at least three — we will come back to this later in this introduction.

Apart from that, we provide a new geometric proof for the fact that no thick spherical buildings of types H_3 and H_4 exist in Section 6.6. In 1976, an article by Jacques Tits was published, showing that Moufang m-gons only exist for $m \in 2, 3, 4, 6, 8$ (cf. [**Tit76**, Théorème 1]). Another result by Tits — contained in ([**Tit74**, Addenda, p. 275]) — shows that thick spherical buildings of types H_3 and H_4 do not exist. To summarize the proof of Tits briefly: suppose thick spherical buildings of types H_3 (or H_4) existed. According to [**Tit74**, Addenda, p. 274], these would have to be Moufang. With that, it would follow that the rank-2 residues of a Moufang spherical building of type H_3, which form generalized pentagons (and which are also contained in H_4), also satisfy the Moufang condition [**Tit74**, Addenda, p. 274]. But that is impossible, because no such generalized pentagons exist according to

[3] At least they were most likely known before. The author could not find a reference for this particular construction but expects that one exists.

[**Tit76**, Théorème1]. Since the proof of [**Tit74**, Addenda, p. 274] relies on the extension theorem, which is, as mentioned before, rather technical, there already have been attempts to find different ways to prove the non-existence. One different way of proving that there are no thick spherical buildings of types H_3 and H_4, without using the extension theorem, was found by Hendrik Van Maldeghem in 1995 (cf. [**Van95**, §5]). Van Maldeghem's proof works by first assuming that thick spherical buildings of type H_3 exist, showing that the rank-2 residues that are generalized pentagons are regular, and showing that regular generalized pentagons do not exist. In our proof, we also do not need the extension theorem. We only assume that thick spherical buildings of type H_3 exist and then give geometric constructions for non-trivial root elations of generalized pentagons, implying the Moufang condition for them, which then, again, leads to a contradiction using [**Tit76**, Théorème1]. This is a more direct and less intricate way of proving the non-existence of thick spherical buildings of types H_3 and H_4, since we do not need the concept of regularity. Furthermore, it completes an alternative geometric proof for Tits's results for buildings of rank three. To make our proof more accessible and potentially more broadly applicable, we provide an elementary axiom system for Lie incidence geometries associated with buildings of type H_3 in 4.10.1.1, and write our arguments purely in the language of point-line geometries, not needing the theory of buildings.

The close connection between root elations and 'certain products of projections' became clear quite early and lead the author, together with Hendrik Van Maldeghem, to wonder whether investigating the latter a bit more thoroughly could be of value. Ultimately, the decision to do so was made. The groups connected to the before mentioned 'certain products of projections' are also called *special projectivity groups*, and we will continue to describe them briefly, since they reveal an important feature of spherical buildings: as spherical buildings consist of triangulated spheres, one not only can — but one can also easily see how one can — define an *opposition relation* between the *simplices* of an apartment. By a simplex we basically just mean an n-dimensional analogue of a triangle; we will give a formal definition for simplicial complexes in Chapter 1. A vertex is the smallest example, and a chamber is the largest example for a simplex in a building in our simplicial view. One can show that the opposite of a chamber in an apartment is unique and it is just exactly the one, that is opposite in the sphere; or, with other words: "on the other side of the sphere," with maximal distance to the original chamber. A vertex is opposite another vertex in an apartment, if — no matter how one divides the apartment into two hemispheres — both vertices either lie in the (never empty) intersection of them or none of the two lie in that intersection. In general, two simplices in an apartment are called opposite, if their union is closed under opposition with regards to their vertices. Note that this is, again, a very rough description, to mainly just give an impression and we will give the details in Chapter 2. One can check that, if two simplices are opposite in one apartment, then they are opposite in all apartments, and this gives us a global definition for opposition in the building. Now, given a non-maximal simplex A in an apartment Σ of our building Δ, we can consider a chamber C in Σ containing A and a simplex A' opposite to A in Σ. Then one can show that there exists a unique chamber C' in Σ such that A' is contained in C' and such that C and C' have minimal gallery distance from each other. In this scenario, we call C' the *projection* of C from A to A' and write $C' = \mathsf{proj}^A_{A'}(C)$. Since we can

do this with all chambers containing A, we obtain a map from the simplices contained in chambers containing A, to the simplices contained in chambers containing A'. The map restricted to all simplices in $\mathsf{Res}_\Delta(A) = \bigcup_{\{C \in \mathscr{C} | A \subset C\}} C \setminus A$, where $\mathscr{C}$ denotes the set of all chambers of Δ, is an isomorphism and called a *perspectivity* from A to A'. We call a product of perspectivities a *projectivity*. If we consider a product of perspectivities, such that we start and end at the same simplex, we call it a *self-projectivity*. A projectivity, which is the product of n perspectivities, will be said to have *length* n. The set of all self-projectivities of some non-maximal simplex in a building is called the *general projectivity group* of that simplex, and the set of all self-projectivities of even length forms a subgroup in it, called the *special projectivity group* of that simplex. In projective geometry, the *groups of projectivities*, or *projectivity groups* have been long known and play an important role in many proofs. For instance, projectivities between lines in a projective plane can be used to define *non-degenerate conics* (Steiner's approach) and prove properties of them. In [**Kna88**], Knarr defined groups of projectivities and groups of even projectivities for generalized polygons and determined them in the finite case. This was further generalized to large infinite classes by Hendrik Van Maldeghem in [**Van98**], where the group of projectivities was called the *general projectivity group* and the group of even projectivities the *special projectivity group* related to a point or line. In the way described above, one obtains a generalization of the definitions to all spherical buildings and the question, in which Hendrik Van Maldeghem and the author were interested, was, whether one can determine all the various general and special projectivity groups of non-maximal simplices in thick spherical buildings. Together with Jeroen Schillewaert, Hendrik Van Maldeghem and the author first determined all special and general projectivity groups occurring for simplices with irreducible residues in thick spherical buildings of rank at least three and *simply laced type* (cf. Theorem M) — which means that there are no double- or triple-bonds appearing in the Coxeter diagram of the building — and then, Hendrik Van Maldeghem and the author continued to do the same for the thick spherical buildings of rank at least three and non-simply laced type (cf. Theorem N and Theorem O). All proofs and results are included in Chapter 10.

While determining all these projectivity groups, we made many interesting observations.

Special projectivities and root elations.

> We showed that, given a Moufang building Δ and some simplex F in Δ, every self-projectivity of the special projectivity group of F is induced by a product of elations, each of which pointwise fixes F, and, given any element of the little projective group of Δ that pointwise fixes F, we can write that element as a self-projectivity of F of even length. Thus, we get a correspondence between the pointwise stabilizers of simplices in the little projective group and the special projectivity groups of the simplices. This is Theorem 2.10.0.3.

Special vs. general projectivity groups.

> For thick spherical buildings of simply laced type, we establish a general diagrammatic rule to decide when exactly the special and general projectivity groups of a simplex coincide or not. This is Theorem F and Theorem G. As

a corollary, we obtain a new combinatorial characterization of the polar type. This is Corollary B. For buildings of types B_n, C_n, and F_4, we have Theorem H and Theorem I, which already tell us something about the relation of the special and general projectivity groups, but they do not yield as clear-cut rules as we have for the simply laced cases.

Reductions.

We show that, under certain conditions, the general projectivity group of a simplex F is generated by the self-projectivities of F of length 3; this is Theorem J, and under certain conditions, we get the same for length 4; this is Theorem K.

Levi subgroups.

In the case that the building Δ is simply laced and of rank at least three, the little projective group of Δ, which we will denote by G now, can be seen as the simple algebraic group — in that case the (adjoint) Chevalley group — attached to Δ. The pointwise stabilizers of simplices in G are the so-called *Levi subgroups* of the *parabolic subgroups* of the Chevalley group. Hence, in the language of algebraic groups, we determined all Levi subgroups of parabolic subgroups of Chevalley groups associated to Moufang buildings of simply laced type. We will explain more about that in Appendix C and also provide an alternative algebraic proof for some of our results. Moreover, an anonymous referee observed that the general and special projectivity groups coincide exactly when the index of the *root lattice* in the *weight lattice* (of a certain *Lie algebra* related to G) is 1.

During our work, we also noticed that it can be incredibly helpful to know when there exists a common opposite to a number of given simplices. In [**Tit74**, Proposition 3.30], Jacques Tits showed that, given two simplices (of the same type) in a thick spherical building Δ, there always exists a simplex of Δ which is opposite to both of them. This has to do with the fact that, in a thick building, every panel is the face of at least three chambers. We will give a rough sketch of the proof: first, we note that it suffices to consider chambers. So, let C and D be two chambers. Then, among all chambers opposite C, we can choose a chamber E with maximal distance to D. If E is also opposite D, then we are done. If E is not opposite D, then there exists an apartment Σ of Δ containing A and E (that is a general property of buildings) and a face A of E, such that every minimal gallery in Σ between D and E goes through A (this is called the so-called *gate property*, and A is also called a *gate*). Now A can be contained in E and some unique chamber not opposite C, but, since Δ is thick, it will also be contained in a third chamber E'. This chamber E' must then still be opposite C, and further away from D than E. But that contradicts the fact that E was at maximal distance from D among the chambers opposite C.

We wanted to obtain a generalization of this result to be able to construct specific projectivities needed for our reductions. For that, we first defined the following: we say that a building has *thickness at least* t if every panel is contained in at least t chambers. One can then, by adding a short induction, immediately obtain the result that, given t simplices (of the same type) in a spherical building Δ of thickness at least $t+1$, there always exists a simplex of Δ which is opposite to all of them. This is Proposition 7.1.0.1. However, this result was not yet strong enough for all our

purposes, and we wondered when it could be the case that, given $t+1$ vertices in a spherical building Δ of thickness at least $t+1$, there always exists a vertex of Δ which is opposite to all of them. This exact question led Hendrik Van Maldeghem and the author to answer it for a variety of different cases, which were not all needed for the reductions that we wanted to make regarding the projectivity groups, but which turned out to be interesting in their own right and could still be helpful for other constructions in the future. We give a more detailed description of all the different use cases and connections for this question in Chapter 7, but will also insert a short summary here.

Blocking sets.

Blocking sets were originally defined for projective planes as a set of points in the plane that every line intersects. They became a notion of great interest in finite geometry, as they have applications in the theory of communication systems, coding theory, and cryptography. It turns out that hidden in our question lies an approach to generalize the notion of a blocking set to all spherical buildings, and the negative answers to our question determine minimal blocking sets in the corresponding building.

Geometric lines.

We will give a proper definition for *geometric lines* in Chapter 7. For now, we just note that they have been defined by Anna Kasikova and Hendrik Van Maldeghem in [**KV13**], and capture a certain phenomenon: sometimes we can have a point-line geometry Δ, all of whose points are embedded into some projective space Γ, but without the property that all lines in Γ between points contained in Δ also form lines in Δ. These specific lines in Γ, all of whose points are contained in Δ, but which do not form a line in Δ, are geometric lines. On first sight, these do not seem to have anything to do with our question. Indeed, for finite geometries, we will be able to see that the definition of a blocking set — in the original version and the generalized version — is much weaker than the definition of a geometric line. However, we are able to show that, in certain finite geometries, the notion of minimal blocking sets coincides with the notion of geometric lines (cf. Theorem C).

Opposition preserving maps.

With our results about finding opposites, we can deduce that certain opposition-preserving maps and transformations of a Lie incidence geometry are actually (bijective) *collineations*. With that, we can characterize automorphisms of exceptional spherical buildings using opposition. We will give the details about this in Section 7.5.

Thus, all in all, in this thesis, we investigate several different groups of automorphisms of higher-rank Moufang buildings, use the special notion of opposition, given for spherical buildings, to construct different building automorphisms, which generate different subgroups of the full automorphism group of the building — like the little projective group and the general and special groups of projectivities — and work a lot inside the corresponding Lie incidence geometries to gain the insight that we truly need to obtain our results about said groups. We give direct, geometric,

and elementary proofs for the fact that thick irreducible spherical buildings of types A_n, B_n, C_n, and D_n are Moufang, that polar spaces of infinite rank are Moufang, and for the fact that thick spherical buildings of types H_3 and H_4 cannot exist. Furthermore, we determine all general and special projectivity groups for all types of simplices with irreducible residues, for all thick spherical buildings, and uncover a correspondence between stabilizers of simplices in the little projective group of a Moufang building, special projectivity groups of those simplices, and Levi subgroups of parabolic subgroups of Chevalley groups associated to the building. Besides that, we answer the question of when it is possible to find a common opposite to a number of given vertices in a given spherical building, for a few more than just the most relevant cases, and describe some surprising connections and applications with regards to blocking sets, geometric lines, and characterizations of building automorphisms of exceptional spherical buildings by using opposition. Along the way, we develop some new methods, gain more insight into the nature of certain automorphisms, groups, and buildings, and make some observations.

0.3. Outline of the Thesis

The first five chapters of this thesis primarily lay the groundwork for most of our main results that we prove later on.

In Chapter 1 we will give the most basic definitions. We will make clear what we mean by a graph, a Cayley graph, a Coxeter diagram, a Dynkin diagram, and a simplicial complex, explain Coxeter groups and Coxeter complexes, as well as finite reflection groups and how they can be viewed as Coxeter groups.

The 2nd chapter introduces buildings and describes both the combinatorial and the simplicial view. We provide examples, show what we mean by the Bourbaki labeling of the vertices, discuss roots, residues, and the notion of opposition, and define the Moufang property, little projective groups, and groups of projectivities. We then move on to BN-pairs and RGD-systems, and provide insight into groups acting on buildings. Subsequently, we describe the subgroup structure of Moufang buildings of simply laced type, briefly discuss Chevalley groups, and explain the connection between certain subgroups of the little projective groups of such buildings and the root and weight lattices of the associated Chevalley groups using the notion of characters. The chapter concludes with a description of Levi subgroups for simply laced buildings, general observations establishing a correspondence between simplex stabilizers in the little projective group and special groups of projectivities, and an algebraic method for determining the special projectivity groups of buildings of simply laced type using the root and weight lattices.

In Chapter 3, we turn to incidence geometry, introducing the various point-line geometries that will serve as the main geometric framework for our results.

Chapter 4 builds on this by linking buildings with Lie incidence geometries: we illustrate, in some examples, how to recover the diagram of a given Lie incidence geometry, how to pass between geometries and buildings, and provide general rules

for extracting geometric information from a diagram. These methods will be applied to study key types of parapolar spaces relevant for later chapters. We also introduce *icosahedral geometries*, show that the maximal singular subspaces of icosahedral geometries are projective planes, that their apartments are icosahedra, and prove that any two singular subspaces of an icosahedral geometry lie in a common apartment. That way, we obtain a purely point-line geometric description for buildings of types H_3, and we use it in Chapter 6 to prove the non-existence of thick buildings of types H_3 and H_4.

Since many concepts from incidence geometry will reappear in our arguments, Chapter 5 delves deeper into the theory. We discuss the classification and properties of polar spaces, introduce structures such as ideal subquadrangles, reguli, ovoids, spreads, conics, nuclei, and equator geometries, and present models for apartments of certain parapolar spaces. We then describe two classes of parapolar spaces, long root geometries and hexagonic geometries, and show some shared properties of the respective parapolar spaces contained in these classes. Finally, we use linear algebra to figure out by which automorphisms the simple adjoint Chevalley groups attached to certain buildings are generated. This involves talking about similitudes and varieties.

The 6th chapter is exactly the one where we give direct geometric constructions for certain root elations, show the Moufang property for thick irreducible spherical buildings of rank at least three of types A_n, B_n, C_n, and D_n, and for polar spaces of infinite rank, and give an elementary, purely geometric proof for the fact that there do not exist thick spherical buildings of types H_3 and H_4.

Then, we included a small interlude to have a chance to explain exactly how we will continue in the next chapters and why, and talk about all the different contexts in which the results that we will prove can be seen and used. We will define blocking sets and geometric lines and describe an application for the results of Chapters 8 and 9 regarding building automorphisms and opposition-preserving maps.

Chapters 8 and 9 focus on characterizing lines in finite Lie incidence geometries. That will be done in two main steps: first, we characterize geometric lines in exceptional hexagonic geometries of rank at least three — that is the content of Chapter 8 — and then we will use these results to obtain the actual line characterization, which is the content of Chapter 9. We will introduce round-up triples (Chapter 8), a way to describe all possible mutual positions of lines in hexagonic geometries, and a combing algorithm that will bring everything together (Chapter 9). At the end, we will discuss generalized hexagons and mention some observations for generalized octagons.

The final chapter of this thesis (Chapter 10) is devoted to the determination of all special and general projectivity groups of simplices with irreducible residues in thick spherical buildings of rank at least three. We will give a thorough introduction to this chapter when it will be the most useful: at its beginning.

Appendix A gathers some basic algebra knowledge, just to make sure that there will be no confusion about which conventions were used to define certain objects. In particular, we will define *forms* and *quadratic alternative division algebras*.

In appendix D, we collect several results of Dieudonné, as we use these in Chapter 10 without giving much context, and we provide English translations for accessibility.

The purpose of Appendices C and L is to explain some of the algebraic background that we skip in Chapter 2. We give a rough outline of the construction of adjoint Chevalley groups, mention their connection to algebraic groups, and state some properties that will hopefully aid the understanding of the algebraic determination of special projectivity groups for Moufang buildings of simply laced type. However, our treatment remains rudimentary, as we omit substantial parts of the general theory and their proofs, since we do not require any of these details. Even the algebraic determination of the special projectivity groups for simply laced Moufang buildings should rather be viewed as a bonus, intended to make our results more accessible and interesting to a wider audience. We provide geometric proofs for all results in Chapter 10.

0.4. Associated Articles

The results in this thesis are contained in several articles that are in preparation, under review, accepted for publication, or already published. The associated articles are listed below.

(i) [**Bus25a**] On the Moufang Condition for Polar Spaces.
Sira Busch, preprint: arXiv:2502.12857, *submitted* (2025).

(ii) [**Bus25b**] A Point-Line Approach to the Nonexistence of Buildings of Types H_3 and H_4.
Sira Busch, preprint: arXiv:2502.12857, *submitted* (2025).

(iii) [**BV25b**] Lines and Opposition in Finite Lie Incidence Geometries of Exceptional Type.
Sira Busch & Hendrik Van Maldeghem,
to be submitted (2025).

(iv) [**BV26**] A Characterization of Lines in Finite Lie Incidence Geometries of Classical Type.
Sira Busch & Hendrik Van Maldeghem,
published in: *Discrete Mathematics* **349** (2026),
no. 2, Paper No. 114711, 15 pp.

(v) [**BSV25**] Groups of Projectivities and Levi Subgroups in Spherical Buildings of Simply Laced Type.
Sira Busch, Jeroen Schillewaert & Hendrik Van Maldeghem,
preprint: arXiv:2407.09226, (2024),
to appear in: *Journal of Group Theory.*

(vi) [**BV25a**] Groups of Projectivities in Spherical Buildings of Non-Simply Laced Type.
Sira Busch & Hendrik Van Maldeghem,
preprint: arXiv:2502.12857, *submitted* (2025).

At the beginning of each chapter we will indicate whether it is based on one of these articles and, if so, specify which one. Chapter 6 includes content of both [**Bus25a**] and [**Bus25b**]. Chapter 8 and Chapter 9 are both based on both [**BV25b**] and [**BV26**]. Approximately, the content of [**BV25a**] is contained in Chapter 10, whereas [**BSV25**] is distributed across Chapter 2, Chapter 5, and Chapter 10.

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1. Graphs, Groups and Complexes

Some concepts that will appear repeatedly throughout this thesis are graphs, simplicial complexes, and Coxeter groups. This chapter gathers the necessary formal definitions. As mentioned in the introduction, a building can be described in two ways: as an edge-colored graph — in the combinatorial view — or as a simplicial complex — in the simplicial view — satisfying certain properties. In either case, a Coxeter group is always associated with it. Thus, in this chapter we will introduce the building blocks of our buildings, which we will then use in the next chapter to define the buildings themselves.

In the introduction, we already noted that — in the combinatorial view — spherical buildings can be constructed by gluing together copies of a *Cayley graph* of a given *Coxeter system* according to certain rules. Hence, we will first discuss graphs, Coxeter groups, and then Cayley graphs. In the simplicial view, by contrast, spherical buildings can be seen as complexes obtained by gluing triangulated spheres, again following specific rules. These complexes are precisely the *Coxeter complexes* that we will introduce in this chapter. Cayley graphs of finite Coxeter groups and spherical Coxeter complexes are, in fact, already examples of spherical buildings: they are *thin spherical buildings*. However, as mentioned earlier, our main focus in this thesis will be on thick buildings.

Finite Coxeter groups are classified, and we will present a list of all finite irreducible Coxeter groups — the ones from which the other finite ones can be assembled. We will also describe the isomorphism between the *symmetric group* $\mathsf{Sym}(X)$, consisting of all permutations of a finite non-empty set X of cardinality n, and Coxeter groups of type A_n. Finally, we explain how to construct *finite reflection groups* and *Weyl groups* from *root systems*, and how these can be viewed as examples of Coxeter groups. This naturally leads to the introduction of *long roots*, *short roots*, and *Dynkin diagrams*.

1.1. Graphs

We begin with a few basic definitions in the context of graphs (most of which can also be found in [**BC13**] or [**Wei04**]), slightly adapted to our purposes.

A *pre-graph* is a pair $\Gamma = (V, E)$, where V is a non-empty set, whose elements are called *vertices*, and E is a set of pairs of distinct vertices, called *edges*. We call a pre-graph a *graph* if it contains no double edges, that is, if E is a set of unordered pairs of distinct vertices. In the literature, this definition is often less restrictive:

what we define as a graph here would usually be called a *simple* or *simplicial* graph, i.e., a graph without loops and without double edges. In this thesis, however, when we talk about a graph, we always want to exclude loops and double edges, so we incorporate this condition directly into the definition. The notion of pre-graphs will later be useful when defining diagrams. Since our edge set only allows pairs of distinct vertices, both pre-graphs and graphs are loopless.

Let $\Gamma = (V, E)$ be a (pre-)graph. Two vertices v and w are called *adjacent* if $\{v, w\}$ is an edge; in this case we also write $v \sim w$ and refer to w as a *neighbor* of v. A *path of length* n in Γ, for some $n \in \mathbb{N} \setminus \{0\}$, from a vertex v to a vertex v' is a sequence

$$\gamma = (v = v_0, v_1, \dots, v_n = v')$$

of $n + 1$ vertices such that $v_{j-1} \sim v_j$ for all $j \in 1, \dots, n$. The *distance* $\mathsf{dist}(v, v')$ between v and v' is the length of a shortest path from v to v', if such a path exists, and ∞ otherwise. A (pre-)graph is called *connected* if $\mathsf{dist}(v, v') < \infty$ for all pairs of vertices v, v'. If a path γ starts and ends at the same vertex, i.e., $\gamma = (v_0, v_1, \dots, v_n = v_0)$, then γ is called *closed*. A path $\gamma = (v = v_0, v_1, \dots, v_n = v')$ is called *non-stammering* if $v_{i-1} \neq v_{i+1}$ for all $i \in 1, 2, \dots, n-1$, and *stammering* otherwise. A non-stammering, closed path is called a *circuit*. The *girth* of a (pre-)graph Γ is the length of its shortest circuit, if one exists, and ∞ otherwise. The *valency of a vertex* in a (pre-)graph is defined as the number of edges incident with that vertex.

A (pre-)graph $\Gamma = (V, E)$ is called

- *bipartite*, if its vertex set can be partitioned into two non-empty subsets V_1 and V_2 such that no edges lie within V_1 or within V_2;
- *multipartite* with partition $(V_j)_{j \in J}$, if V can be partitioned into non-empty subsets V_j, for $j \in J$, such that, for each $j \in J$, no edges lie within V_j;
- *complete*, if Γ is a graph and every pair of distinct vertices is connected by an edge;
- *finite*, if it has only finitely many vertices;
- *connected*, if a path in Γ can always be found between any two of its vertices.

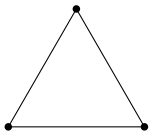
For a (pre-)graph $\Gamma' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$, we say that Γ' is a *(pre-)subgraph* of Γ. If $A \subseteq V$, then the *(pre-)subgraph of* (V, E) *induced on* A is the pair $(A, E \cap (A \times A))$. If this induced (pre-)subgraph is a complete graph, then A is often referred to as a *clique*.

An *edge coloring* of a graph is a surjective map from the edge set E to a set I, whose elements are regarded as *colors*, assigning to each edge a color $i \in I$. The set I is called the *index set of the edge-colored graph*. If Γ is such an edge-colored graph, v and v' are vertices of Γ, and v, v' is an edge of color i, then we also say that v and v' are *i-adjacent*, and write $v \sim_i v'$.

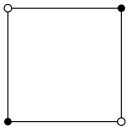
In a similar way, a *vertex coloring* of a (pre-)graph is a surjective map $V \to I'$, where the elements of I' are interpreted as colors, with the condition that no two adjacent vertices share the same color. The set I' is then called the *index set of the vertex-colored graph*. Note that bipartite (pre-)graphs can be seen as vertex-colored graphs with two colors, while multipartite (pre-)graphs with partition $(V_j)_{j \in J}$ can be viewed as vertex-colored (pre-)graphs with $|J|$ colors.

Examples 1.1.0.1.

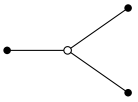
A complete, finite, connected, not bipartite graph



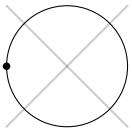
A finite, connected, bipartite, not complete graph



A finite, connected, bipartite, not complete graph

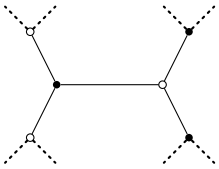


Not a graph and also not a pre-graph



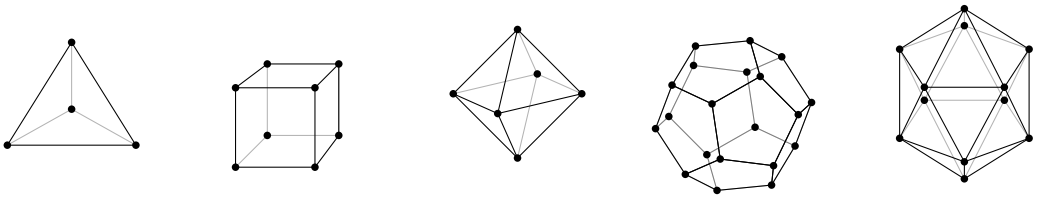
A finite, connected, bipartite pre-graph

An infinite, connected, bipartite, not complete graph



1.2. Coxeter Groups

Coxeter groups offer a unified framework for studying the symmetry groups of regular polygons, permutation groups, and the symmetry groups of the Platonic solids.



Tetrahedron | Cube | Octahedron | Dodecahedron | Icosahedron

Table 1. The five Platonic solids.

In 1934, Harold Scott MacDonald Coxeter introduced an approach to generalizing the aforementioned groups that led to the definition of Coxeter groups (cf. [**Cox34**]). One year later, he also classified all finite Coxeter groups (cf. [**Cox35**]); the groups thus bear his name.

In his 1934 article (cf. [**Cox34**]), he defines groups denoted $[k]$ and $[k_1, k_2]$ as follows:

$$S = \{s_1, s_2\},\ [k] = \langle S \mid s_1^2 = s_2^2 = (s_1 s_2)^k = 1\rangle,\ \text{for } k \geq 1,$$
$$T = \{t_1, t_2, t_3\},\ [k_1, k_2] = \langle T \mid (t_1 t_2)^{k_1} = (t_1 t_3)^2 = (t_2 t_3)^{k_2} = 1\rangle,$$
$$\text{for } k_1 \geq 2,\ k_2 \geq 2,\ 1/k_1 + 1/k_2 > 1/2.$$

For $k \geq 3$ and $k_1 = 3$, $k_2 = 3, 4, 5$, the groups $[k]$ and $[k_1, k_2]$ are the symmetry groups of the regular polygons and polyhedra, respectively.

He then writes:

"*These groups can be made vividly comprehensible by using actual mirrors for the generating reflections. It is found that a candle makes an excellent object to reflect. By hinging two vertical mirrors at an angle* π/k *(*$k = 2, 3, 4, \dots$*), we easily see* $2k$ *candle flames, in accordance with the group* $[k]$*. To illustrate the groups* $[k_1, k_2]$*, we hold a third mirror in the appropriate positions*[sic]*. In the present paper, we generalize the above results to space*[sic] *of any number of dimensions, and to groups which, though infinite, are still free from infinitesimal operations.*"

— H. S. M. Coxeter, 1934, cf. [**Cox34**].

With this imagery in mind, we now turn to the formal definitions.

Definition 1.2.0.1 (**Coxeter matrix**)**.** Let I be an index set. A *Coxeter matrix* is a symmetric array $[m_{ij}]$, with $i, j \in I$, such that for all $i, j \in I$, m_{ij} is either a positive integer or ∞, and $m_{ij} = 1$ if and only if $i = j$.

Definition 1.2.0.2 (**Coxeter group**)**.** Let $[m_{ij}]$ be a Coxeter matrix for some index set I. A *Coxeter group* is a group of the form:

$$W := \langle S \mid (s_i s_j)^{m_{ij}} = 1,\ \text{for all } i, j \in I,\ m_{ij} \neq \infty\rangle,$$

where S is a set of generators $\{s_i \mid i \in I\}$ indexed by I. We call (W, S) a *Coxeter system*.

Definition 1.2.0.3 (**Coxeter diagram**, **fully labeled**, **reductively labeled**)**.** Let $[m_{ij}]$ be a Coxeter matrix for some index set I. The *fully labeled Coxeter diagram* for $[m_{ij}]$ is a pre-graph that is obtained as follows.

- We draw a vertex for every element of I,
- we draw edges between i, j, if $m_{ij} > 2$,
- and we label all edges with m_{ij}.

The *reductively labeled Coxeter diagram* for $[m_{ij}]$ is the same pre-graph, except that we only label the edges between two vertices i and j if $m_{ij} > 4$, and we draw a double edge if $m_{ij} = 4$. A fully labeled Coxeter diagram is always a graph, whereas a reductively labeled Coxeter diagram might not be. If a Coxeter diagram is a graph, we can view the set of all entries m_{ij} of the Coxeter matrix as a set of colors, and with that, the Coxeter diagram is an edge-colored graph. If we just talk about a *Coxeter diagram* in the following, then it does not matter whether or not it is fully or reductively labeled in that context. Note that most authors work with either the reductively labeled version of a Coxeter diagram or with a version in between, where they take the fully labeled Coxeter diagram but omit all edge labels equal to 3. For finite irreducible Coxeter groups of rank ≥ 3 that have no odd entries > 3 in their Coxeter matrix, it is also easily possible to consider *unlabeled Coxeter diagrams*,

while retaining all information, if we draw a triple edge, whenever $m_{ij} = 6$. This is apparent by considering the classification of these groups, which we will show in Table 2. Having these different versions of depicting the diagram will make it a bit less complicated for us later in Chapter 4, to describe certain rules.

Given a Coxeter diagram, we can reconstruct the Coxeter matrix. All information about a Coxeter group is encoded in both the Coxeter matrix and the Coxeter diagram. If W is a Coxeter group (or (W, S) is a Coxeter system), whose structure is encoded in a Coxeter diagram Π, then we say that W (or (W, S)) has *type* Π.

Definition 1.2.0.4 (**Ranks**)**.** For a set X, we denote by $|X|$ its cardinality. We call $|I| = |S|$ the *rank* of

- the Coxeter system (W, S),
- the Coxeter group W with generating set S,
- the Coxeter diagram Π with index set I.

Remark 1.2.0.5. We can always view a Coxeter diagram Π for a Coxeter system (W, S) as a vertex-colored pre-graph if we take I as the set of vertex colors.

Example 1.2.0.6 (The symmetry group of a regular triangle)**.** We consider the following group:

$$W := \langle s, t \mid s^2 = t^2 = (st)^3 = \mathsf{id} \rangle = \{\mathsf{id}, s, t, st, ts, sts = tst\}$$

Then W is a Coxeter group where

$$\bullet\!\!-\!\!\bullet \text{ and } \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

are the respective reductively labeled Coxeter diagram and Coxeter matrix. Its type is referred to as A_2. Coxeter groups of rank 2 are sometimes called *dihedral groups* and are exactly the *symmetry groups* of *regular polygons*.[1] An element of a symmetry group is also called a *symmetry*. These include *rotations* and *reflections*. We used many words in the last sentences that we did not define yet. We do not need to know the details about these concepts at the moment. Heuristically, they are what one would expect. We will talk about reflections and finite reflection groups more formally in Section 1.5. For now, we would just like to give some rough context and intuition for this example. It turns out that the group W that we defined here is the symmetry group of a regular triangle, and we will describe how we can see that in the following.

First, we fix some arbitrary regular triangle Δ in Euclidean space and call its vertices a_1, a_2, and a_3. Let m_i be the midpoint (with respect to Euclidean distance) of the line containing a_i and a_{i+1}, for $i \in \mathbb{Z}/3$. Connect each m_i with the vertex of Δ that m_i is not collinear with (that is a_{i+2}). Then all lines $m_i a_{i+2}$ intersect in a point m, which has equal distance to each a_i. We can see a division of Δ into six subtriangles and call that the *barycentric subdivision* of Δ (cf. Fig. 1).

Note that the interior angles of Δ are equal to $\pi/3 = 60°$. The interior angles of each triangle of the barycentric subdivision of Δ are 30° at a_i, 60° at m, and 90°

[1] By *regular polygons* we mean ordinary n-gons, for $n \in \mathbb{N}_{\geq 3}$, in Euclidean space, for which all edges have the same Euclidean length and all vertex angles are equal.

at m_i. We choose two of the new lines (say a_1m_2 and m_1a_3) and denote by s and t the reflections of Δ along these lines, respectively (cf. Fig. 2). We claim that we can write every symmetry of Δ as a product of s and t, and obtain the group W, if we consider the set of all such symmetries with composition as the group operation.

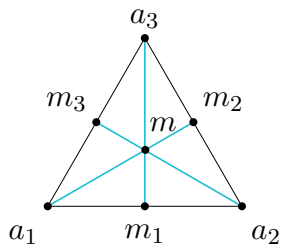


FIG. 1. A regular triangle and its barycentric subdivision.

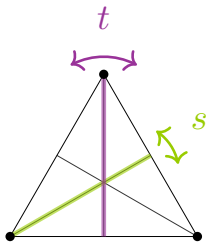


FIG. 2. The reflections s and t.

For a map that interchanges two vertices a_i and a_j of Δ, while fixing the others, we will write $(i, j) = (j, i)$ for $i \neq j$, with $i, j \in 1, 2, 3$. In this notation, we can write s as $(2, 3)$ and t as $(1, 2)$.

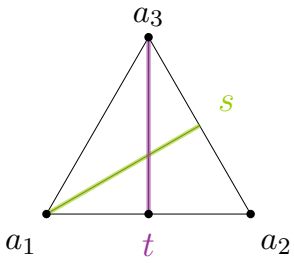


FIG. 3. The regular triangle Δ.

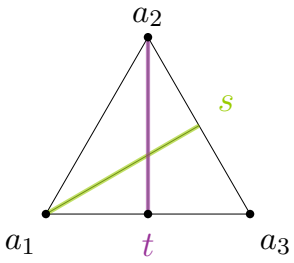


FIG. 4. The image of Δ under s.

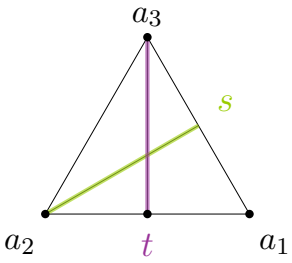


FIG. 5. The image of Δ under t.

First, let us consider the compositions st and sts. The reflection s interchanges a_2 and a_3, and then t interchanges a_1 and a_3 (cf. Fig. 6). If we apply s once more, then we interchange a_1 and a_2 (cf. Fig. 7).

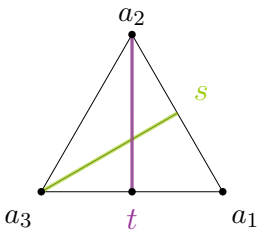


FIG. 6. The image of Δ under st.

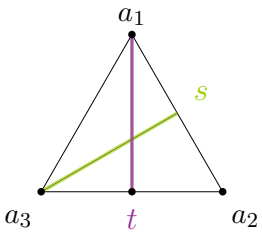


FIG. 7. The image of Δ under sts.

We can see that st corresponds to a counterclockwise rotation by $2\pi/3 = 120°$. Similarly, ts corresponds to a clockwise rotation by $2\pi/3 = 120°$. Note that applying the

rotation st and then the reflection s results in the same image as first applying the reflection t and then the rotation st to Δ. Thus, $sts = tst$ (cf. Fig. 9).

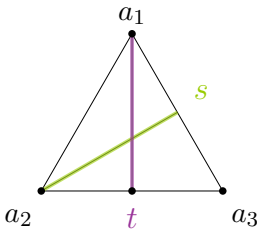


FIG. 8. The image of Δ under ts.

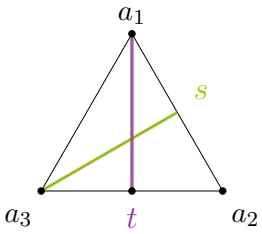


FIG. 9. The image of Δ under tst.

It should now be quite clear — or at least straightforward to verify — that every symmetry can be written as a product of s and t, and moreover, that the relations $s^2 = t^2 = (st)^3 = \mathsf{id}$ hold.

Fact 1.2.0.7. For $n \in \mathbb{N}$, groups of the form

$$W := \langle s, t \mid s^2 = t^2 = (st)^n = \mathsf{id} \rangle,$$

are precisely the symmetry groups of the regular n-gons.

Next, we introduce some terminology that will be used repeatedly throughout this thesis.

Definition 1.2.0.8 (**Simply-laced**)**.** We call a reductively labeled Coxeter diagram *simply-laced* if it contains neither labeled edges nor double edges. Equivalently, a fully labeled Coxeter diagram is simply-laced if it does not contain any edge with a label greater than 3, or, in the unlabeled version, if it does not contain double or triple bonds.

Definition 1.2.0.9 (**Spherical**)**.** A Coxeter diagram or Coxeter system is called *spherical* if the corresponding Coxeter group is finite.

Definition 1.2.0.10 (**Irreducible**)**.** A Coxeter group or Coxeter system is called *irreducible* if the corresponding Coxeter diagram is connected.

As mentioned above, the finite Coxeter groups were classified in [**Cox35**]. We will list all finite irreducible Coxeter groups in Table 2, noting that every finite Coxeter group is a direct product of finite irreducible ones.

Another class of Coxeter groups that is classified are the *affine irreducible Coxeter groups*. We only need them in Definition & Facts 2.3.0.8 for defining the *polar types* of a building, and they will not be relevant in this thesis at any other point. We present the diagrams in Table 3, and we provide more explanation about affine Coxeter groups in Note 1.5.0.26, after having introduced root systems.

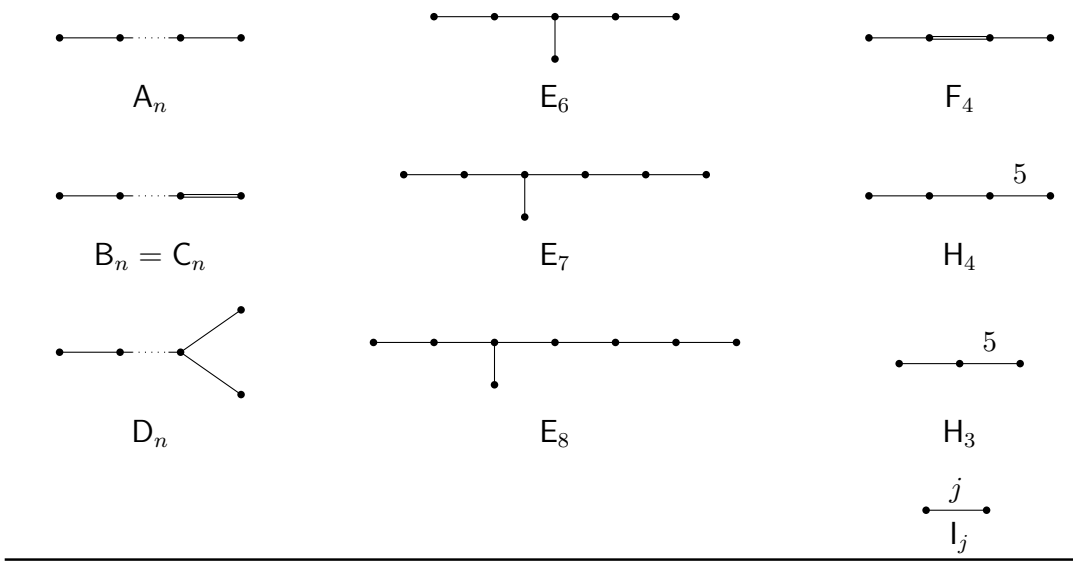


TABLE 2. The finite irreducible Coxeter groups (cf. [**Cox35**]).

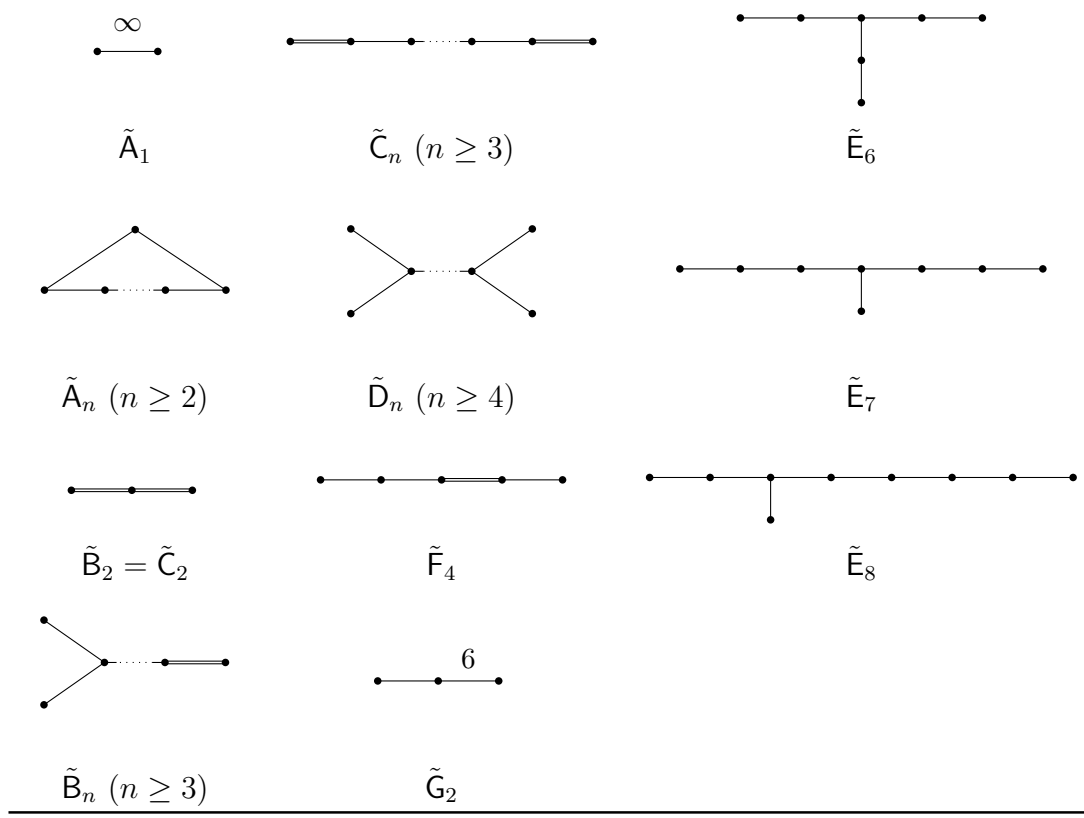


TABLE 3. The irreducible affine Coxeter groups (cf. [**Cox35**]).

1.2.1. Symmetric Groups

In this subsection we describe the isomorphism between Coxeter groups of type A_n for $n \geq 2$, and *permutation groups* or *symmetric groups* on a set of cardinality n.

Definition 1.2.1.11 (**Group actions**)**.** Let G be a group and let X be a non-empty set. A *group action of G on X* is a map $G \times X \to X$, $(g, x) \mapsto gx$, such that

(i) $\mathsf{id}x = x$ for all $x \in X$, and
(ii) $g(hx) = (gh)x$ for all $x \in X$, $g, h \in G$.

If G is a group and X is a set such that there exists a group action of G on X, then we also say that G *acts on* X.

Definition 1.2.1.12 (**Symmetric groups**, **permutations**)**.** Let X be a finite, non-empty set. We define

$$\mathsf{Sym}(X) := \{\alpha \colon X \to X \mid \alpha \text{ bijective}\}$$

and call it the *symmetric group on* X. If $n = |X|$, then we also write $\mathsf{Sym}(n)$ instead of $\mathsf{Sym}(X)$. The elements of $\mathsf{Sym}(X)$ are called *permutations*. A *transposition* is a permutation that exchanges two elements and fixes all others. For example, let $a, b \in X$ with $a \neq b$; then

$$a \mapsto b,\ b \mapsto a,\ c \mapsto c \text{ for all c } \in X \setminus \{a, b\}$$

is a transposition, which we denote by (a, b). If G is a subgroup of a symmetric group, then we also call G a *permutation group*.

Note 1.2.1.13. For the following we refer to [**CM17**, pp. 6–7].

- It is possible to define symmetric groups on infinite sets, but these are not relevant here.
- Every permutation can be written as a product of transpositions. In fact, a generating set for $\mathsf{Sym}(n)$ is given by $\{(i, i+1) | i \in 1, \ldots, n-1\}$.
- The transpositions $(i, i+1)$ are sometimes denoted by σ_i.
- Let X be a finite, non-empty set and let $G = \mathsf{Sym}(X)$. For each $g \in G$, let φ_g denote the map $x \mapsto g(x)$. Define $\alpha \colon \mathsf{Sym}(X) \times X \to X$ by

  $$\alpha(g, x) = \varphi_g(x).$$

  Then α is a group action, and hence $\mathsf{Sym}(X)$ acts on X.

Note 1.2.1.14. Let $I := \{1, \ldots, n-1\}$. If we view $\mathsf{Sym}(n)$ as the group generated by $\{\sigma_i = (i, i+1) \mid i \in I\}$, then we can observe the following relations:

- $\sigma_i^2 = \mathsf{id}$
- $\sigma_i \sigma_j = \sigma_j \sigma_i$, for $|i - j| > 1$
- $(\sigma_i \sigma_{i+1})^3 = \mathsf{id}$

In words: transpositions have order 2; any two non-consecutive transpositions commute; while multiplying two consecutive transpositions yields a 3-cycle. If we denote

the order of $\sigma_i\sigma_j$ by m_{ij} for all $i, j \in I$, then we can write

$$\mathsf{Sym}(n) = \langle \{\sigma_i \mid i \in I\} \mid (\sigma_i\sigma_j)^{m_{ij}} = \mathsf{id} \rangle.$$

Thus $\mathsf{Sym}(n)$ is isomorphic to a Coxeter group of type A_n.

Definition 1.2.1.15 (Permutation equivalence). Let $G_i \subseteq \mathsf{Sym}(X_i)$ for $i \in 1, 2$ be two permutation groups. Then G_1 and G_2 are called *permutation equivalent* if there exists a bijection $\alpha\colon X_1 \to X_2$ and an isomorphism $f\colon G_1 \to G_2$ such that $\alpha(gx) = f(g)\alpha(x)$, for all $g \in G_1$ and $x \in X_1$.

1.3. Cayley Graphs

There are several different ways to define Cayley graphs. We adopt a semi-restrictive definition, since only this form will be needed in the course of this thesis.

Definition 1.3.0.1. Let G be a finitely generated group and let $S \subseteq G$ be a finite generating set of G. The *Cayley graph* $\mathsf{Cay}(G, S)$ of G with respect to S is the following edge-colored graph:

- The vertex set V consists of the elements of G.
- Two vertices $g, h \in G$ are connected by an edge if $h = gs$ for some $s \in S$. We do not draw multiple edges. Thus the edge set is the set of unordered pairs $\{g, gs\}$ with $g \in G,\ s \in S$, and we may also refer to an edge as the pair $(g, gs) = (gs, g)$.
- Each edge (g, gs) is assigned the color s.

Note that $\mathsf{Cay}(G, S)$ is always connected.

Example 1.3.0.2. We consider the group W with presentation $\langle g, h \mid g^2 = h^2 = (gh)^3 = \mathsf{id} \rangle$. Then the set of elements of W is $\{\mathsf{id}, g, h, gh, hg, hgh = ghg\}$. The Cayley graph $\mathsf{Cay}(W, S)$, where $S = \{g, h\}$, and where we associate g with the color green and h with the color purple, is depicted below, in Fig. 10.

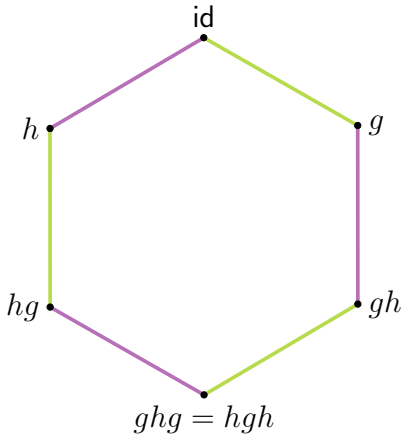


Fig. 10. The Cayley graph for the symmetry group of a regular triangle.

1.4. Complexes

The goal of this section is to introduce Coxeter complexes. To do so, we first need to clarify what we mean by a *simplicial complex*. We assume that the reader is already familiar with the concept, but we will nevertheless recall it briefly.

As described in [**Hat02**, §2.1], all closed surfaces can be constructed from triangles by identifying edges. Simplicial complexes generalize this idea to higher dimensions, using the notion of an *n-simplex*, the n-dimensional analogue of a triangle. A vertex is a 0-simplex, an edge a 1-simplex, a triangle a 2-simplex, and a tetrahedron a 3-simplex. In general, an n-simplex consists of $n+1$ vertices and can be built from $(n-1)$-simplices (for $n > 1$).

Formally, a *simplicial complex* consists of a set X_0 of vertices, together with sets X_n of n-simplices, where each n-simplex is an $(n+1)$-element subset of X_0, such that each $(k+1)$-element subset of the vertices of an n-simplex in X_n is a k-simplex in X_k (cf. [**Hat02**, p. 107]).

The *dimension of an n-simplex* is n. If the highest dimension of any simplex in a simplicial complex Δ is n, then we call n the *dimension of* Δ. If Δ is n-dimensional, its $(n-1)$-simplices are said to have *codimension* 1, while the n-simplices are called *maximal simplices* or *facets*. For a simplex A of dimension m, the $(m-1)$-simplices contained in A are called the *faces* of A, and the union of all these faces is the *boundary* of A.

Definition 1.4.0.1 (**Morphisms between complexes**)**.**
A *morphism* $\varphi\colon \Delta \longrightarrow \Delta'$ between two simplicial complexes Δ and Δ' is a map of the underlying sets such that every simplex of Δ is mapped bijectively onto a simplex of Δ'. That is, the restriction of φ to each simplex is bijective, although φ itself need not be (cf. [**Tit74**, §1.1]).

Definition 1.4.0.2 (**Thick**, **thin**)**.**

- A simplicial complex is called *thick* if every codimension-1 simplex is a face of at least three maximal simplices.
- A simplicial complex is called *thin* if every codimension-1 simplex is a face of exactly two maximal simplices.

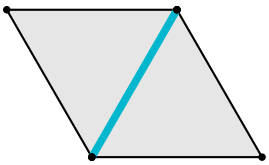
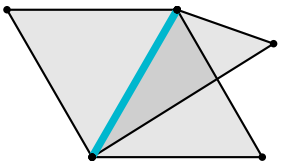
Thin:	Thick:
In a thin simplicial complex of dimension 2, every edge is contained in exactly two 2-simplices.	In a thick simplicial complex of dimension 2, every edge is contained in at least three 2-simplices.
	

Fig. 11. A visualization for thickness and thinness in simplicial complexes of dim. 2.

1.4.1. Coxeter Complexes

In this subsection, we construct a simplicial complex for a given Coxeter system, called the *Coxeter complex*. We will not be overly formal here and refer to [**AB08**] (in particular §3.1) for details. In the following, for a group $\langle\{s_j \mid j \in J\}\rangle$, we will also write $\langle s_j \mid j \in J\rangle$ for short.

Definition 1.4.1.3 (**Parabolic subgroups**, **standard cosets**)**.**
Let $|I| \geq 2$ and let W be the Coxeter group with generating set $S = \{s_i \mid i \in I\}$. For $J \subsetneq I$, we call

$$W_J := \langle s_j \mid j \in J\rangle$$

a *parabolic subgroup of* W.

A parabolic subgroup W_J is called *maximal* if $|J| = |I| - 1$, and *minimal* if $J = \varnothing$. A *standard coset in* W is a coset of the form wW_J with $w \in W$.

We now construct a simplicial complex associated with a Coxeter system (W, S) with index set I. We proceed as follows:

- Draw a vertex for each coset of a maximal parabolic subgroup (if two cosets coincide, we draw only one vertex).
- Draw a maximal simplex (called a *chamber*) for each coset of the minimal parabolic subgroup. (Since every group element $w \in W$ gives such a coset, each chamber corresponds to exactly one element of W.)
- Add simplices corresponding to the cosets of the remaining parabolic subgroups in the natural way.

The resulting simplicial complex is denoted by $\Sigma(W, S)$ and is called the *Coxeter complex associated with* (W, S).

Example 1.4.1.4 (A Coxeter complex for A_2)**.** Let $S = \{g, h\}$, and

$$W = \langle g, h \mid g^2 = h^2 = (gh)^3 = \mathsf{id}\rangle = \{\mathsf{id}, g, h, gh, hg, hgh = ghg\}.$$

Define $J_g := \{g\}$ and $J_h := \{h\}$. Then

$$W_{J_g} = \langle g\rangle = \{\mathsf{id}, g\} \quad \text{and} \quad W_{J_h} = \langle h\rangle = \{\mathsf{id}, h\}$$

are the maximal parabolic subgroups. The minimal parabolic subgroup is $\langle\varnothing\rangle$.

The cosets of the minimal parabolic subgroup are:

$$\begin{aligned}
\mathsf{id}\langle\varnothing\rangle &= \{\mathsf{id}\},\\
g\langle\varnothing\rangle &= \{g\},\\
h\langle\varnothing\rangle &= \{h\},\\
hg\langle\varnothing\rangle &= \{hg\},\\
gh\langle\varnothing\rangle &= \{gh\},\\
hgh\langle\varnothing\rangle &= \{hgh\}.
\end{aligned}$$

These correspond to chambers. The cosets of the maximal parabolic subgroups are:

$$\begin{gathered}
\mathsf{id}W_{J_g} = \mathsf{id}\langle g\rangle = \mathsf{id}\{\mathsf{id}, g\} = \{\mathsf{id}, g\},\\
gW_{J_g} = g\langle g\rangle = g\{\mathsf{id}, g\} = \{g, \mathsf{id}\},\\
hW_{J_g} = h\langle g\rangle = h\{\mathsf{id}, g\} = \{h, hg\},\\
hgW_{J_g} = hg\langle g\rangle = hg\{\mathsf{id}, g\} = \{hg, h\},\\
ghW_{J_g} = gh\langle g\rangle = gh\{\mathsf{id}, g\} = \{gh, ghg\},\\
hghW_{J_g} = hgh\langle g\rangle = hgh\{\mathsf{id}, g\} = \{hgh, hghg = gh\}.
\end{gathered}$$

and

$$\begin{gathered}
\mathsf{id}W_{J_h} = \mathsf{id}\langle h\rangle = \mathsf{id}\{\mathsf{id}, h\} = \{\mathsf{id}, h\},\\
gW_{J_h} = g\langle h\rangle = g\{\mathsf{id}, h\} = \{g, gh\},\\
hW_{J_h} = h\langle h\rangle = h\{\mathsf{id}, h\} = \{h, \mathsf{id}\},\\
hgW_{J_h} = hg\langle h\rangle = hg\{\mathsf{id}, h\} = \{hg, hgh\},\\
ghW_{J_h} = gh\langle h\rangle = gh\{\mathsf{id}, h\} = \{gh, g\},\\
hghW_{J_h} = hgh\langle h\rangle = hgh\{\mathsf{id}, h\} = \{hgh, hg\}.
\end{gathered}$$

Notice that $\mathsf{id}W_{J_g} = gW_{J_g}$, $hW_{J_g} = hgW_{J_g}$, $ghW_{J_g} = hghW_{J_g}$, $\mathsf{id}W_{J_h} = hW_{J_h}$, $gW_{J_h} = ghW_{J_h}$ and $hgW_{J_h} = hghW_{J_h}$. In total we get 6 different cosets of maximal parabolic subgroups. These correspond to vertices.

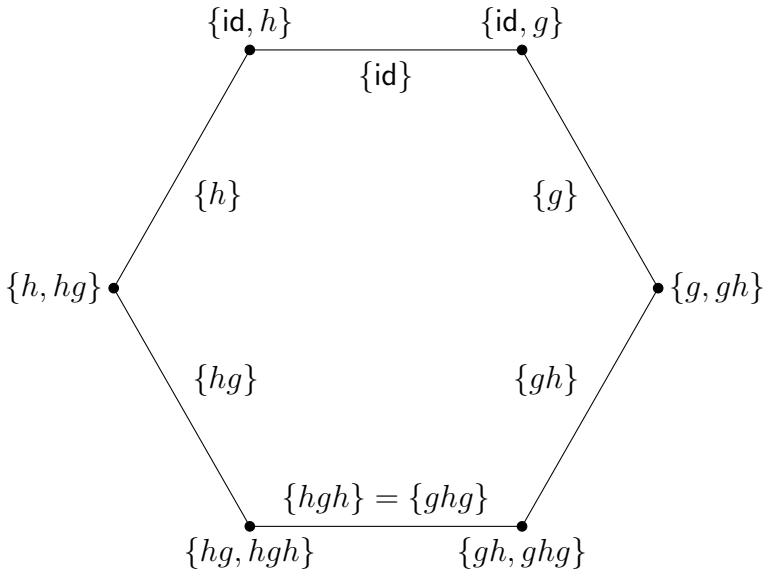


FIG. 12. The Coxeter complex for the symmetry group of a regular triangle.

1.5. Groups from Roots

We now turn to root systems and finite reflection groups.[2] Given a Euclidean space, we can reflect along its hyperplanes. Our goal is to construct finite groups generated by such reflections and to associate so-called *roots* with them. The groups we are interested in are called *Weyl groups*. As we will see, every Weyl group is a Coxeter group, but not every finite Coxeter group is a Weyl group — although most of them are. This explains the motivation for this section. We will use the description via root systems for certain finite Coxeter groups in Appendix C. At the end of this section, we will introduce *Dynkin diagrams*, which are similar to Coxeter diagrams but contain slightly more information. For now, we begin with some basic definitions.

Let V be a finite-dimensional vector space over $\mathbb{R}$ equipped with a bilinear form $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{R}$ such that for all $x, y, z \in V$ and $a, b \in \mathbb{R}$, the following hold:

(i) For $x \neq 0$, one has $\langle x, x \rangle \in \mathbb{R}$ and $\langle x, x \rangle > 0$.
(ii) $\langle x, y \rangle = \langle y, x \rangle$.
(iii) $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$.
(iv) $\langle x, ay + bz \rangle = a\langle x, y \rangle + b\langle x, z \rangle$.

Then V is called a *Euclidean vector space* and $\langle \cdot, \cdot \rangle$ is called a *scalar product*, or an *inner product* on V. For any vector $v \in V$, we define the *length of* v as

$$\|v\| := \sqrt{\langle v, v \rangle}.$$

Notation 1.5.0.1. We always mean the standard inner product, whenever we write $\langle \cdot, \cdot \rangle$ in this subsection. If we talk about a Euclidean vector space, we always consider it equipped with $\langle \cdot, \cdot \rangle$.

A *hyperplane* of V is a subspace of codimension 1. The *reflection* along a hyperplane H is the linear transformation $s_H \colon V \to V$ that acts as the identity on H and as multiplication by -1 on the *orthogonal complement* of H:

$$H^\perp := \{v \in V \mid \langle u, v \rangle = 0 \text{ for all } u \in H\}.$$

If $\alpha \in V$ is a non-zero element of $H^\perp$ with $H = \alpha^\perp$, we also write s_α instead of s_H.

[2] We chose the section title 'Groups from Roots' instead of 'Finite Reflection Groups' or something similar, so that all sections in this chapter begin with a 'G' or a 'C'. It looks nicer in the table of contents.

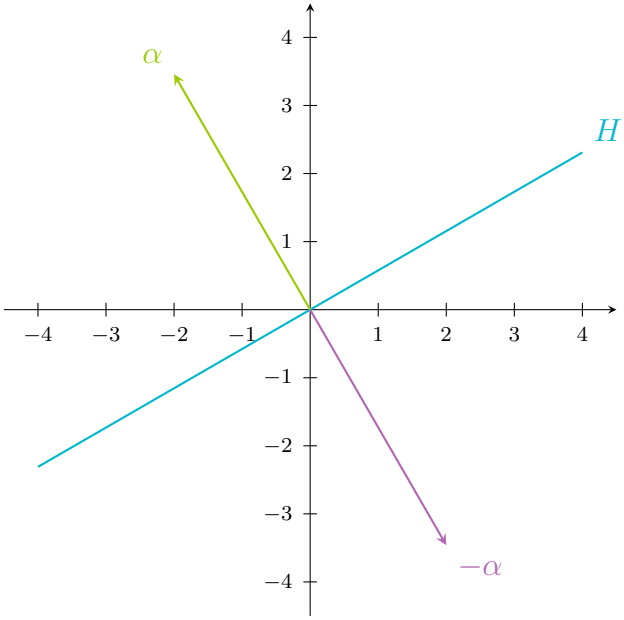


FIG. 13. A visualization of a hyperplane H in a 2-dimensional Euclidean vector space with scalar product, and an element α of the orthogonal complement of H and its image $-\alpha$ under the reflection along H.

For this subsection our main references are [**AB08**], [**Car72**], [**Hum72**], [**Hum90**], [**Tit77**], and the thesis *Root Graded Groups* by Torben Wiedemann [**Wie24**].

Suppose we are given such an α and an element $x \in V$. Then we can write $x = h + \lambda\alpha$ with $h \in H$ and $\lambda \in \mathbb{R}$. Using the properties of the inner product, we obtain the following (cf. [**AB08**, §1.1]):

$$
\begin{aligned}
h + \lambda\alpha &= x && \text{(We take the inner product of both sides with } \alpha.\text{)} \\
\langle h + \lambda\alpha, \alpha\rangle &= \langle x, \alpha\rangle && \text{(We use property (iii).)} \\
\langle h, \alpha\rangle + \lambda\langle \alpha, \alpha\rangle &= \langle x, \alpha\rangle && \text{(Since } h \in H,\ \alpha \in H^{\perp},\ \langle h, \alpha\rangle = 0\text{)} \\
\lambda\langle \alpha, \alpha\rangle &= \langle x, \alpha\rangle \\
\lambda &= {}^{\langle x,\alpha\rangle}\!/_{\langle \alpha,\alpha\rangle}
\end{aligned}
$$

Furthermore, since $h + \lambda\alpha = x$, we also have $h = x - \lambda\alpha$. Thus we can write:[3]

$$
\begin{aligned}
s_\alpha(x) &= s_\alpha(h + \lambda\alpha) = h - \lambda\alpha = (x - \lambda\alpha) - \lambda\alpha = x - 2(\lambda\alpha) \\
s_\alpha(x) &= x - 2({}^{\langle x,\alpha\rangle}\!/_{\langle \alpha,\alpha\rangle})\alpha
\end{aligned}
$$

Definition 1.5.0.2 (**Finite reflection groups**)**.** Let V be a Euclidean vector space. A *finite reflection group* is a finite group W of invertible linear transformations of V, generated by reflections s_H , where H ranges over a set of hyperplanes.

Example 1.5.0.3. We have already encountered a reflection group in Example 1.2.0.6: Let V be a 2-dimensional Euclidean vector space. Then the hyperplanes in V are lines. Choose two lines in V that intersect at an angle of ${}^{\pi}\!/_{3}$ (or, more generally, at an angle of ${}^{\pi}\!/_{m}$ for $m \in \mathbb{N}_{\geq 2}$, cf. Fact 1.2.0.7). Let s and t be the corresponding reflections. Then the group generated by s and t is a finite reflection

[3] $s_\alpha(h + \lambda\alpha) = h - \lambda\alpha$, because s_α is a reflection. Since $h = x - \lambda\alpha$, this equals $(x - \lambda\alpha) - \lambda\alpha$.

group: the symmetry group of a regular triangle (or, more generally, the symmetry group of a regular m-gon). For further examples, see [**AB08**, §1.2].

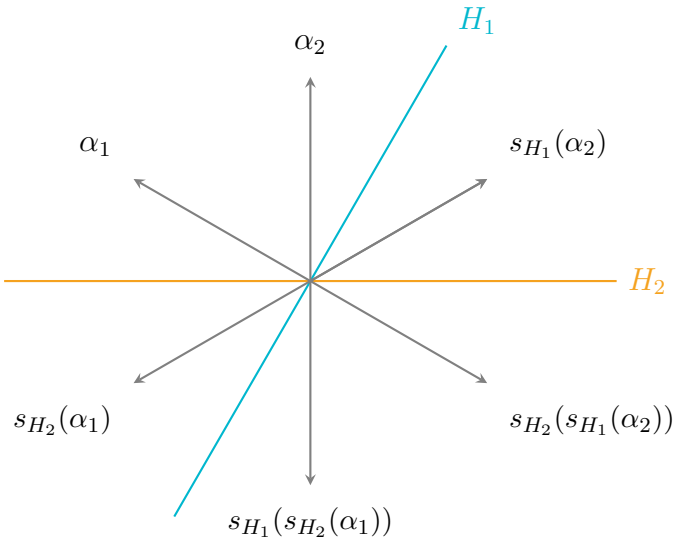


FIG. 14. A visualization of two hyperplane H_1 and H_2 intersecting in an angle of $\pi/3$ in a 2-dimensional Euclidean vector space with scalar product. Drawn in are elements α_i, for $i = 1, 2$, of the orthogonal complement of H_i and some images under the reflections along H_i.

Definition 1.5.0.4 (**Root systems**, **roots**, **opposite roots**)**.** Let V be a Euclidean vector space and let ϕ be a finite non-empty set of non-zero vectors in V. Then we call ϕ a *root system* if the following axioms are satisfied:

(i) ϕ spans V.
(ii) If $\alpha, \beta \in \phi$, then $s_\alpha(\beta) \in \phi$.
(iii) If $\alpha, \beta \in \phi$, then $2^{\langle \alpha, \beta \rangle}/_{\langle \alpha, \alpha \rangle}$ is an integer.
(iv) If $\alpha, \lambda\alpha \in \phi$ with $\lambda \in \mathbb{R}$, then $\lambda = \pm 1$.

Note that for $\alpha \in \phi$, we have

$$s_\alpha(\alpha) = \alpha - 2\frac{\langle \alpha, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha = -\alpha,$$

and therefore $-\alpha \in \phi$. We call the elements of ϕ the *roots* of ϕ, and for every $\alpha \in \phi$, we call $-\alpha$ the *opposite root* of α in ϕ.

Note 1.5.0.5. What we defined as a *root system* is often referred to as a *crystallographic reduced root system*, because of conditions (iii) and (iv), respectively (cf. [**Hum72**, Ch. III, §9.2, p. 43], [**Wie24**, §1.2; in particular Definitions 1.2.10 & 1.2.27]).

Example 1.5.0.6. Let $V \cong \mathbb{R}^2$ with scalar product. Define

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathscr{B} = \{x, y\}.$$

Then $\mathscr{B}$ is the standard basis of V. The scalar product $\langle a, b \rangle$ of two vectors a and b can be expressed as the product of the length of a with the length of b and cos of the angle between a and b. For a vector a, we denote its length by $\|a\|$. Whenever

two vectors a and b are orthogonal, the angle between them is $\pi/2 = 90°$, and since $\cos(\pi/2) = 0$, their scalar product is 0. We set

$$-x = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad -y = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \phi = \{x, y, -x, -y\}.$$

We then compute:

$$\begin{aligned} s_y(x) &= x - 2\frac{\langle x, y\rangle}{\langle y, y\rangle} y = x - 2\frac{0}{\|y\|^2 \cos(0)} y = x, \\ s_x(x) &= x - 2\frac{\langle x, x\rangle}{\langle x, x\rangle} x = x - 2\frac{\|x\|^2 \cos(0)}{\|x\|^2 \cos(0)} x = x - 2x = -x, \\ &\text{and similarly } s_x(y) = y \text{ and } s_y(y) = -y. \end{aligned}$$

Thus ϕ is a root system in V.

Definition 1.5.0.7. A root system ϕ is called *indecomposable* (cf. [**Car72**, §3.5, p. 41]) or *irreducible* (cf. [**Wie24**, Definition 1.2.47, Remark 1.2.48]) if it cannot be decomposed into two non-empty complementary subsets ϕ_1 and ϕ_2 such that the inner product $\langle \alpha_1, \alpha_2 \rangle$ equals 0 for all $\alpha_1 \in \phi_1$ and $\alpha_2 \in \phi_2$.

Two root systems ϕ_1 and ϕ_2 are *isomorphic* if there exists a bijection $\varphi\colon \phi_1 \to \phi_2$ such that

$$\langle \varphi(\alpha), \varphi(\beta) \rangle = \lambda \langle \alpha, \beta \rangle$$

for all $\alpha, \beta \in \phi_1$, where λ is a positive real number independent of α and β (cf. [**Car72**, §3.5]).

Definition 1.5.0.8 (**Weyl groups**)**.** Let V be a Euclidean vector space and ϕ a root system in V. The group generated by the reflections s_α for $\alpha \in \phi$,

$$W_\phi := \langle \{ s_\alpha \mid \alpha \in \phi \} \rangle,$$

is called the *Weyl group of* ϕ.

Lemma 1.5.0.9. *Let V be a Euclidean vector space, ϕ a root system in V, and W_ϕ the associated Weyl group. Then W_ϕ acts faithfully on the subset ϕ of V and is a finite reflection group.*

Reference. See [**Car72**, §2.1, pp. 12–13] and [**Hum90**, Part I, §1.2, pp. 6–7]. □

Notation & Definition 1.5.0.10 (**Co-roots**, **dual root systems**)**.** For a root α, instead of $2\alpha/\langle \alpha, \alpha \rangle$ we also write $\alpha^\vee$. Using this notation, we obtain

$$\begin{aligned} s_\alpha(x) &= x - 2\frac{\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha \\ &= x - \alpha^\vee \langle x, \alpha \rangle, \\ \langle \alpha^\vee, \alpha \rangle &= 2. \end{aligned}$$

For a root α, $\alpha^\vee$ is called the *co-root* of α. The set of all co-roots of the roots in a root system ϕ forms a root system as well, called the *dual root system* of ϕ (or sometimes the *inverse root system* of ϕ), and is denoted by $\phi^\vee$ (cf. [**Car72**, Proposition 3.6.1]).[4]

[4]Carter denotes the co-root of a root r by h_r and writes ϕ^* instead of $\phi^\vee$, see [**Car72**, p. 49].

Construction 1.5.0.11. We now describe a way to 'divide' our vector space V into parts. For this, let V^+ be a subset of V satisfying the following conditions:

(i) If $v \in V^+$ and $\lambda > 0$, then $\lambda v \in V^+$.
(ii) If $v_1, v_2 \in V^+$, then $v_1 + v_2 \in V^+$.
(iii) For each $v \in V$, either $v \in V^+$, $-v \in V^+$, or $v = 0$.

Such a subset always exists. For example, if $\mathscr{B} = \{e_1, e_2, \ldots, e_n\}$ is a basis of V, then

$$\Big\{ \sum_{i=1}^{n} \lambda_i e_i \;\Big|\; e_i \in \mathscr{B},\ \lambda_i \in \mathbb{R},\ \lambda_i > 0 \Big\}$$

is an instance of a set $V^+ \subset V$.

If we fix a set $V^+ \subset V$, we can define an order $>$ by

$$v_1 > v_2 \iff v_1 - v_2 \in V^+, \quad v_1, v_2 \in V.$$

This is a total order on V compatible with addition and scalar multiplication by positive real numbers.

A subset of ϕ is called a *positive system of roots* if it has the form $\phi \cap V^+$ for some V^+ as described above.

Definition & Fact 1.5.0.12 (Fundamental systems). The set ϕ contains a subset Π satisfying the following conditions:

(i) Π is linearly independent.
(ii) Every root in ϕ is a linear combination of roots in Π with coefficients that are either all non-negative or all non-positive.

Such a subset Π is called a *fundamental system of roots*.

Proposition 1.5.0.13. *Every positive system of roots in ϕ contains exactly one fundamental system of roots.*

Reference. This is [**Car72**, Proposition 2.1.2 and Proposition 2.1.3]. □

Notation 1.5.0.14. In the following, V will always denote a Euclidean vector space, ϕ a root system in V, and ϕ^+ a positive system of roots in ϕ, corresponding to a fundamental system Π of ϕ.

Definition 1.5.0.15 (Positive roots, negative roots, simple roots). We call the roots contained in ϕ^+ *positive roots* and the roots in $\phi \setminus \phi^+$ *negative roots*. The set of negative roots is denoted by ϕ^-. Roots contained in Π are called *fundamental roots* or *simple roots*.

Proposition 1.5.0.16. *Each root in ϕ is a linear combination of roots in Π with integer coefficients.*

Reference. This is [**Car72**, Proposition 2.1.6]. □

Definition & Facts 1.5.0.17 (Chambers, walls, fundamental chamber). Let ϕ be an irreducible root system. Let H_α be the hyperplane in V orthogonal to α, for

each root $\alpha \in \phi$. The connected components of $V \setminus \bigcup_{\alpha\in\phi} H_\alpha$ are called *chambers*. Let C be any chamber in V and ∂C the boundary of C. Then the hyperplanes

$$H_\alpha, \text{ such that } H_\alpha \cap \partial C \text{ is not contained in any proper subspace of } H_\alpha,$$

are called the *walls of* C. If $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a fundamental system in ϕ, then the set C of vectors $x \in V$ with $\langle \alpha_i, x\rangle > 0$, for $i = 1, 2, \ldots, n$, is a chamber. The walls of C are $H_{\alpha_1}, H_{\alpha_2}, \ldots, H_{\alpha_n}$ (cf. [**Car72**, p. 21]). The roots orthogonal to the walls of a chamber and pointing into the chamber, form a fundamental system. Moreover, every fundamental system arises in this way from some chamber (cf. [**Car72**, Proposition 2.3.1]). If we are considering a fixed fundamental system, the corresponding chamber will be called the *fundamental chamber*.

Definition & Fact 1.5.0.18. As we have stated, we can write every root $\alpha \in \phi$ as a linear combination $\sum_{\alpha_i\in\Pi} \lambda_i\alpha_i$; we define the *height of a root* α as the sum over all λ_i that appear in that expression. For every positive root that is not fundamental, there exists another positive root in ϕ of smaller height; the positive roots of minimal height are the fundamental roots. Fundamental roots have height 1 (cf. [**Car72**, p. 16]). We call the map that maps every root to its height the *height function*.

Note 1.5.0.19. We noted before that we can always choose a fundamental system of roots Π, for some root system ϕ in some Euclidean vector space V, and that we can then write every root $\alpha \in \phi$ as $\sum_{\alpha_i\in\Pi} \lambda_i\alpha_i$, where $\lambda_i \in \mathbb{Z}$ and $\alpha_i \in \Pi$. Furthermore, every root system spans V, and thus, for every $x \in V$, we can write x as $\sum_{j=1}^{n} \eta_j\beta_j$, where $\eta_j \in \mathbb{R}$ and $\beta_j \in \Pi$.

For every root α, we can consider the corresponding reflection $s_\alpha \in W_\phi$. For every $x \in V$, $s_\alpha(x)$ is given by $x - 2(\langle x,\alpha\rangle/\langle \alpha,\alpha\rangle)\alpha$. To understand s_α, we only need to check how s_α maps the fundamental roots of Π. Let β be a fundamental root, then we have

$$s_\alpha(\beta) = \beta - 2\frac{\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha,$$

and with Definition 1.5.0.4 (ii), $2\langle \beta,\alpha\rangle/\langle \alpha,\alpha\rangle$ is an integer for every i.

Definition 1.5.0.20 (**Cartan matrix**)**.** For any fundamental system of roots Π, the matrix

$$\Big(\frac{2\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle}\Big)_{\alpha,\beta\in\Pi}$$

is called the *Cartan matrix* of Φ (with respect to Π) (cf. [**Wie24**, Definition 1.2.26]). The Cartan matrix does not depend on the choice of Π up to a permutation of the rows and columns.

Proposition 1.5.0.21.

(i) Each element of W_ϕ is represented by a matrix with integer coefficients,
(ii) every root in ϕ is the image of some root in Π under some element of W_ϕ,
(iii) and W_ϕ is generated by the reflections s_α, for $\alpha \in \Pi$.

Reference. This is [**Car72**, Corollary 2.1.7 and Proposition 2.1.8]. □

Fact 1.5.0.22. Every Weyl group is a Coxeter group. Not all finite Coxeter groups are Weyl groups, but most of them are (cf. [**Bou68**], [**Car72**, p. 25]). Finite reflection groups have been classified (cf. [**AB08**, §1.3], [**Bou68**], [**Hum90**]).

It turns out that there are very limited options for the integers $a_{\alpha,\beta} := 2^{\langle\alpha,\beta\rangle}/_{\langle\alpha,\alpha\rangle}$, for roots α, β of an irreducible root system ϕ.

Lemma 1.5.0.23. *Let α and β be elements of an irreducible root system ϕ. Then*

$$a_{\alpha,\beta} \in \{0, \pm 1, \pm 2, \pm 3\}.$$

More specifically, up to permutation of α and β (such that $\|\beta\| \geq \|\alpha\|$), exactly one of the following occurs:

$a_{\alpha,\beta}$	$a_{\beta,\alpha}$	*Angle between α and β*	*Order of $s_\alpha s_\beta$*	$\|\beta\|/\|\alpha\|$
1	1	$\pi/3$	3	1
1	2	$\pi/4$	4	$\sqrt{2}$
1	3	$\pi/6$	6	$\sqrt{3}$
0	0	$\pi/2$	2	
-1	-1	$2\pi/3$	3	1
-1	-2	$3\pi/4$	4	$\sqrt{2}$
-1	-3	$5\pi/6$	6	$\sqrt{3}$
2	2	0	1	1
-2	-2	π	1	1

Reference. This is [**Bou68**, Ch. VI, §1, 3.] or [**Jan19**, Lemma 7.1.2]. □

This implies there can be at most two different root lengths in an irreducible root system; if, for example, a root system ϕ contains roots α_1, α_2, α_3 such that the angle between α_1 and α_2 is π/n and the angle between α_1 and α_3 is π/m with $n \neq m$ and $n, m \in \{2, 3, 4, 6\}$, then ϕ cannot be irreducible.

Thus, given the length $\|\alpha\|$ of a root α in an irreducible root system ϕ, any other root β in ϕ can only have

(i) the same length as α, or
(ii) either $\|\beta\| = \sqrt{2}\|\alpha\|$ or $\|\beta\| = \sqrt{3}\|\alpha\|$.

Note that if $a_{\alpha,\beta} = a_{\beta,\alpha} = 2$, then $\alpha = \beta$, and if $a_{\alpha,\beta} = a_{\beta,\alpha} = -2$, then $\alpha = -\beta$.

Furthermore, we have the following.

$a_{\alpha,\beta}a_{\beta,\alpha}$	Order of $s_\alpha s_\beta$
0	2
1	3
2	4
3	6

Definition & Fact 1.5.0.24 (**Dynkin diagrams**)**.** Let ϕ be an irreducible root system and Π a fundamental system in ϕ. If we draw

- a vertex for each element in Π,
- an edge between two vertices corresponding to roots α, and β in Π, whenever $a_{\alpha,\beta}a_{\beta,\alpha} \geq 1$,
- and label that edge by the order of $s_\alpha s_\beta$, for $a_{\alpha,\beta}a_{\beta,\alpha} \geq 2$,

then we obtain precisely the fully labeled Coxeter diagram of the Weyl group W_ϕ. If we replace every edge labeled 4 by a double edge and omit the label, we obtain the reductively labeled version. If we furthermore replace every edge labeled 6 by a triple edge and omit the label, we obtain the unlabeled version of the Coxeter diagram.

If, in addition, we draw

- an arrow tip on an edge pointing from a vertex corresponding to a longer root to a vertex corresponding to a shorter root, if roots of different lengths exist in Π,

then the resulting graph is called the *Dynkin diagram* of ϕ (cf. [**Tit55**] and [**Tit57**, footnote 4]).

Fact 1.5.0.25. Every Dynkin diagram of a Weyl group of an irreducible root system is one of the ones in Table 4 below.

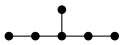
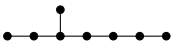
Type	Diagram	Type	Diagram
A_n		E_6	
B_n		E_7	
C_n		E_8	
D_n		F_4	
		G_2	

TABLE 4. Dynkin diagrams for Weyl groups of irreducible root systems.

To end this chapter, we provide some additional context for affine Coxeter groups.

Note 1.5.0.26 (**Affine Coxeter Groups**)**.** Let (W, S), $S = \{s_1, \ldots, s_n\}$, be a spherical Coxeter system of type X_n corresponding to the Weyl group of an irreducible crystallographic root system Φ in $\mathbb{R}^n$ and fundamental basis Π. Let α be the highest positive root and let r be the corresponding reflection. For each $s_i \in S$, let $m_{i,0} = m_{0,i}$ be the order of $s_i r$. Then $(\widetilde{W}, S \cup \{s_0\})$ is a Coxeter system of affine type $\widetilde{\mathsf{X}}_n$ if $\widetilde{W}$ has presentation $\langle s_0, s_1, \ldots, s_n \mid (s_i s_j)^{m_{i,j}} = s_i^2 = 1, \forall i, j, i \neq j \rangle$.

A representation as automorphism group of Euclidean space $\mathbb{R}^n$ is obtained by letting s_0 correspond to the reflection in $\mathbb{R}^n$ about a non-trivial affine translate (a translate not containing the origin) of the hyperplane perpendicular to the root α.

If, given an irreducible crystallographic root system, we build a diagram with the same rules as for the Dynkin diagram, but now using a fundamental system completed with the corresponding highest negative root, then we obtain one of the *extended Dynkin diagrams* in Table 5.

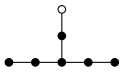
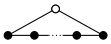
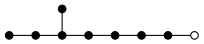
Type	Diagram	Type	Diagram
$\widetilde{\mathsf{A}}_1$		$\widetilde{\mathsf{E}}_6$	
$\widetilde{\mathsf{A}}_n$		$\widetilde{\mathsf{E}}_7$	
$\widetilde{\mathsf{B}}_n$		$\widetilde{\mathsf{E}}_8$	
$\widetilde{\mathsf{C}}_n$		$\widetilde{\mathsf{F}}_4$	
$\widetilde{\mathsf{D}}_n$		$\widetilde{\mathsf{G}}_2$	

TABLE 5. Extended Dynkin diagrams for Weyl groups of irreducible root systems.

2. Buildings

One can find many different ways of defining a *'building'* in the literature. However — as we have mentioned — there are two main approaches: one starts with an edge-colored graph, and one starts with a simplicial complex. Originally, Jacques Tits used the second approach when he gave the first-ever definition of a building. He was also the person who developed and established the first approach (cf. [**Tit81**]). We will describe both points of view briefly before we focus on the second one, which we will use for the rest of this thesis. The book *The Structure of Spherical Buildings* by Richard Weiss contains a full description of the first approach (cf. [**Wei04**]). A full description of the second approach can be found in the book *Buildings* by Peter Abramenko and Kenneth Brown (cf. [**AB08**]).

2.1. Buildings as Chamber Systems

Definition 2.1.0.1 ($\{i\}$**-residues**)**.** Let Δ be an edge-colored graph with color set I. For $i \in I$, let Γ_i be the subgraph of Δ obtained from Δ by discarding all the edges whose color is not i. Then a connected component of Γ_i is called an $\{i\}$*-residue* of Δ.

Recall that, given a Coxeter system (W, S), we also call the elements $w \in W$ *words of* W over the *alphabet* S, and elements $s \in S$ *letters*. Since S is the generating set of W, every word can be written with letters in S. If we write a word with the minimal amount of letters possible, then we call it *reduced*. For details, we refer to [**Wei04**].

Definition 2.1.0.2 (**Buildings**)**.** Let Π be a Coxeter diagram with vertex set I, $S = \{s_i \mid i \in I\}$ a set of generators, and let (W, S) be a Coxeter system of type Π.

A *building* Δ *of type* Π is a pair (Δ, δ), where

- Δ is an edge-colored graph with color set I, and
- δ is a map $\delta\colon \Delta \times \Delta \to W$

such that

(i) every $\{i\}$-residue is a complete subgraph with at least two vertices, and

(ii) for each reduced word $w = s_1 s_2 \dots s_n \in W$, and for every ordered pair (x, y) of vertices,

$$\delta(x, y) = w \Leftrightarrow \exists \text{ path in } \Delta \text{ of type } i_1 i_2 \dots i_n \text{ from } x \text{ to } y.$$

Definition 2.1.0.3. A building (Δ, δ) is called

- *spherical*, if the corresponding Coxeter group W is finite.
- *thick*, if for every $i \in I$, every $\{i\}$-residue contains at least 3 vertices.
- *thin*, if for every $i \in I$, every $\{i\}$-residue contains exactly 2 vertices.
- *irreducible*, if the corresponding Coxeter diagram is connected.

Note & Notation 2.1.0.4. In the following, we use both the Coxeter system (W, S) and the Coxeter diagram Π to refer to the type of a building. Sometimes for a building (Δ, δ) we only write Δ for short. Vertices in Δ will be called *chambers*, and paths in Δ will be called *galleries*. The map δ is also called the *Weyl-distance map*. If we measure the Weyl-distance between two chambers and then take the word length, we obtain a numerical distance between the chambers. In this way, we can view buildings as metric spaces.

Proposition 2.1.0.5. *Let (W, S) be a Coxeter system. Then the Cayley graph of W with respect to the generating set S is the only thin building of type (W, S). Every thick building of type (W, S) is a union of thin buildings of type (W, S).*

Reference. This is [**Wei04**, Proposition 8.11] □

Definition 2.1.0.6 (**Apartments**)**.** Let Δ be a building of type (W, S). Then we call the thin buildings isomorphic to $\mathsf{Cay}(W, S)$ contained in Δ the *apartments* of Δ.

Example 2.1.0.7 (A thick spherical building of type A_2)**.** The following is an explicit example for Example 7.4 described in [**Wei04**]. We will construct a graph that is a thick irreducible spherical building of type A_2.

Let V be the vector space $\mathbb{F}_2^3$ over the field $\mathbb{F}_2$. Let X denote the set of all proper non-trivial subspaces of V with respect to the standard basis. Then the set X contains the following elements.

$$\langle \begin{pmatrix}1\\0\\1\end{pmatrix} \rangle, \langle \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \rangle, \langle \begin{pmatrix}0\\1\\0\end{pmatrix} \rangle, \langle \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \rangle, \langle \begin{pmatrix}0\\0\\1\end{pmatrix} \rangle, \langle \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \rangle, \langle \begin{pmatrix}1\\1\\1\end{pmatrix} \rangle, \langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \rangle,$$
$$\langle \begin{pmatrix}0\\1\\1\end{pmatrix} \rangle, \langle \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \rangle, \langle \begin{pmatrix}1\\1\\0\end{pmatrix} \rangle, \langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \rangle, \langle \begin{pmatrix}1\\0\\0\end{pmatrix} \rangle, \langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \rangle$$

Let S be the set of all subsets $\{Y_1, Y_2\}$ of X such that $Y_1 \subset Y_2$. We write (Y_1, Y_2) instead of $\{Y_1, Y_2\}$ to indicate the natural ordering of the Y_i by inclusion. Then the elements of S are also called *flags*, and they are exactly the following.

$$(\langle \begin{pmatrix}0\\0\\1\end{pmatrix} \rangle, \langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \rangle), (\langle \begin{pmatrix}1\\0\\0\end{pmatrix} \rangle, \langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \rangle), (\langle \begin{pmatrix}1\\0\\1\end{pmatrix} \rangle, \langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \rangle),$$
$$(\langle \begin{pmatrix}1\\0\\1\end{pmatrix} \rangle, \langle \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \rangle), (\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \rangle, \langle \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \rangle), (\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \rangle, \langle \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \rangle),$$
$$(\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \rangle, \langle \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \rangle), (\langle \begin{pmatrix}0\\0\\1\end{pmatrix} \rangle, \langle \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \rangle), (\langle \begin{pmatrix}0\\1\\1\end{pmatrix} \rangle, \langle \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \rangle),$$

$$(\langle\begin{pmatrix}0\\0\\1\end{pmatrix}\rangle,\langle\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\rangle),(\langle\begin{pmatrix}1\\1\\1\end{pmatrix}\rangle,\langle\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\rangle),(\langle\begin{pmatrix}1\\1\\0\end{pmatrix}\rangle,\langle\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\rangle),$$
$$(\langle\begin{pmatrix}1\\1\\1\end{pmatrix}\rangle,\langle\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\rangle),(\langle\begin{pmatrix}0\\1\\1\end{pmatrix}\rangle,\langle\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\rangle),(\langle\begin{pmatrix}1\\0\\0\end{pmatrix}\rangle,\langle\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\rangle),$$
$$(\langle\begin{pmatrix}0\\1\\1\end{pmatrix}\rangle,\langle\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\rangle),(\langle\begin{pmatrix}1\\1\\0\end{pmatrix}\rangle,\langle\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\rangle),(\langle\begin{pmatrix}1\\0\\1\end{pmatrix}\rangle,\langle\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\rangle),$$
$$(\langle\begin{pmatrix}1\\1\\0\end{pmatrix}\rangle,\langle\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\rangle),(\langle\begin{pmatrix}1\\0\\0\end{pmatrix}\rangle,\langle\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\rangle),(\langle\begin{pmatrix}0\\1\\0\end{pmatrix}\rangle,\langle\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\rangle)$$

The corresponding building has the elements of S as its chambers, and two chambers $\{Y_1, Y_2\}$ and $\{Y_1', Y_2'\}$ are i-adjacent for some $i \in \{1, 2\}$ if $Y_i = Y_i'$ and $Y_j \neq Y_j'$ for $i \neq j \in \{1, 2\}$. The set S has 21 elements. For every element $y = (Y_1, Y_2)$, there are two other elements that agree with y in the first component, and two that agree with y in the second component. The information we can extract from that — for the graph that we are about to draw — is that every chamber is a vertex contained in two circuits of length 3 (ordinary triangles). We call the following graph Γ.

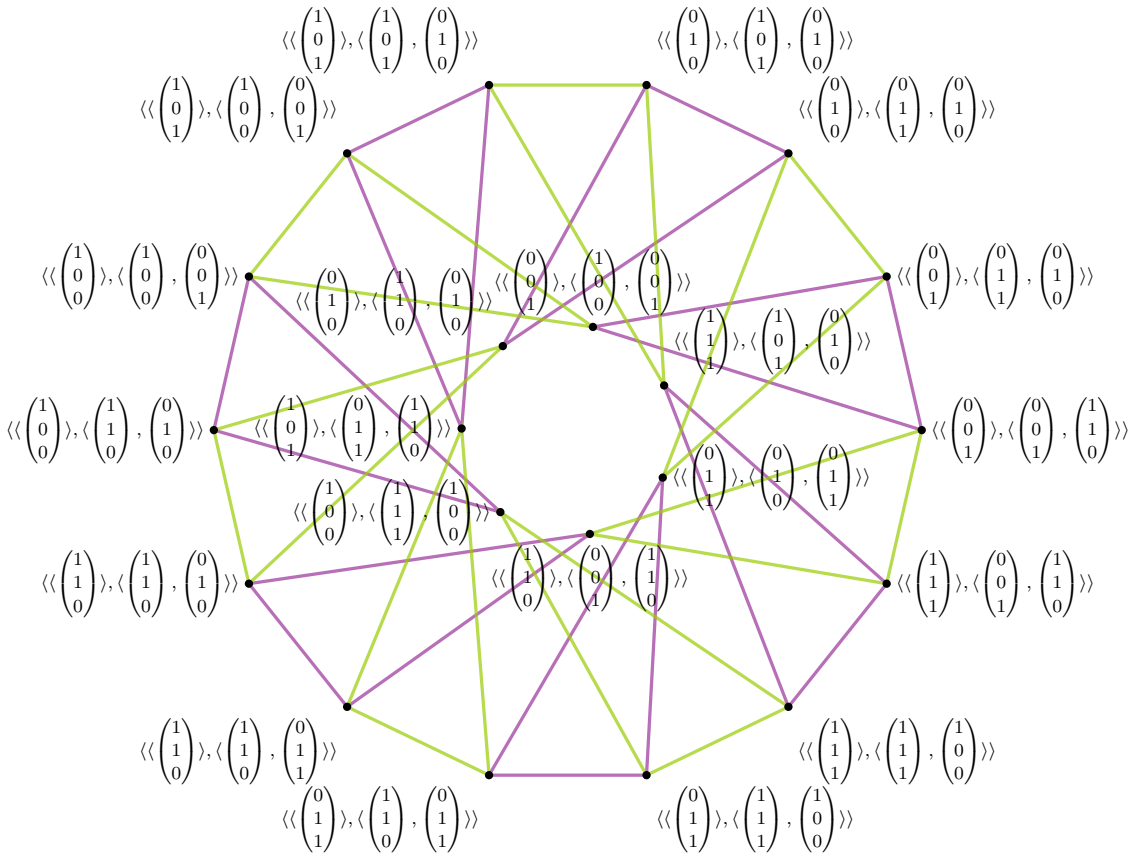


FIG. 1. An example of a thick irreducible spherical building of type A_2 as an edge-colored graph.

Let g represent the color of the edges that are drawn in green, and let h represent the color of the edges that are drawn in purple. Then the graph Γ is a building — as

we will further describe — with the following corresponding Coxeter system (W, S) and diagram Π.

$$S = \{s_g, s_h\}, \quad W = \langle s_g, s_h \mid s_g^2 = s_h^2 = (s_g s_h)^3 = \mathsf{id} \rangle, \quad \Pi = \bullet\!\!-\!\!\!-\!\!\bullet \ .$$

It has 7 residues of type g and 7 residues of type h.

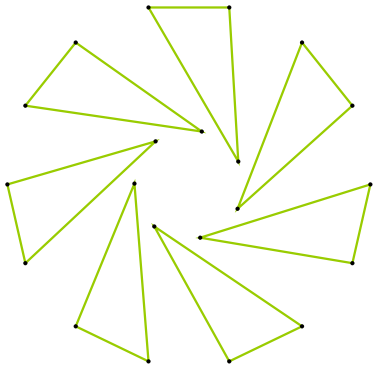


FIG. 2. The 7 residues of Γ of type g.

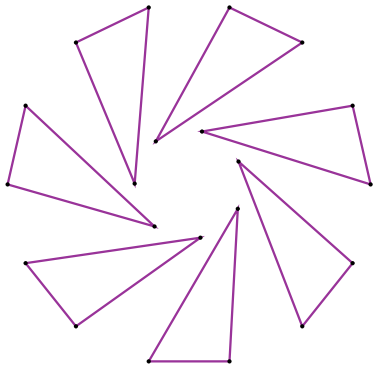


FIG. 3. The 7 residues of Γ of type h.

We can see that Γ is thick because every $\{g\}$-residue and every $\{h\}$-residue contains 3 chambers (an ordinary triangle has 3 vertices).

The Weyl-distance map δ, associated with Γ, is defined as follows. Let $y = (Y_1, Y_2)$ and $y' = (Y_1', Y_2')$ be two chambers of Γ. Let $Y_0 = Y_0' = \{0\}$ and $Y_3 = Y_3' = V$. We define

$$\sigma(i) := \min\{j \in [1,3] \mid Y_i \subseteq Y_{i-1} + Y_j'\}, \text{ for all } i \in [1,3].$$

One can verify that σ is a permutation of the interval $[1,3]$, set $\delta(y, y') = \sigma$, and see that this defines $\delta \colon \Gamma \times \Gamma \to W$ as desired (cf. [**Wei04**, p. 48] for details).

Since Γ is a union of apartments isomorphic to $\mathsf{Cay}(W, S)$, we can find many subgraphs isomorphic to $\mathsf{Cay}(W, S)$ in Γ.

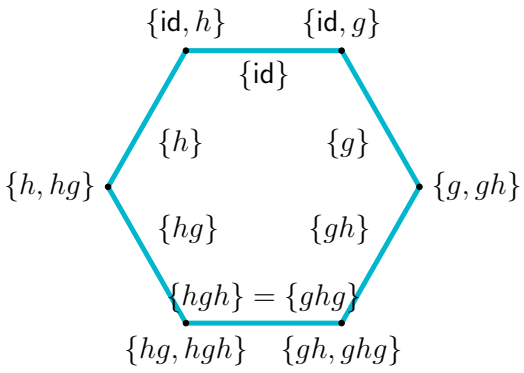


FIG. 4. The Cayley graph of W w.r.t. S.

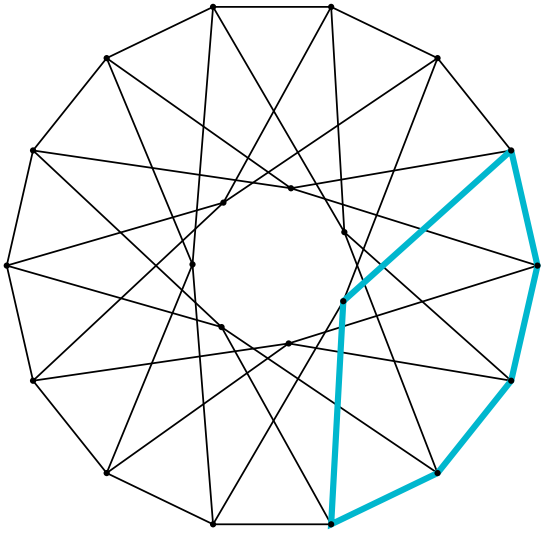


FIG. 5. The Cayley graph of W w.r.t. S as an apartment in Γ.

Note 2.1.0.8. As described in [**Wei04**, Example 7.4], buildings of type A_n can be obtained for any vector space V of dimension $n+1$ ($n \geq 1$) over a field or skew field, using a procedure similar to the one presented in Example 2.1.0.7. For that, we always first consider the set X of all proper subspaces, and then we define the set S of all nonempty subsets $\{Y_1, Y_2, \ldots, Y_m\}$ of X such that $Y_1 \subset Y_2 \subset \cdots \subset Y_m$. The elements of S are always called *flags* in this context.

2.2. Buildings as Simplicial Complexes

After our brief excursion into buildings as certain edge-colored graphs, or *chamber systems*, we now turn to the simplicial perspective, where we will view them as certain *chamber complexes*. Even if we focus on this specific approach, there are different subapproaches to defining a building. One particularly elegant one was given by Jacques Tits in his lecture notes *Buildings of Spherical Type and Finite BN-Pairs* (cf. [**Tit74**, §3.1]). However, for this one, it is important to keep in mind that it always demands buildings to be thick simplicial complexes. In the general literature, buildings that are thick simplicial complexes are commonly called *thick buildings*. We will first present the definition from Tits's lecture notes and use it for defining thick buildings, and afterwards we will give a more general definition for the term *building*.

Definition 2.2.0.1 (**Chamber complexes**, **chambers**, **galleries**, **panels**)**.**

A simplicial complex is a *chamber complex* if

- all maximal simplices (*chambers*) have the same dimension, and
- for two maximal simplices C and C', we can always find a finite sequence of chambers $(C = C_0, C_1, \ldots, C_n = C')$ (a *gallery*) such that $C_i \cap C_{i+1}$ is a simplex of codimension 1 (a *panel*).

Example 2.2.0.2. Let Δ be a chamber complex in which the chambers are 2-simplices, and let $C_0, \ldots, C_4$ be chambers as depicted in Fig. 6. Then the sequence $(C_0, C_1, \ldots, C_4)$ is a gallery from C_0 to C_4.

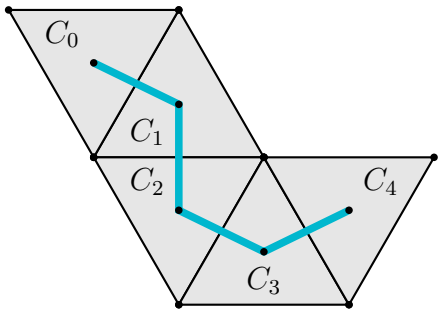


Fig. 6. A visualization for a gallery from C_0 to C_4.

Definition 2.2.0.3 (**Thick buildings**)**.** A *thick building* is a thick simplicial chamber complex Δ such that

- Δ is the union of thin chamber subcomplexes, and we call these *apartments*,
- for any two simplices in Δ, there always exists an apartment in Δ containing both of them, and
- every two apartments in Δ are isomorphic through an isomorphism that fixes every vertex in their intersection.

Theorem 2.2.0.4. *The thin chamber subcomplexes of a thick building are Coxeter complexes.*

Reference. This is [**Tit74**, Theorem 3.7]. □

The reason why the definition from Tits's lecture notes is so elegant is that it only uses simplicial chamber complexes. We will see how this differs from other definitions next.

Definition 2.2.0.5 (**Buildings**)**.** A *building* is a simplicial complex Δ such that:

- Δ is the union of Coxeter complexes, and we call these *apartments*,
- for any two simplices in Δ, there always exists an apartment in Δ containing both of them, and
- every two apartments in Δ are isomorphic through an isomorphism that fixes every vertex in their intersection.

If we use this definition, then a building is called *thick* if it is a thick simplicial complex, and *thin* if it is a thin simplicial complex. There exist buildings that are neither thick nor thin.

Comparing with Tits's lecture notes, what we call *thick buildings* here is what Tits calls *buildings* in his notes. There, buildings that are not thick are referred to as *weak buildings*. For weak buildings, it is not necessarily the case that they are unions of Coxeter complexes. This is why we cannot simply take Tits's lecture notes definition of buildings and omit the condition that the simplicial complex has to be thick. If we want to allow simplicial complexes that are not thick, we must add the condition that all apartments are Coxeter complexes in order to obtain the desired definition of a building here.

In this thesis — even though we will almost exclusively be concerned with thick buildings — we will, for the sake of clarity, always explicitly say *thick building* instead of just *building* whenever we are dealing with a building that is thick. In the following, we will still give a few further basic definitions for buildings in general.

Since the apartments of a building are always Coxeter complexes and always isomorphic, every building comes with an associated Coxeter system, just as in the previous section.

Definition 2.2.0.6 (**Type of a building**)**.** The *type of a building* Δ with associated Coxeter system (W, S) is the type of (W, S). However, we also sometimes refer to the type of Δ as just (W, S).

Definition 2.2.0.7 (**Weyl groups**)**.** If for a Coxeter group W there exists a building of type (W, S) for some generating set S of W, then we also call W a *Weyl group*.

Definition 2.2.0.8 (**Thickness at least** t)**.** We say that a building has *thickness at least* t if every panel is contained in at least t chambers. That means a building is thick if it has thickness at least 3.

Definition 2.2.0.9 (**Spherical**)**.** A building Δ of type (W, S) is *spherical* if the apartments have a finite number of vertices. This is equivalent to saying that W is finite.

Proposition 2.2.0.10. *If a building is spherical, its apartments have the form of triangulated spheres. In other words, a Coxeter complex associated to a Coxeter system (W, S) where W is a finite Coxeter group has the form of a triangulated sphere.*

Reference. See [**AB08**, Proposition 1.108] and [**AB08**, Section 3.1, Page 115]. □

The next definition is the same as Definition 2.1.0.3, but we need it in this setting as well.

Definition 2.2.0.11 (**Irreducible**)**.** A building Δ is called *irreducible* if the corresponding Coxeter diagram is connected.

Notation 2.2.0.12. For a building Δ associated to a Coxeter system (W, S) with Coxeter diagram Π, we call $|S|$ the *rank* of Δ. The rank coincides with the number of vertices of Π (cf. Definition 1.2.0.4). If I is the index set of Π, then we also call I the *type set* of Δ and say that Δ *is indexed by* I.

Note 2.2.0.13. As we have seen, irreducible finite Coxeter groups are classified. One could ask the questions, whether also all thick irreducible spherical buildings are classified, and whether there exist thick irreducible spherical buildings for every irreducible finite Coxeter group.

The answer to both questions is no.
However, all thick irreducible spherical buildings of rank at least 3 are classified (cf. [**Tit74**]), and all thick irreducible spherical *Moufang* buildings of rank 2 are classified (cf. [**TW02**], [**MV99**]). We will talk about what it means for a building to be Moufang in Section 2.4.

There exist thick irreducible spherical buildings for every irreducible finite Coxeter group, except for H_3 and H_4, as we will see in Section 6.6. This non-existence was first proven by Tits in [**Tit77**].

Definition 2.2.0.14 (**Gallery distance**, **type of a gallery**)**.**

The *length of a gallery* $C_0, C_1, \ldots, C_n$ in a building Δ is the integer n (i.e., the number of chambers contained in the sequence minus one). The minimal length of a gallery connecting two chambers C and C' is called the *gallery distance* between C and C'. A gallery of minimal length between two chambers C and C' is called a *minimal gallery* from C to C'.

Recall that, in a Coxeter complex, every chamber corresponds to a word of the

associated Coxeter group. We call two chambers *adjacent* if they intersect in a panel.

Let A be a chamber in a building Δ of type (W, S) corresponding to $w \in W$, and let B be a chamber in Δ adjacent to A corresponding to $w' \in W$. Then there exists an $s_i \in S = \{s_i \mid i \in I\}$ such that $ws_i = w'$. We say that A and B are *s_i-adjacent*, or *i-adjacent* for short.

If $\gamma = (C = C_0, C_1, \ldots, C_n = C')$ is a gallery, then the type $\mathsf{typ}(\gamma)$ of γ is the sequence $s := (s_1, \ldots, s_n)$ such that C_{i-1} is s_i-adjacent to C_i, for $i = 1, \ldots, n$.

Proposition 2.2.0.15. *Let Δ be a building of type (W, S). We denote the set of chambers of Δ by $\mathsf{Ch}(\Delta)$. There exists a map*

$$\delta\colon \mathsf{Ch}(\Delta) \times \mathsf{Ch}(\Delta) \longrightarrow W$$

with the following properties.

- *Given a minimal gallery $\gamma = C_0, C_1, \ldots, C_n$ of type*
$$\mathsf{typ}(\gamma) = (s_1, \ldots, s_n),$$
$\delta(C_0, C_n)$ is the element $w = s_1 \ldots s_n$ represented by $\mathsf{typ}(\gamma)$.
- *Let C and C' be chambers, and let $w = \delta(C, C')$. The function $\gamma \mapsto \mathsf{typ}(\gamma)$ gives a one-to-one correspondence between minimal galleries from C to C' and reduced decompositions of w.*
- *Let C and C' be chambers, and suppose there is a gallery from C to C' whose type $(s_1, \ldots, s_n)$ is a reduced word. Then $\delta(C, C') = s_1 \ldots s_n$.*

Reference. This is [**AB08**, Proposition 4.81 and Proposition 4.84]. □

Definition 2.2.0.16 (**Weyl-distance**)**.** We call

$$\delta\colon \mathsf{Ch}(\Delta) \times \mathsf{Ch}(\Delta) \longrightarrow W$$

of Proposition 2.2.0.15 the *Weyl-distance map* and the image $\delta(C, D)$, for two chambers $C, D \in \mathsf{Ch}(\Delta)$, the *Weyl-distance* between C and D in Δ.

If we consider the word length for $\delta(C, D) \in W$, then the integer we obtain coincides with the gallery distance between C and D.

Definition 2.2.0.17 (**Chamber graph**)**.** For a building Δ, we define a graph whose vertex set is isomorphic to $\mathsf{Ch}(\Delta)$, where two vertices are joined by an edge if and only if the corresponding chambers are adjacent in Δ, and we call this the *chamber graph* of Δ. The gallery distance $\delta(C, C')$ between two chambers C and C' is then exactly the distance in the chamber graph between the vertices corresponding to C and C'.

Example 2.2.0.18 (A thick spherical building of type A_2)**.** The following example is described in [**AB08**, §4.2], and we will use it to compare the two different approaches of defining a building. Let V be the vector space $\mathbb{F}_2^3$ over the field $\mathbb{F}_2$, let X denote the set of all proper non-trivial subspaces of V, with respect to the standard basis, and let S be the set defined as in Example 2.1.0.7. We draw a purple vertex for each element of X that is generated by a single vector, and we draw a green vertex for each element of X that is generated by two vectors. We connect two vertices by an edge if there exists an element of S that contains both of the corresponding subspaces. Then we obtain the following simplicial complex.

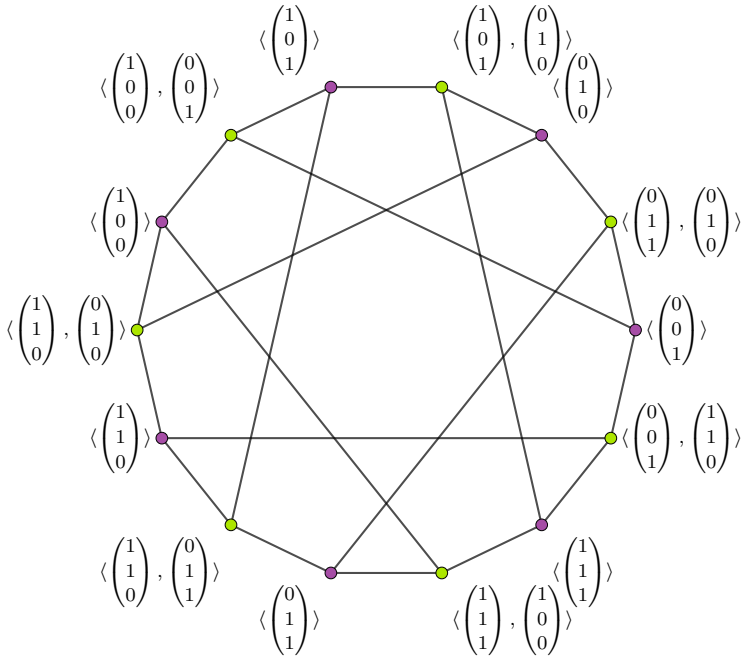


FIG. 7. An example of a thick irreducible spherical building of type A_2 as a simplicial complex.

This is a thick irreducible spherical building of type A_2, just as Example 2.1.0.7. Since its maximal simplices only have dimension 1, it can be viewed as a graph as well.[1]

Indeed, Fig. 7 is also called the *Heawood graph*, and can be viewed as the incidence graph of the Fano plane (which is exactly the minimal example of a projective space that is a Lie-incidence geometry of type A_2; see Example 3.2.0.2).

It has 14 vertices and 21 edges. If we draw a vertex for every edge of the Heawood graph, we obtain the 21 vertices of Γ in Example 2.1.0.7; see Fig. 8.

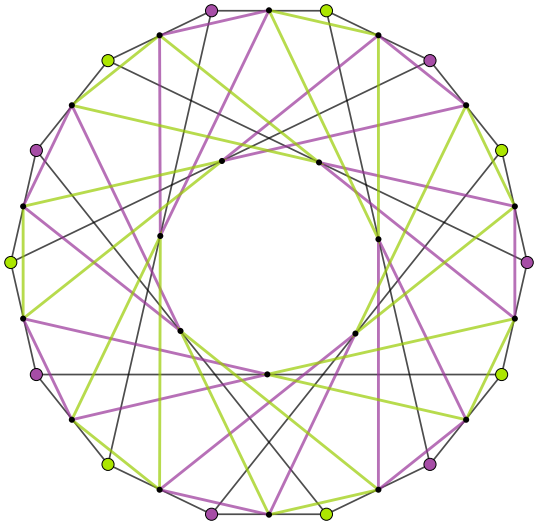


FIG. 8. The graph Γ of Example 2.1.0.7 and the Heawood graph on top of each other.

[1]It would be very difficult to draw an example of a thick building that has higher-dimensional simplices.

In order to understand how to translate between the combinatorial and the simplicial points of view, and why it is legitimate to do so, we refer to [**Tit81**] and [**AB08**, §5.6] for a detailed description. It is proved there that both approaches are equivalent: given a building of a certain type as a simplicial complex, we can obtain a building as an edge-colored graph of the same type such that both are canonically isomorphic — chambers correspond to chambers, apartments correspond to apartments, distances agree, and so on, and vice versa. A building in the simplicial view is thick/irreducible/spherical/... if and only if it is thick/irreducible/spherical/... in the combinatorial view.

We now return to the simplicial view, in which we will remain for the rest of the thesis.

2.2.1. Types & Bourbaki Labeling

Definition 2.2.1.19 (**Type function, type of a simplex**)**.** Let Δ be a chamber complex of rank n, and let I be a set with n elements. A *type function* on Δ with values in I is a function typ that assigns to each vertex v an element $\mathsf{typ}(v) \in I$, in such a way that the vertices of every chamber are mapped bijectively onto I.

We say that a chamber complex Δ is *colorable*, or *vertex colorable*, if it admits a type function (cf. [**AB08**, §A.1.3]).

Given a colorable chamber complex Δ with a type function onto I, the elements of I can be thought of as colors. Then

- each vertex is colored,
- vertices that are joined by an edge have different colors,
- the total number of colors is equal to the rank of Δ, and
- a chamber has exactly one vertex of each color.

However, given a vertex v, we will usually call $\mathsf{typ}(v)$ the *type* of v (instead of the color of v). If F is a simplex of a colorable chamber complex Δ, then we define $\mathsf{typ}(F)$ as the set of types of its vertices.

Fact 2.2.1.20. Buildings are colorable (cf. [**AB08**, §1.6.2]) and the type function on them is essentially unique: any two type functions on a given building with values in sets I and I' differ only by a bijection $I \cong I'$. Equivalently, the partition of vertices into types is unique. To see this, fix an arbitrary chamber C; we will also refer to C as the *fundamental chamber*. Once we assign types to the vertices of C, the types of vertices of all adjacent chambers are determined, and once these are determined, the types of the chambers adjacent to those are determined, and so on (cf. [**AB08**, p. 665]). (It turns out that there is a canonical choice for the colors of the vertices of C; see [**AB08**, p. 60].)

Example 2.2.1.21. As an example, we can look at Fig. 7 again. Fig. 7 shows a rank 2 building, which has the form of a bipartite graph. Vertices are drawn in the colors green and purple. We identify g with the color green and h with the color purple. Then $I = \{g, h\}$ and every vertex has either type g or type h.

2.2.1.1. The Bourbaki Labeling of the Vertices

Let Δ be a building with associated Coxeter system (W, S) and Coxeter diagram Π. Since the cardinality of S is the rank of Δ, which is equal to the number of vertices of Π, and since the rank of Δ is also equal to the cardinality of the type set of Δ, we usually associate each type with a vertex of Π. We will describe one way to do this in the following.

Suppose we are given a spherical Coxeter diagram Π of rank n for a Coxeter group W with index set I. View Π as a vertex-colored pre-graph and I as a set of vertex colors. We name these colors $1, 2, \ldots, n$ in the following way:

- We look up Π in Table 2. The first vertex, reading from left to right, receives color 1.
- If Π is not of type E_6, E_7, or E_8, then the second vertex receives color 2, and we continue naming the colors of the vertices from left to right such that i is adjacent to $i+1$, either until we are done or until we reach a vertex with three neighbors. In the latter case, Π has type D_n, and two vertices remain unlabeled. The upper one receives color $n-1$ and the lower one color n.
- If Π is of type E_6, E_7, or E_8, then the vertex that shares an edge with the vertex of valency 3 but with no other vertex of Π receives color 2, and the vertex adjacent to the first vertex receives color 3. We then continue naming the remaining colors from left to right such that i is adjacent to $i+1$, until we are done.

This is called the *Bourbaki labeling* of the vertices of a spherical Coxeter diagram, first described in [**Bou68**], and shown explicitly in Table 1.

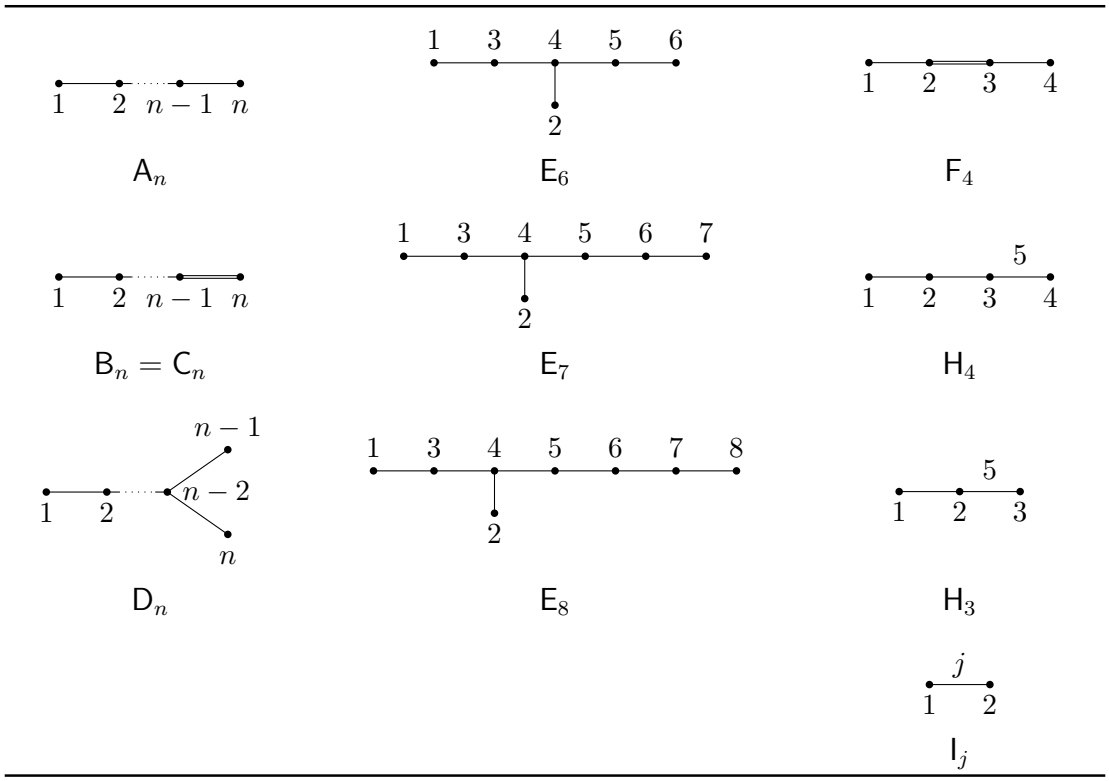


TABLE 1. The Bourbaki labeling for spherical Coxeter diagrams.

So, if Δ is a building of type Π indexed by I, we regard I as the type set, and then we can view the type function as a mapping that maps every vertex of Δ to a vertex of Π.

2.3. Roots, Residues & Opposition

In this subsection, Δ always denotes a thick spherical building of type (W, S), Σ an apartment, and we take the simplicial point of view.

Definition 2.3.0.1 (**Homomorphism**, **automorphisms**, **type-preserving**, ...)**.** Let Δ and Δ' be two buildings with respective index sets I and I' and respective sets of chambers $\mathsf{Ch}(\Delta)$ and $\mathsf{Ch}(\Delta')$. A map

$$\phi\colon \mathsf{Ch}(\Delta) \longrightarrow \mathsf{Ch}(\Delta')$$

is called a *homomorphism* from Δ to Δ' if there exists a map

$$\sigma\colon I \longrightarrow I'$$

such that for all C, $D \in \mathsf{Ch}(\Delta)$

$$C \sim_i D \iff \phi(C) \sim_{\sigma(i)} \phi(D).$$

In words: For all $i \in I$, ϕ sends each i-panel of Δ to some $\sigma(i)$-panel of Δ'; i.e., $\phi(x)$ and $\phi(y)$ are $\sigma(i)$-adjacent chambers of Δ', whenever x and y are i-adjacent chambers of Δ. If we want to emphasize the role of the map σ, we also call ϕ a *σ-homomorphism* (cf. [**Wei04**]).

- A homomorphism is called *special* or *type-preserving* if $I = I'$ and σ is the identity map, denoted id.
- We call Δ and Δ' *isomorphic* if both σ and ϕ are bijections.
- An *endomorphism* of Δ is a homomorphism from Δ to itself.
- By an *automorphism* of Δ we mean an isomorphism from Δ to itself.
- The set of automorphisms of Δ forms a group, and we will denote it by $\mathsf{Aut}(\Delta)$.
- The set of special automorphisms forms a subgroup of $\mathsf{Aut}(\Delta)$, and we denote it by $\mathsf{Aut}^\circ(\Delta)$. It is also often denoted by $\mathsf{Spe}(\Delta)$ in the literature.

Note 2.3.0.2. Isomorphisms of buildings are isometries: they map galleries to galleries of the same length and thus preserve the gallery distance between chambers.

Definition 2.3.0.3. Let Σ be an apartment of Δ. An endomorphism ρ of Σ is called *idempotent* if $\rho^2 = \rho$ (cf. [**AB08**, p. 129]).

Definition & Fact 2.3.0.4 (**Foldings**)**.** Let C and C' be two adjacent chambers in Δ. Let Σ be an apartment containing both of them. Then there exists exactly one type-preserving, idempotent endomorphism ρ of Σ that maps C' to C such that each chamber in the image of ρ has two chambers in its preimage. For each pair of adjacent chambers, we will call such a morphism a *folding* (cf. [**AB08**, §3.4.1]).

Definitions 2.3.0.5 (**Roots, opposite roots, walls, boundaries**)**.**
Let C and C' be two adjacent chambers in an apartment Σ of Δ. Let ρ be the folding of Σ for C and C' that maps C' to C.

- Let α be the image of ρ. Then α is also called a *root*.
- The *opposite folding* of ρ is the folding that maps C to C'.
- The root associated to the opposite folding is called the *opposite root*, denoted by $-\alpha$.
- The intersection $\alpha \cap (-\alpha)$ is called a *wall*.
  It is also referred to as the *boundary* of α, denoted $\partial\alpha$, and the boundary of $-\alpha$.

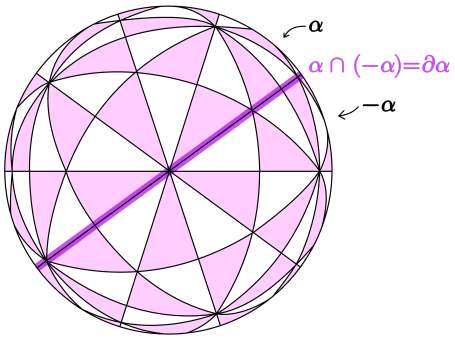


FIG. 9. An example of two roots in a Coxeter complex.[2]

Definition & Facts 2.3.0.6 (**Opposite simplices**)**.** Let Σ be an apartment of Δ.

- Two vertices v and v' are called *opposite in* Σ if every wall containing v also contains v' (and then automatically every wall containing v' contains v). For every vertex in Σ there exists a unique opposite vertex in Σ.
- *Opposite simplices* of Σ are two simplices A and B with the property that the vertex opposite to any vertex in A is contained in B, and vice versa. Again, for every simplex in Σ there exists a unique opposite simplex in Σ.
- If two simplices are opposite in some apartment, then the building axioms guarantee that they are opposite in every apartment in which they are both contained. Thus, we say that two simplices are opposite in Δ if they are opposite in some apartment Σ of Δ.

Notation & Definition 2.3.0.7 ($\equiv$, **self-opposite**)**.** Opposition defines a permutation, also denoted by $\equiv$, of order at most 2 on the type set I. With that (recall Section 2.2.1.1), $\equiv$ induces an automorphism of the corresponding Coxeter diagram. A subset $J \subseteq I$ is called *self-opposite* if $J^{\equiv} = J$.

Definition & Facts 2.3.0.8 (**Centers of roots**, **polar types**)**.** Every root α contains a unique simplex fixed by every (not necessarily type-preserving) automorphism of Σ that preserves α.[3] This simplex is called the *center of the root* α. It is the simplex whose image under the folding onto α coincides with the image of its opposite simplex in Σ.

If Δ is irreducible, the type of a simplex that is the center of a root is called a *polar type* of Δ. In the reducible case, the polar types of the connected components will be called *polar types* of the building Δ.

The polar type in the simply-laced and irreducible case is unique. It is the set

[2]For the drawing of the Coxeter complex, we took a figure of [**Kle90**] (p. 106) as a model that we found as Fig. 1.11 in [**AB08**].

[3]This is regarded as "well-known". However, we could not find a reference, and therefore include a sketch of proof in Lemma 10.0.0.2.

of nodes to which the additional generator is joined when constructing the affine Coxeter diagram out of the given one (cf. Table 3, Table 2). If we denote the type set of Δ by $I = \{1, 2, \dots, r\}$, where r is the rank of Δ, then it is $\{1, r\}$ in case A_r, it is 2 in case of D_r, and $2, 1, 8$ for E_6, E_7, E_8, respectively, with respect to the Bourbaki labeling.

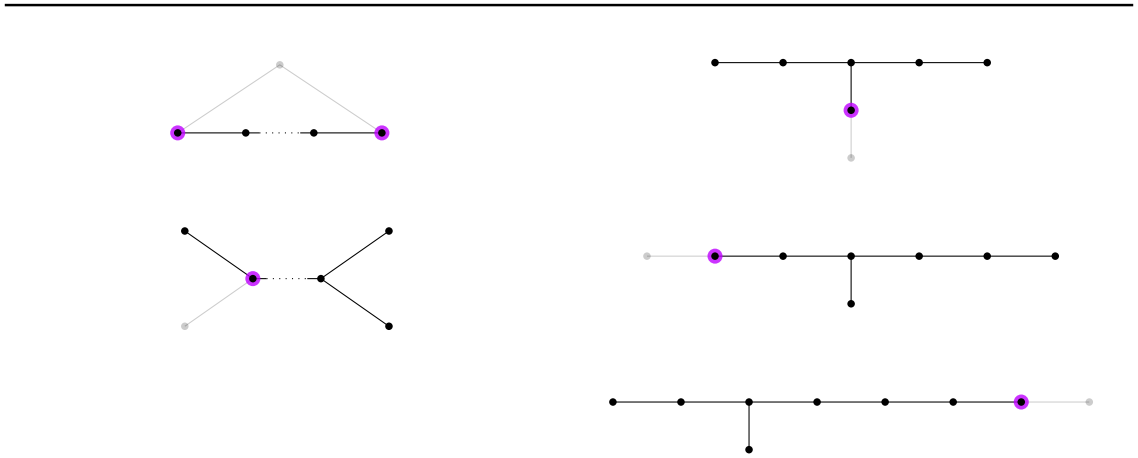
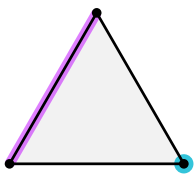


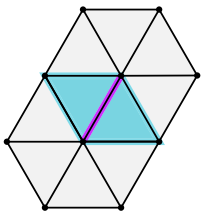
TABLE 2. The polar types in the simply-laced irreducible cases are marked in purple. The affine Coxeter diagrams are hinted at in sheer gray.

Definition 2.3.0.9 (**Residues**)**.**

- We say that a vertex v and a simplex F are *joinable* if $v \notin F$ and $F \cup \{v\}$ is a simplex. In that case, we write $v \sim F$.
- The simplicial complex induced on the vertices joinable to a given simplex F of a building Δ forms a building, called the *residue of* F *in* Δ and denoted by $\mathsf{Res}_\Delta(F)$.[4]



A vertex is joinable to an edge if both are contained in a common higher-dimensional simplex.



The residue of the edge marked in purple is given by the simplicial complex induced on the simplices colored in blue.

Note 2.3.0.10. The Coxeter diagram of a residue is obtained from the Coxeter diagram of the building by deleting the vertices with type in $\mathsf{typ}(F)$.

Definitions 2.3.0.11 (**Local opposition**, **cotype**, **type of a residue**)**.** The opposition relation in $\mathsf{Res}_\Delta(F)$ will be denoted by $\equiv_F$ (also on the types). Two simplices

[4]In the literature, what we defined as the residue of F is also sometimes called the *link of* F, or rarely the *star of* F. However, usually the star of a simplex F is defined in a way such that it contains F itself and does not form a building.

of $\mathsf{Res}_\Delta(F)$ that are opposite in $\mathsf{Res}_\Delta(F)$ are called *locally opposite at* F. The *cotype* $\mathsf{cotyp}(F)$ *of a simplex* F is defined as $I \setminus \mathsf{typ}(F)$. The *type of the residue* $\mathsf{Res}_\Delta(F)$ is the cotype of F.

We now state Tits's famous rigidity and extension theorems. We will use the rigidity theorem, for example, in Section 2.9.1 and Chapter 6.

Theorem 2.3.0.12 (**Tits's rigidity theorem** & **Tits's extension theorem**). *Let Δ and Δ' be two thick irreducible spherical buildings of rank at least three. For $A \in \Delta$ (or Δ'), and $i \in \mathbb{N}$ such that $i \geq \mathsf{codim}(A)$, we denote by $E_i(A)$ the set of all chambers having a face of codimension i in common with A. Let $\Sigma \subset \Delta$ be an apartment, let $C \in \Sigma$ be a chamber, and let $C' \in \Delta'$ be a chamber.*

(*i*) *Any isomorphism $\varphi\colon \Delta \to \Delta'$ is entirely determined by its restriction to $E_1(C) \cup \Sigma$.*
(*ii*) *Any adjacency-preserving bijection $\psi\colon E_2(C) \to E_2(C')$ extends to an isomorphism of Δ onto Δ'.*

Reference. Item (i) is [**Tit74**, Theorem 4.1.1], and item (ii) is [**Tit74**, Theorem 4.1.2]. □

2.3.1. A connectivity theorem

A useful property that we will prove here and use later is the *connectivity of the subgeometry of a Lie incidence geometry of type* $\mathsf{E}_{6,2}$, $\mathsf{E}_{7,1}$, *or* $\mathsf{E}_{7,3}$ *induced by the points opposite two given points of the geometry.* This can be shown geometrically inside certain *parapolar spaces* that we will define in Chapters 3 and 5. We will give a more general, purely building-theoretic proof here. The aim of this subsection is therefore to establish the following proposition, which holds for all irreducible spherical buildings of simply-laced type (and also for *twin buildings* of simply-laced type—but since we have not defined these, we do not include them here).[5]

Proposition 2.3.1.13. *Let Δ be an irreducible spherical building of simply-laced type such that each panel is contained in at least four chambers. Let C, C' be two arbitrary chambers of Δ. Then the subgraph $\Gamma^{C,C'}$ of the chamber graph Γ (cf. 2.2.0.17) induced on the set of chambers opposite both C and C' is connected.*

The proof we present is rather similar to the proof of [**MR95**, Theorem 5.1], which is essentially the case $C = C'$ of Proposition 2.3.1.13. So, we first verify the latter for rank 2 residues.

Lemma 2.3.1.14. *Let Δ be a spherical building of type $\mathsf{A}_1 \times \mathsf{A}_1$ or A_2 such that each panel is contained in at least four chambers. Let C, C' be two arbitrary chambers of Δ. Then the subgraph $\Gamma^{C,C'}$ of the chamber graph Γ induced on the set of chambers opposite both C and C' is connected.*

[5] We are grateful to the anonymous referee of our article [**BSV25**] for bringing this more general approach to our attention.

PROOF. For $C = C'$, this is straightforward; hence assume that $C \neq C'$.

For type $\mathsf{A}_1 \times \mathsf{A}_1$, the chamber graph is a grid, and hence the said subgraph is a subgrid, which is always connected.

Suppose now Δ stems from a projective plane with at least 4 points per line. First suppose each line has exactly 4 points. Then one verifies easily that, if C and C' are adjacent, then $\Gamma^{C,C'}$ is the chamber graph of a 3×3 grid. If C and C' have distance 2 in the chamber graph, then one verifies that $\Gamma^{C,C'}$ is a cycle of length 18, hence connected. Finally, if C and C' are opposite, then $\Gamma^{C,C'}$ consists of four 3-cliques $\{a_{-2}, a_{-1}, a_0\}$, $\{a_0, a_1, a_2\}$, $\{b_{-2}, b_{-1}, b_0\}$, and $\{b_0, b_1, b_2\}$ and edges $\{c_i, b_i\}$ and $\{c_i, a_i\}$, $i \in \{-2, -1, 1, 2\}$.

So, we may suppose that each line has at least five points. Let D, D' be two chambers both opposite both C and C'. We observe that, if D and D' have distance 2, then the unique chamber E adjacent to both D and D' is also opposite both C and C'. (This is most easily seen considering chambers as flags of the corresponding projective plane. Indeed, then E consists of a point of either D or D', and a line of either D' or D, respectively. Hence, since C is opposite both D and D', its elements are not incident and do not coincide with any of the elements of D and D', and hence neither with the elements of E.) Hence we may assume that D and D' are opposite. Let S be the set of chambers sharing their point with D and let S' be the set of chambers sharing their line with D'; note that S and S' are opposite panels. For each chamber $B \in S$ there exists a unique chamber $B' \in S'$ at distance 2 in the chamber graph. Let C_B be the unique chamber adjacent to both B and B'. Then we observe (with a similar proof as our first observation above) that any chamber opposite both D and D' is not opposite at most two members of $U = \{C_B \mid B \in S\}$. In particular, C is not opposite at most two members of U, and C' is not opposite at most two members of U. Consequently, there is at least one member of U opposite both C and C', and our first observation now implies that there is a path from D to D' inside $\Gamma^{C,C'}$. □

We can now prove Proposition 2.3.1.13. Let D and D' be two chambers belonging to $\Gamma^{C,C'}$. Let γ be a path in $\Gamma^{C,C}$ connecting D with D' such that, among all such paths, the minimal distance d from C' to any member of Γ is the highest, and the number n of chambers attaining that minimal distance is smallest. We claim that d is the diameter of Γ, which shows that γ is inside $\Gamma^{C,C'}$. Indeed, suppose there are elements of γ not opposite C', and let D_1 be the first chamber of γ having distance d to C'. Let D_0 be the element of γ preceding D_1 and D_2 the one following D_1. Then our assumptions imply that $F = D_0 \cap D_1 \cap D_2$ has corank 2, and so $R := \mathsf{Res}_\Delta(F)$ is a rank 2 building corresponding to either a generalized digon or a projective plane. Since all elements of γ are opposite C, the projection C_R of C onto R is opposite all of D_0, D_1, D_2. Let C'_R be the projection of C' onto R. Let n_i, $i = 0, 1, 2$, be the distance in the chamber graph of R from C'_R to D_i. Since projections of chambers onto panels are unique, it is straightforward to find paths γ_i, $i = 1, 2$, of length $2 - n_i$ (if R corresponds to a generalized digon) or $3 - n_i$ (if R corresponds to a projective plane) connecting D_i with a chamber of R opposite C'_R. Then, using Lemma 2.3.1.14, we obtain a path in R connecting D_0 with D_2 having one chamber less at distance d_1 from C'_R, and all other chambers have distance at least $d_1 + 1$. Replacing D_0, D_1, D_2 by this path, we obtain a path in $\Gamma^{C,C}$ with either higher

minimal distance n or fewer chambers at that minimal distance, a contradiction. The proposition is proved. □

2.4. Little Projective Groups

Again, Δ will always denote a thick irreducible spherical building. We will now introduce root groups, elations, the Moufang condition, and little projective groups. Recall that several of the main goals of this thesis are to construct root elations directly, to provide alternative geometric proofs that thick irreducible spherical buildings of rank 3 are Moufang, and to make new observations concerning the associated little projective groups.

Definition 2.4.0.1 (**Root elations**, **root groups**)**.** Let α be a root of Δ. Let U_α be the group of automorphisms of Δ pointwise fixing every chamber that has a panel in $\alpha \setminus \partial\alpha$. The elements of U_α are called *(root) elations*, and U_α itself is called a *root group*. An element of the root group U_α is called a *central elation* if it belongs to U_β for each root β having the same center as α.

Definition 2.4.0.2 (**Moufang**)**.** If a root group U_α acts transitively on the set of apartments containing α, then we say that α is *Moufang*. If every root is Moufang, then we say that Δ is Moufang (or that Δ *has the Moufang property* or *satisfies the Moufang condition*, compare [**Wei04**, Definition 11.2]).

Even though a *Moufang building* for us here will always be thick, irreducible, and spherical, we often speak of *Moufang spherical buildings*, where the word *Moufang* is understood as a specification of the spherical buildings.

Definition 2.4.0.3 (**Little projective group**)**.** If Δ is Moufang, then the subgroup of $\mathsf{Aut}(\Delta)$ that is generated by the root elations is denoted by $\mathsf{Aut}^\dagger(\Delta)$ and called the *little projective group of* Δ.

Note 2.4.0.4. Every spherical building Δ of rank at least three is Moufang (cf. [**Tit74**, p. 274]). If Δ is simply-laced and U_α is a root group associated with a root α in Δ, then U_α only depends on the center of α; that is, if β is any root in Δ that has the same center as α, then, by Tits's rigidity theorem (2.3.0.12), each member of U_α pointwise fixes each chamber of Δ that has a panel in $\beta \setminus \partial\beta$.

2.5. Groups of Projectivities

Again, Δ will always denote a thick spherical building, and $\mathsf{Ch}(\Delta)$ the set of chambers of Δ. In this subsection, we will discuss projections in buildings, which will play a major role in every part of this thesis. We will use them for constructions of root elations and to define projectivities and groups of projectivities. As noted earlier,

a major objective of this work is to determine all special and general projectivity groups of non-maximal simplices with irreducible residues inside thick spherical buildings.

Definition & Fact 2.5.0.1 (Projections). Let F and F' be two opposite simplices in Δ. Let $C \in \mathsf{Ch}(\Delta)$ be a chamber such that $F \subset C$. Then there exists a unique chamber $C' \supset F'$ at minimal gallery distance from C (cf. [**Tit74**, Proposition 2.29]). The chamber C' is called the *projection of C from F onto F'* and denoted $\mathsf{proj}^F_{F'}(C)$.

Fact 2.5.0.2. The mapping $\mathsf{proj}^F_{F'}(C)$ is a bijection from the set of chambers through F to the set of chambers through F' and preserves adjacency in both directions. It thus defines a unique isomorphism from $\mathsf{Res}(F)$ to $\mathsf{Res}(F')$ (cf. [**Tit74**, Theorem 3.28]), which we denote by $\mathsf{proj}^F_{F'}$.

Note & Notation 2.5.0.3. We note that $\mathsf{proj}^F_{F'}$ is a special case of the projection operator as defined in 3.19 of [**Tit74**]. When the context makes F clear, we sometimes remove the F from the notation and simply write $\mathsf{proj}_{F'}$. We will use both the phrasing *projection from the simplex F to the simplex F'* and the phrasing *projection from the residue of F to the residue of F'* for $\mathsf{proj}^F_{F'}$ in the following.

This projection has the following property.

Proposition 2.5.0.4. *Let F and F' be opposite simplices of a spherical building Δ. Let v be a vertex of Δ adjacent to each vertex of F, and set $i := \mathsf{typ}(v) \in I$. Then the type i' of the vertex $\mathsf{proj}^F_{F'}(v)$ is the opposite in $\mathsf{Res}(F')$ of the opposite type of i in Δ, that is, $i' = (i^{\equiv})^{\equiv_{F'}}$. Also, vertices $v \sim F$ and $v' \sim F'$ are opposite in Δ if and only if $v' \equiv_{F'} \mathsf{proj}^F_{F'}(v)$.*

Reference. This is [**Tit74**, Proposition 3.29]. □

There is an immediate corollary that we will use later on.

Corollary 2.5.0.5. *Let F be a simplex of a spherical building Δ. Then a collection T of vertices in $\mathsf{Res}(F)$ admits an opposite in $\mathsf{Res}(F)$ (viewed as spherical building on its own) if and only if it admits an opposite vertex in Δ.*

Proof. If v is a vertex in $\mathsf{Res}(F)$ opposite each vertex of T in $\mathsf{Res}(F)$, then, for any simplex F' opposite F, the vertex $\mathsf{proj}^F_{F'}$ is opposite every member of T by Proposition 2.5.0.4. Conversely, let v' be a vertex of Δ opposite each member of T. Select $t \in T$. Applying Proposition 2.5.0.4, we find a simplex F' in $\mathsf{proj}^t_{v'}(F)$ opposite F. Then, again by Proposition 2.5.0.4, $\mathsf{proj}^{F'}_F(v')$ is, in $\mathsf{Res}(F)$, opposite every member of T. □

Definitions 2.5.0.6 (Perspectivity, projectivity, projectivity groups, ...).

- Let F and F' be two opposite, non-maximal simplices in Δ. Then we call the isomorphism $\mathsf{proj}^F_{F'}$ a *perspectivity (between residues)* and write $F \barwedge F'$.
- If $F_0, F_1, \ldots, F_\ell$ is a sequence of consecutively opposite simplices, then the isomorphism $\mathsf{Res}(F_0) \to \mathsf{Res}(F_\ell)$ given by
$$\mathsf{proj}^{F_{\ell-1}}_{F_\ell} \circ \cdots \mathsf{proj}^{F_1}_{F_2} \circ \mathsf{proj}^{F_0}_{F_1}$$
is called a *projectivity (of length ℓ)*.

- If ℓ is even, it is called an *even* projectivity, and if $F_0 = F_\ell$, it is called a *self-projectivity*.
- The set of all self-projectivities of a simplex F forms a group, denoted $\Pi(F)$, which we call the *general projectivity group of* F or, equivalently, the *general projectivity group of* $\mathsf{Res}(F)$.
- Similarly, the set of all even self-projectivities of a simplex F constitutes another group, denoted $\Pi^+(F)$, referred to as either the *special projectivity group of* F or the *special projectivity group of* $\mathsf{Res}(F)$.

Note that $\Pi(F) = \Pi^+(F)$ as soon as $(\mathsf{typ}(F))^{\equiv} \neq \mathsf{typ}(F)$.

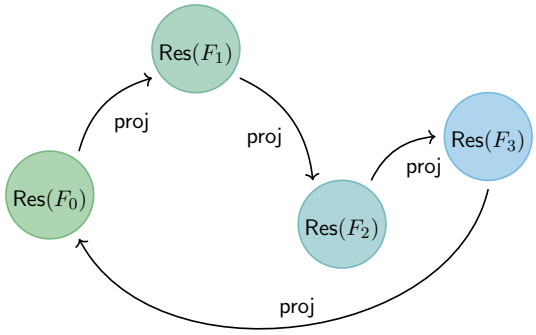


FIG. 10. A visualization of an even self-projectivity of length 4.

Note & Notation 2.5.0.7 ($n(J)$). Let $\Pi(F)$ be the general projectivity group of a non-maximal simplex F in a spherical building Δ. As an abstract permutation group, $\Pi(F)$ depends only on the type of F. The same holds for the special projectivity group $\Pi^+(F)$. We have the natural inclusion $\Pi^+(F) \trianglelefteq \Pi(F)$ with index at most 2, i.e., $[\Pi(F) : \Pi^+(F)] \leq 2$, and put

$$n(J) := [\Pi(F) : \Pi^+(F)],$$

where J is the type of F. We have $n(J) = n(J^{\equiv})$, since $n(J) = 1$ whenever $J^{\equiv} \neq J$.

Observation 2.5.0.8. Let Δ be a spherical building over the type set I, and let $J \subseteq I$ be self-opposite. Let F be a simplex of type J. Then $n(J) = 1$ if and only if the identity in $\Pi(F)$ can be written as the product of an odd number of perspectivities.

PROOF. If the identity in $\Pi(F)$ can be written as the product of an odd number of perspectivities, then, by composing this product with any even projectivity, we see that we can write any putative member of $\Pi(F) \setminus \Pi^+(F)$ as a product of an even number of perspectivities, that is, as a member of $\Pi^+(F)$ — a contradiction. We conclude $\Pi^+(F) = \Pi(F)$ in this case.

Conversely, if $\Pi^+(F) = \Pi(F)$, then consider any odd projectivity θ. Our assumption implies that we can write θ^{-1} as an even projectivity. Composing those two products of perspectivities, we obtain the identity written as the product of an odd number of perspectivities. □

2.6. Groups Acting on Buildings

We mentioned in the introduction that every thick irreducible spherical building of rank at least 3 not only comes with an associated Coxeter group, but also with a second group, which may be classical, algebraic, or mixed. Up to this point, we have only considered the Coxeter groups. In this section, we begin preparing the ground for discussing the attached classical, algebraic, and mixed groups. For a given building Δ, this attached group *acts* on Δ, and we will say what that means next.

Definition 2.6.0.1 (***G*-actions**)**.** Let Δ be a building and let G be a group. Then we say that G *acts on* Δ if G acts as a permutation group on the set of chambers $\mathsf{Ch}(\Delta)$ of Δ and consists of morphisms $\Delta \to \Delta$ (as in 1.4.0.1). We then also call this a *G-action* on Δ.

Example 2.6.0.2. Let Δ be a thin building (i.e., a Coxeter complex) with associated Coxeter system (W, S). Then we can identify the chambers of Δ with words in W and obtain a group action $W \times \mathsf{Ch}(\Delta) \longrightarrow \mathsf{Ch}(\Delta)$ of W on Δ, where W acts on Δ by left multiplication. Left multiplication by an arbitrary element of W is a special automorphism.

Definition 2.6.0.3 (***G*-invariant**, **strongly transitive**)**.** Let Δ be a building and let G be a group acting on Δ such that the action of G is type-preserving. We say that a set of apartments $\mathscr{A}$ of Δ is *G-invariant* if for every $\Sigma \in \mathscr{A}$, the image $g\Sigma$ is again an apartment contained in $\mathscr{A}$.

We say that the G-action is *strongly transitive with respect to* $\mathscr{A}$ if G acts transitively on the set of pairs (Σ, C) consisting of an apartment $\Sigma \in \mathscr{A}$ and a chamber $C \in \Sigma$.

This is equivalent to saying that the action is chamber transitive and that the stabilizer of a given chamber C is transitive on the set of apartments in $\mathscr{A}$ containing C.

Alternatively, it is equivalent to saying that G is transitive on $\mathscr{A}$ and that the stabilizer of a given apartment $\Sigma \in \mathscr{A}$ is transitive on $\mathsf{Ch}(\Sigma)$.

If $\mathscr{A}$ is the full set of apartments of Δ and a G-action is strongly transitive with respect to $\mathscr{A}$, then we also just say that the G-action is *strongly transitive* (on the apartments of Δ).

Notation 2.6.0.4. Let Δ be a building and let G be a group acting on Δ such that the action of G is type-preserving and strongly transitive. We fix an arbitrary apartment Σ of Δ and an arbitrary chamber $C \in \Sigma$ and call them the *fundamental apartment* and the *fundamental chamber*, respectively. We define the following three subgroups of G.

$$B := \{g \in G \mid gC = C\},$$
$$N := \{g \in G \mid g\Sigma = \Sigma\},$$
$$T := \{g \in G \mid g \text{ fixes } \Sigma \text{ pointwise}\}.$$

Note 2.6.0.5 (Fundamental chamber and fundamental apartment)**.** It has happened twice now that we considered an arbitrary but fixed chamber C in a building

Δ and called it the *fundamental chamber*. In the following, we will continue to use the expressions *fundamental chamber* and *fundamental apartment* for an arbitrary but fixed chamber in Δ, and an arbitrary but fixed apartment in Δ, respectively, whenever it is not necessary to specify this any further. Sometimes, we might want to refine our usage of these expressions a bit more: if we want to bring our building theoretic concepts together with our descriptions from Section 1.5, then of course the fundamental chamber should match with the fundamental chamber as described in Definition & Facts 1.5.0.17.

Fact 2.6.0.6. The subgroup T is the kernel of the homomorphism $f\colon N \to W$ induced by the action of N on Σ, and thus T is a normal subgroup of N. The map f is surjective, and thus W and N/T are isomorphic (cf. [**AB08**, 6.1.1]). Hence, we can identify W with N/T and will do that in the following. Furthermore, $T = B \cap N$.

Proposition 2.6.0.7. *Let Δ be a building and let G be a group acting on Δ such that the action of G is type-preserving and strongly transitive. Let B and N be as in Notation 2.6.0.4. Then $G = BNB$.*

Reference. This is [**AB08**, Proposition 6.3]. □

Definition 2.6.0.8 (**Weyl-transitivity**)**.** We say that the action of a group G on a building Δ is *Weyl-transitive* if for each $w \in W$, the action is transitive on the set of ordered pairs (C, D) of chambers of Δ with $\delta(C, D) = w$, where δ denotes the Weyl-distance map (cf. [**AB08**, Definition 6.10]).

Proposition 2.6.0.9. *Let Δ be a spherical building and let G be a group acting on Δ such that the action of G is type-preserving. Then the following conditions are equivalent.*

(i) G acts strongly transitively.
(ii) G acts Weyl transitively.
(iii) The action of G is transitive on ordered pairs (C, C') of opposite chambers.

Reference. This is [**AB08**, Proposition 6.15]. □

2.7. BN-Pairs & RGD-Systems

In order to understand the attached classical, algebraic, or mixed group of a spherical Moufang building, we present some additional definitions and facts from building theory in this section and introduce the algebraic notions of a *BN-pair* and an *RGD-system*. A better understanding of this algebraic side of buildings will not only assist us in determining certain special groups of projectivities later, but will also allow us to interpret our findings in a broader context.

Roughly speaking, BN-pairs can be used to prove that many groups of Lie type are simple modulo their centers. This is known as *Tits's simplicity theorem*, and was proven by him in [**Tit64**].

More importantly for us, every group with a BN-pair gives rise to a building.

In our context, RGD-systems can be seen as 'refinements' of BN-pairs: while BN-pairs give rise to buildings, RGD-systems give rise to Moufang buildings (cf. [**AB08**, §7.8]). Furthermore, a group determined by an *RGD-system of spherical type* can be either classical, algebraic, or mixed (cf. [**Tit92**], [**Tit74**]).

2.7.1. BN-Pairs

Definition 2.7.1.1. A pair of subgroups B and N of a group G is a *BN-pair* if B and N generate G, the intersection $T := B \cap N$ is normal in N, and the quotient $W := N/T$ admits a set of generators S such that the following two conditions hold:

(BN1): For $s \in S$ and $w \in W$,

$$sBw \subseteq BswB \cup BwB.$$

(BN2): For $s \in S$,

$$sBs^{-1} \nsubseteq B.$$

The group W is also referred to as the *Weyl group associated to* (B, N), and the quadruple (G, B, N, S) is called a *Tits system*. (In the theory of algebraic groups, the group B is sometimes called a *Borel subgroup* and T is a *maximal torus*.)

Every BN-pair gives rise to a building. However, not all buildings are associated to a BN-pair, and different BN-pairs may give rise to the same building. In the case of thick irreducible spherical buildings of rank at least three, we have that every such building has an attached BN-pair (cf. [**Tit74**, VI]). We have the following theorem.

Theorem 2.7.1.2.

(i) Given a BN-pair in G, the generating set S is uniquely determined, and (W, S) is a Coxeter system. There is a thick building $\Delta = \Delta(G, B)$ that admits a strongly transitive G-action such that
- *B is the stabilizer of a fundamental chamber, and*
- *N stabilizes a fundamental apartment and is transitive on its chambers, and*
- *moreover, $T = B \cap N$ pointwise fixes the fundamental apartment.*

(ii) Conversely, suppose a group G acts strongly transitively on a thick building Δ with fundamental apartment Σ and fundamental chamber C. Let B be the stabilizer of C, and let N be a subgroup of G that stabilizes Σ and is transitive on the chambers of Σ. Then (B, N) is a BN-pair in G, and Δ is canonically isomorphic to $\Delta(G, B)$.

The building $\Delta(G, B)$ associated to a BN-pair only depends on (G, B).

Reference. This is [**Ron09**, Theorem 5.2, Theorem 5.3], or [**AB08**, Theorem 6.56]. □

Definition 2.7.1.3. Let (B, N) be a BN-pair and $T = B \cap N$. We call (B, N) *saturated* if

$$\bigcap_{w \in W} wBw^{-1} = T.$$

Note 2.7.1.4. A BN-pair is saturated if N is the full stabilizer of the fundamental apartment of $\Delta(G, B)$. Saturated BN-pairs are essentially in one-to-one correspondence with strongly transitive group actions on thick buildings (cf. [**AB08**, §6.2.5, §6.2.6]).

Facts 2.7.1.5. Let (G, B, N, S) be a Tits system, W the Weyl group associated to (B, N), and $T = B \cap N$. We will state some other facts about BN-pairs, which all can be found in [**AB08**, §6.2.6].

(1) G has a *Bruhat decomposition*, which is a bijection

$$C\colon W \xrightarrow{\sim} B\backslash G/B,$$
$$w \mapsto BwB.$$

In other words, G is the disjoint union $\bigsqcup_{w\in W} BwB = BWB$. Given $s \in S$ and $w \in W$, we have

$$C(s)C(w) = \begin{cases} C(sw), & \text{if } \ell(sw) > \ell(w), \\ C(sw) \cup C(w), & \text{if } \ell(sw) < \ell(w). \end{cases}$$

(2) The subgroups of G containing B are precisely the subgroups

$$P_J = BW_JB,$$

for $J \subseteq S$. They are called the *standard parabolic subgroups* of G. They are self-normalizing, and no two of them are conjugate.

(3) The building $\Delta = \Delta(G, B)$, arising from a group G with a BN-pair, has several equivalent descriptions.

- For the simplicial point of view, we will present two different ways of describing Δ. Firstly, we can view Δ as the set of standard parabolic cosets gP ($g \in G$, $P \supseteq B$), ordered by reverse inclusion. Then, again, G acts on Δ by left translation and the fundamental apartment is
$$\Sigma = \{wP \mid w \in W,\ P \supseteq B\}.$$
- In the second description for the simplicial point of view, we consider Δ to be the poset of parabolic subgroups, ordered by reverse inclusion. Then G acts on Δ by conjugation and the fundamental apartment is
$$\Sigma = \{wPw^{-1} \mid w \in W,\ P \supseteq B\}.$$
- If we view Δ as a chamber system,[6] then we can view $C = G/B$ as the set of chambers and $\delta\colon G/B \times G/B \to W$ as the Weyl-distance map, where
$$\delta(gB, hB) = w \iff Bg^{-1}hB = C(w).$$
Then G acts on Δ by left translation and the fundamental apartment is
$$\Sigma = \{wB \mid w \in W\}.$$

(4) If (B, N) is a saturated BN-pair and $N' \subseteq N$ is a subgroup, then (B, N') is a BN-pair if and only if $N'T = N$ (see [**AB08**, Lemma 6.59]).

[6] This is described in more detail on page 308 of [**AB08**] and also in [**AB08**, Remarks 6.58].

(5) If (B, N) is a saturated BN-pair such that $\Delta(G, B)$ is spherical, and $N' \subseteq N$ is a subgroup such that (B, N') is also a saturated BN-pair, then

$$N' = bNb^{-1}$$

for some $b \in B$ (see [**AB08**, Exercise 6.60]).

Definition 2.7.1.6. Let (G, B, N, S) be a Tits system. We say that a Bruhat decomposition for (G, B) is *thick* if the building $\Delta(G, B)$ is thick. A subgroup B of a group G is called a *Tits subgroup* if (G, B) admits a thick Bruhat decomposition.

Note 2.7.1.7 (Simplicity results)**.** Let G be a group with a Tits subgroup B such that $\Delta(G, B) = \Delta$ is thick and irreducible of type (W, S). Then G acts on Δ, and we denote the kernel of that group action by Z. With that, we have

$$Z = \bigcap_{g \in G} gBg^{-1},$$

and in many examples, Z is the center of G.

If H is a normal subgroup of G, then either $H \subseteq Z$, or $HB = G$. In other words, either H acts trivially on Δ, or H acts chamber-transitively on Δ (cf. [**AB08**, Lemma 6.61]).

If G is perfect and B is solvable, then every proper normal subgroup of G is contained in Z, and hence G/Z is a simple group in that case (cf. [**AB08**, Theorem 6.62]).

Let U be a normal subgroup of B such that $B = UT$, and let G' be the normal closure of U in G. Assume that G' is perfect and that U is solvable. Then every subgroup of G that is normalized by G' either is contained in Z or contains G'. In particular, $G'/(G' \cap Z)$ is a simple group (cf. [**AB08**, Theorem 6.64]).

2.7.2. RGD-Systems

After having seen BN-pairs, we now give a brief description of RGD-systems. The letters R, G, and D stand for *Root Group Data*. RGD-systems have been introduced by Jacques Tits in [**Tit92**].

Let Σ be a thin spherical building of type (W, S). For any $s \in S$, let α_s denote the set of all chambers in Σ such that the words in W corresponding to these chambers become longer when multiplied with s. Let ϕ be the set of all subsets of the form $w\alpha_s$, for $w \in W$ and $s \in S$. Then ϕ is the set of all roots of Σ. Let C be the chamber in Σ corresponding to the identity element 1 in W. We define

$$\phi^+ = \{\alpha \in \phi \mid C \in \alpha\}, \quad \text{and} \quad \phi^- = \{\alpha \in \phi \mid C \notin \alpha\},$$

and we call the elements of ϕ^+ and ϕ^- *positive roots* and *negative roots*, respectively. The roots α_s are called *simple roots*.

For a pair $\{\alpha, \beta\}$ of non-opposite roots, we set

$$[\alpha, \beta] := \{\gamma \mid \gamma \in \phi,\ \alpha \cap \beta \subset \gamma,\ (-\alpha) \cap (-\beta) \subset \gamma\},$$
$$(\alpha, \beta) := [\alpha, \beta] \setminus \{\alpha, \beta\}.$$

For each $\alpha \in \phi$, there is a unique element of W conjugate to some element of S and permuting α and $-\alpha$ by left translation. Said element is called the *reflection associated to* α and denoted by s_α. For instance, if $\alpha = \alpha_s$, then $s_\alpha = s$.

Let G be a group, T a subgroup, and $(U_\alpha)_{\alpha\in\phi}$ a family of subgroups of G. Then $(G, (U_\alpha)_{\alpha\in\phi}, T)$ is called an *RGD-system of type* (W, S) if

- **(RGD0)** for all $\alpha \in \phi$, U_α is not trivial,
- **(RGD1)** for all α and β with $\alpha \neq \pm\beta$ $[U_\alpha, U_\beta] \subseteq U_{(\alpha,\beta)}$,
- **(RGD2)** if $s \in S$ and $u \in U_{\alpha_s} \setminus \{1\}$, there exist elements u', u'' of $U_{-\alpha_s} \setminus \{1\}$ such that, for all $\beta \in \phi$, the product $m(u) := u'uu''$ conjugates U_β onto $U_{s_\alpha(\beta)}$,
- **(RGD3)** if $s \in S$, then $U_{-\alpha_s}$ is not contained in $U^+ := \langle U_\alpha \mid \alpha \in \phi^+\rangle$,
- **(RGD4)** $G = T\langle U_\alpha \mid \alpha \in \phi\rangle$,
- **(RGD5)** T normalizes U_α for each $\alpha \in \phi$, i.e.,

$$T \subseteq \bigcap_{\alpha\in\phi} N_G(U_\alpha).$$

Sometimes we speak of *spherical RGD-systems* if we want to emphasize that we consider RGD-systems of type (W, S), where W is a finite Coxeter group; which will always be the case in this thesis.

Heuristically, a BN-pair gives us a building Δ on which a group G acts, and an RGD-system gives us the root groups in addition, enabling us to prove that Δ is Moufang.

2.8. Subgroups of $\mathsf{Aut}(\Delta)$

We record some observations concerning subgroups of $\mathsf{Aut}(\Delta)$, where Δ is a Moufang spherical buildings Δ of simply-laced type. Note that, since Δ is simply-laced, the ground field is well-defined.

Note 2.8.0.1. The full automorphism group $\mathsf{Aut}(\Delta)$ is typically larger than G. It may include

- type-preserving linear automorphisms (including homologies not necessarily in G),
- diagram automorphisms (from symmetries of the Dynkin diagram),
- and possibly field automorphisms of the ground field.

Definition 2.8.0.2 (Full linear group of Δ). We denote by $\mathsf{Aut}^\circ(\Delta)$ the group of all type-preserving automorphisms of Δ. This group is also sometimes denoted by $\mathsf{Spec}(\Delta)$. Then

$$G \subseteq \mathsf{Aut}^\circ(\Delta) \subseteq \mathsf{Aut}(\Delta).$$

We will refer to the subgroup of $\mathsf{Aut}^\circ(\Delta)$, which consists of all automorphisms of $\mathsf{Aut}^\circ(\Delta)$, that do not involve field automorphisms, as the *full linear type-preserving group* of Δ . If we consider the full linear type-preserving group of Δ and add all

diagram automorphisms not involving field automorphisms, then we will refer to this new group as the *full linear group* of Δ .

Note 2.8.0.3. The names stem from the fact that, among the groups acting on Δ, the full linear type-preserving group is the largest type-preserving linear algebraic group, and the full linear group is the largest linear algebraic group. The full type-preserving linear group is generated by all *homologies*, which are the automorphisms of Δ that fix an apartment and a panel pointwise.

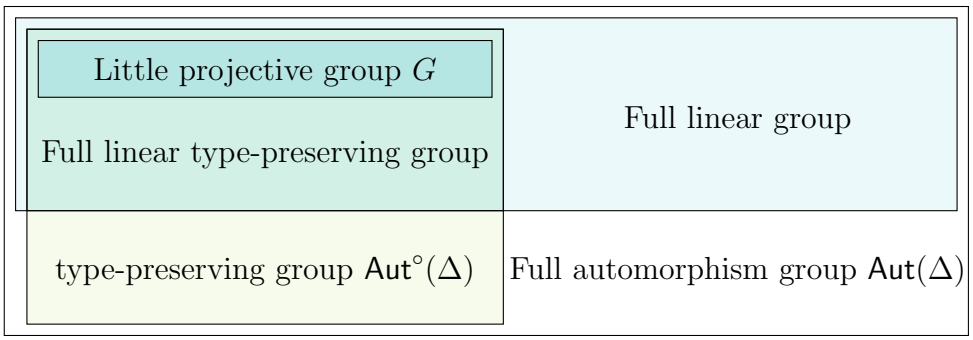


FIG. 11. A possible constellation of the previously mentioned subgroups of $\mathsf{Aut}(\Delta)$ under the assumption of minimal intersections.

2.9. Chevalley Groups & Levi Subgroups

The theory of buildings emerged after Chevalley constructed analogues of exceptional simple Lie groups over arbitrary fields, thereby discovering a new family of finite simple groups, now known as *Chevalley groups*. Tits subsequently developed the theory of buildings by attaching geometric structures to these groups. Chevalley groups defined over arbitrary fields are *groups of Lie type*. Such groups possess BN-pairs, and are therefore associated with buildings (cf. [**Car72**, Proposition 8.2.1]); moreover, they can be described as groups of automorphisms of spherical buildings (cf. [**AB08**, §6.2.6]). Chevalley groups are always simple, except in the cases $\mathsf{A}_1(\mathbb{F}_2)$, $\mathsf{A}_1(\mathbb{F}_3)$, $\mathsf{B}_2(\mathbb{F}_2)$, and $\mathsf{G}_2(\mathbb{F}_2)$ (cf. [**Car72**, Theorem 11.1.2]).

One of the main results of this thesis is the determination of all special and general projectivity groups associated with spherical buildings. In cases where the little projective group is a Chevalley group, the special projectivity groups can sometimes be determined rather quickly by algebraic means. We explain this algebraic approach in Section 2.11. To understand it, knowledge about the Levi decomposition for Chevalley groups is required. This will be discussed in Section 2.9.2, following some more general explanations about Chevalley groups and buildings.

When we talk about Chevalley groups in this chapter, we always mean adjoint Chevalley groups that appear as the little projective group of some Moufang spherical building of rank ≥ 3 and simply-laced type. In particular, every Chevalley group will be simple.

Since we only use the concept of Chevalley groups without actually working inside them, their details will not concern us. However, we provide some definitions, results, a rough description of their construction, and many references in Appendix C. There, we define Chevalley groups as groups of automorphisms of certain *Lie algebras* (cf. Definition C.1.0.1, Appendix C.2).

Chevalley groups can also be defined in a very different way in the context of algebraic groups (e.g. as sets of fixed points of algebraic groups under certain morphisms.)

These are two structurally different approaches. Still, we will make use of the fact that every Chevalley group can be viewed as a *connected $\mathbb{F}$-reductive linear algebraic group*. We provide some definitions and results in the context of linear algebraic groups in Appendix L. The split forms mentioned in L.5.0.9 correspond to finite Chevalley groups. For the general case, it is possible to define Chevalley groups as linear algebraic groups over arbitrary fields using *Galois descent*.

2.9.1. Descriptions of some Groups of Automorphisms

Let Δ be a spherical Moufang building of rank ≥ 3 and simply-laced type. Let G be its little projective group. Then G is a simple adjoint Chevalley group over some division ring. We will only consider the case where this division ring is a field $\mathbb{K}$. Let $\mathfrak{L}$ be the Lie algebra that determines G together with $\mathbb{K}$ (cf. C.2.0.6). Let $\mathfrak{H}$ be a Cartan subalgebra of $\mathfrak{L}$ (cf. Definition C.1.0.17), ϕ a root system of $\mathfrak{L}$ with respect to $\mathfrak{H}$ (cf. Definition C.1.0.19), $\Phi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ a fundamental system in ϕ^+, C_0 the fundamental chamber associated to Φ and let $\mathfrak{L}_{\mathbb{K}}$ and $\mathfrak{H}_{\mathbb{K}}$ be the corresponding Lie algebra and Cartan subalgebra over $\mathbb{K}$ (as in C.2.0.3). Let

$$\mathscr{C} := \{\alpha_j^\vee \mid \alpha_j \in \Phi\} \cup \{e_{\beta_i} \mid \beta_i \in \phi\}$$

be a Chevalley basis for $\mathfrak{L}$ (cf. Appendix C.2). Let $\mathbb{Z}\phi$ be the root lattice and $\mathbb{Z}\hat{\phi}$ be the weight lattice of ϕ (as defined in C.3.0.9 and C.3.0.12).

Put $W := \langle s_{\alpha_i} \mid \alpha_i \in \Phi\rangle$ and $S := \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}\}$. Then Δ is of type (W, S). For s_{α_i} we will just write s_i in the following. There is a unique element $w_0 \in W$ such that $w_0(\phi^+) = \phi^-$ (cf. [**Car93**, Proposition 2.2.6], [**Car93**, p. 20] or [**AB08**, p. 420]). Let $s_{i_1}, s_{i_2}, \ldots, s_{i_n}$ be n different elements of S. Then for each element $w \in W$ of the form $s_{i_1}s_{i_2}\ldots s_{i_n}$, we have that the order of w is given by $|\phi|/|\Phi|$. Furthermore, $|\phi|/|\Phi| - 1$ is the maximal height of any root in ϕ (cf. [**Car93**, p. 20]).

Let α be a root, $-\alpha$ its opposite root and U_α and $U_{-\alpha}$ the corresponding root subgroups. Let $H \subseteq G$ be the group generated by the images of the diagonal matrices under the homomorphism $\varphi_\alpha \colon \mathsf{SL}_2(\mathbb{K}) \to \langle U_\alpha, U_{-\alpha}\rangle$ (cf. Theorem C.3.0.8). We define

$$H_\alpha := \left\langle y_{\alpha,\lambda} \;\middle|\; y_{\alpha,\lambda} = \varphi_\alpha \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},\ \lambda \in \mathbb{K} \right\rangle \subseteq H \subseteq G, \text{ for all } \alpha \in \phi.$$

Then

$$H = \langle \{H_\alpha \mid \alpha \in \phi\}\rangle, \quad H_\alpha \subseteq \langle U_\alpha, U_{-\alpha}\rangle \quad \text{and} \quad H_\alpha = H \cap \langle U_\alpha, U_{-\alpha}\rangle.$$

Each element of H is an automorphism of $\mathfrak{L}_{\mathbb{K}}$, which acts trivially on $\mathfrak{H}_{\mathbb{K}}$ and transforms each root vector e_β into a multiple of itself. The coefficients arising in this way define naturally a character of the root lattice $\mathbb{Z}\phi$ (cf. Theorem C.3.0.11).

Let N be the subgroup generated by H and by the images of the monomial matrices under the homomorphism $\varphi_\alpha\colon \mathsf{SL}_2(\mathbb{K}) \to \langle U_\alpha, U_{-\alpha}\rangle$, for all $\alpha \in \phi$ (cf. C.3.0.19). Let U be the subgroup generated by the root subgroups U_α, for $\alpha \in \phi^+$, and let B be the subgroup UH of G. Then B together with N forms a BN-pair for G (cf. Proposition C.3.0.21), and ${}^{N}\!/_{H} \cong W$ (cf. Theorem C.3.0.19).

Recalling some facts about BN-Pairs (cf. [**Ron09**], Theorem 2.7.1.2), we have the following.

- $H = B \cap N$ pointwise fixes an apartment Σ of Δ,
- N stabilizes Σ,
- B stabilizes a chamber of Σ,
- a parabolic subgroup P_J of G is the stabilizer of the face of C_0 of type J (cf. [**Ron09**, p. 77]) and P_J can be written as a semi-direct product $U_J L_J$, where
$$U_J = \langle \{U_\alpha \mid \Sigma_J \subseteq \alpha \in \phi\}\rangle, \quad L_J = \langle H, \{U_\alpha \mid \alpha \in \phi_J\}\rangle,$$
- L_J is the subgroup fixing the face of C_0 of type J and the simplex of Σ opposite to it pointwise (cf. [**Ron09**, p. 77]).

Beyond that, apartments of Δ correspond to maximally split tori in the Chevalley group G (cf. [**Tit79**, §2]).

The fact that H acts trivially on $\mathfrak{H}_{\mathbb{K}}$ reflects the fact that H pointwise fixes an apartment Σ of Δ. We will regard H as a maximal torus in G in the following. The chamber C of Σ stabilized by B, is, in fact, the chamber of Σ such that every positive root of Σ contains C, and with that C is the fundamental chamber C_0 (cf. [**Ron09**, p. 76], [**AB08**, p. 403]). The unique element w_0, whose action on Σ interchanges the positive and negative roots, permutes the walls containing a panel of the fundamental chamber C_0 (cf. [**AB08**, p. 420]). The action of W is simply transitive on the set of chambers of Σ and there is a one-to-one correspondence between W and the chambers of Σ given by $w \leftrightarrow wC_0$, where C_0 is the fundamental chamber (cf. [**AB08**, p. 36]). The positive roots of Δ can be defined as the ones containing the fundamental chamber (cf. [**AB08**, p. 466]), [**Car72**, §8.4]). Then the subgroup B can be described as the semi-direct product of $\langle \{U_\alpha \mid C_0 \in \alpha\}\rangle$ and H (cf. [**Ron09**, p. 77]).

Now, we consider the character group $X := \mathsf{Hom}(H, \mathbb{K}^\times)$ of the maximal torus H. From the algebraic groups perspective, we know that every root gives rise to a root subgroup U_α of G and each U_α corresponds to an element of X (this is described in §1.9 of [**Car93**], especially on pp. 18-19). Since G is adjoint, X is just the root lattice $\mathbb{Z}\phi$.

From the buildings-theoretic perspective, recall that every fundamental root corresponds to a vertex in the Coxeter diagram of the Coxeter system (W, S). Let α be a fundamental root and i the type of the corresponding vertex in the Coxeter diagram. Then α contains the fundamental chamber C_0. Let C be the chamber corresponding to $s_i \in S$ (i.e., C is s_i-adjacent to the fundamental chamber C_0). Then there is a

unique folding ρ of Σ such that α is the image of ρ, and ρ maps C to C_0. The wall containing $C \cap C_0$ defines the boundary of α.

In the following, we want to define a group of automorphisms of Δ containing H, which is not necessarily a subgroup of G. Since every chamber in Δ contains a vertex of every type, we can call the vertices of the fundamental chamber $v_1, v_2, \ldots, v_n$, where i denotes the type of the vertex v_i in each case. Let $I = \{1, 2, \ldots, n\}$ be the associated index set with $|I| = |S|$. Let $J \subseteq I$ be such that J corresponds to a connected subdiagram of the Coxeter diagram of Δ. For each $J \subset I$, let $\bar{J} := I \setminus J$, and let R_J be the J-residue of the simplex of type $\bar{J}$ contained in C_0. In the case that $J = \{i\}$, we also denote $R_{\{i\}}$ by R_i. Then:

$$R_{\bar{i}} = R_{\{\bar{i}\}} = R_{I\setminus\{i\}} = \mathsf{Res}(v_i).$$

Now, for every $i \in I$, let T_i denote the group of type-preserving automorphisms of Δ that fix the apartment Σ and the residue $\mathsf{Res}(v_i)$ pointwise. Then T_i is not necessarily contained in G (Example 2.9.1.3). Note that, if $T_i \not\subseteq G$, then – because $T_i \subseteq \mathsf{Aut}^\circ(\Delta)$ – the elements of T_i not contained in G corresponds to homologies that are not products of root elations. We set $T := \langle \{T_i \mid i \in I\} \rangle \subseteq \mathsf{Aut}^\circ(\Delta)$.

Before we move on to an example, we want to make an observation about the action of the groups T_i on the sets

$$R_i^* := R_i \setminus \Sigma.$$

Definition 2.9.1.1 (**Free group action**)**.** The action of a group G on a set X is called *free* if the only member of G fixing some member of X is the identity.

Note 2.9.1.2. Each group T_i acts freely on the set $R_i^* = R_i \setminus \Sigma$, as we will explain in the following.

Let i and j be two adjacent vertices in the Coxeter diagram of Δ. The subdiagram containing only the vertices i and j is the diagram of a projective plane. We consider the corresponding residue $R_{i,j} = \mathsf{Res}(C_0 \setminus \{v_i, v_j\})$, and call the corresponding projective plane π. In π, points and lines are dual, so we may assume v_i corresponds to a line L, and v_j to a point p of π on L. The apartment Σ intersects $R_{i,j}$ in an apartment of $R_{i,j}$ that corresponds to an ordinary triangle τ in π.

We note that

$$\begin{aligned} R_{i,j} &\mathrel{\hat{=}} \text{ the set of all chambers in } \Delta \text{ containing } C_0 \setminus \{v_i, v_j\}, \\ R_i &\mathrel{\hat{=}} \text{ the set of all chambers in } \Delta \text{ containing } C_0 \setminus \{v_i\}, \end{aligned}$$

and see that the set of all chambers in Δ containing $C_0 \setminus \{v_i\}$ is the same as the set of all chambers containing v_j in the set of all chambers containing $C_0 \setminus \{v_i, v_j\}$. In other words: $R_i = \mathsf{Res}_{R_{i,j}}(v_j)$.

Note that every automorphism of Δ pointwise fixing all of $E_1(C_0) \cup \Sigma$ is the identity by Tits's rigidity theorem Theorem 2.3.0.12.

Consequently, we have to show that T_i acts freely on the set of lines of π through p not contained in τ. In Δ, the group T_i fixes $\Sigma \cup \mathsf{Res}(v_i)$ pointwise. Hence, T_i pointwise fixes τ and L in π. But if T_i now additionally fixes a line M through p not contained in τ, then it follows that T_i must be the identity, as we have a collineation with at least two centers (namely, the point of τ not contained in L and the intersection of M with the line of τ not through p). That proves the free action.

Example 2.9.1.3. We consider a projective plane of order 7 (i.e., every line contains 8 points and every point is on 8 lines). The fundamental chamber C_0 of Δ corresponds to a flag $\{c, L\}$, consisting of a point c and a line L. The apartment Σ corresponds to an ordinary triangle containing c and L. Let M be the line of the ordinary triangle that is disjoint from c. Consider the homologies with center c and axis M. These homologies act transitively on the points of L distinct from c and $L \cap M$. It is known that these homologies can not be described as products of root elations. However, their third powers can be described as such. A projective plane corresponds to a building of type A_2. According to Table 1, the index $[\mathbb{Z}\hat{\phi} : \mathbb{Z}\phi]$ is exactly 3 in this case.

Note 2.9.1.4. Since Δ is of simply-laced type, the group T defined above is associated to the weight lattice $\mathbb{Z}\hat{\phi}$. We will give a short explanation in the following, and write U_i for U_{α_i}. Note that we have isomorphisms

$$x_i \colon (\mathbb{K}, +) \to U_i, \quad t \mapsto x_i(t),$$

and we have isomorphisms $\theta_i \colon \mathbb{K}^\times \to T_i$, which can be chosen in such a way that

$$\theta_i(\lambda) x_i(t) \theta_i(\lambda)^{-1} = x_i(\lambda t)$$

for all $\lambda \in \mathbb{K}^\times$ and all $t \in \mathbb{K}$; The existence of the automorphisms $\theta_i(\lambda)$ follows from Lemma 58 in [**Ste68**]. With that, each T_i acts transitively on R_i^*. Now we have (in combination with Note 2.9.1.2) that the action of T_i on R_i^* is sharply transitive.

Note that the conjugation action of T_i on U_j is trivial for $j \neq i$ by the definition of T_i.

Furthermore, we have canonical projections $\pi_i \colon T \to T_i$, and the (componentwise) action of T by conjugation on $\prod_{i \in I} U_i$ is faithful. For $i \in I$ we set

$$\chi_i := \theta_i^{-1} \circ \pi_i \colon T \to \mathbb{K}^\times, \quad \text{and} \quad \omega_i := \iota_{T_i}^T \circ \theta_i \colon \mathbb{K}^\times \to T.$$

We observe that

$$\bigcap_{i \in I} \ker(\chi_i) = 1,$$

and that $\Omega = (\omega_i)_{i \in I}$ is a base of $X^\vee(T)$ with respect to the system $(\chi_i)_{i \in I}$ (cf. Definition L.2.0.8).

Let $(A_{ij})_{i,j \in I}$ be the Cartan matrix of ϕ (cf. Definition 1.5.0.20, Observation C.3.0.14). For each $i \in I$ there is a homomorphism $h_i \colon \mathbb{K}^\times \to T$ such that $h_i(\mathbb{K}^\times) = H_i$. Moreover, we have

$$h_i(\lambda)\, x_j(\mu)\, h_i(\lambda)^{-1} = x_j(\lambda^{A_{ji}} \mu)$$

for all $j \in I$, $\lambda \in \mathbb{K}^\times$ and all $\mu \in \mathbb{K}$ (cf. [**Ste68**, p. 23]). By the faithfulness of the conjugation action of T on $\prod_{i \in I} U_i$ we obtain

$$\chi_j \circ h_i(\lambda) = \lambda^{A_{ji}}$$

for all $j, i \in I$ and all $\lambda \in \mathbb{K}^\times$. Thus $h_i \in X^\vee(T)$ and

$$h_i = \sum_{j \in I} A_{ji}\, \omega_j$$

for each $i \in I$.

Let $|\mathbb{K}| = \infty$. We can identify the cocharacters $(\omega_i)_{i \in I}$ with the fundamental weights of $\Phi^\vee$ so that $X^\vee(T) \cong \mathbb{Z}\hat{\phi}^\vee$, and the cocharacters $(h_i)_{i \in I}$ with $(\alpha_i^\vee)_{i \in I}$. Since Φ is

simply-laced, we can identify Φ with $\Phi^\vee$ (cf. Note C.3.0.15). Thus, in this case, and in the way described above, T corresponds to the weight lattice $\mathbb{Z}\hat{\phi}$ of Φ, and the subgroup H of T corresponds to the root lattice $\mathbb{Z}\phi$ of Φ.

Note 2.9.1.5. The full linear type-preserving group of Δ coincides with GT and with the group $\hat{G}$ of Proposition C.3.0.22.

Note 2.9.1.6. We have ${}^{GT}/_{G} \cong {}^{T}/_{H}$ with Proposition C.3.0.22, but in general ${}^{GT}/_{G} \ncong \mathbb{Z}\hat{\phi}/\mathbb{Z}\phi$ and ${}^{T}/_{H} \ncong \mathbb{Z}\hat{\phi}/\mathbb{Z}\phi$. The correpsondence between T and $\mathbb{Z}\hat{\phi}$ (H and $\mathbb{Z}\phi$ respectively) is not an isomorphism and should not be understood pointwise; It works via homomorphisms and depends on the field:

In the case that $\mathbb{Z}\hat{\phi} \neq \mathbb{Z}\phi$, take a fundamental weight β with $\beta \notin \mathbb{Z}\phi$. Then β corresponds to one of the homomorphisms $\omega_i \colon \mathbb{K}^\times \to T$ from above for some $i \in I$. Since $\mathbb{Z}\hat{\phi}/\mathbb{Z}\phi$ is always finite, there exists a multiple of β contained in $\mathbb{Z}\phi$. For the purpose of an example, suppose $2\beta \in \mathbb{Z}\phi$. Then 2β corresponds to $2\omega_i \colon \mathbb{K}^\times \to T$. We have

$$\omega_i(\lambda^2) = \omega_i(\lambda)^2 = 2\omega_i(\lambda).$$

If $\mathbb{K}^\times = (\mathbb{K}^\times)^2$, then

$$\omega_i(\lambda) = t = 2\omega_i(\sqrt{\lambda}),$$

and $\mathsf{Im}(2\omega_i) = \mathsf{Im}\,\omega_i$. Since $2\beta \in \mathbb{Z}\phi$, $\mathsf{Im}(2\omega_i) \subseteq H$. But then $\mathsf{Im}\,\omega_i \subseteq H$, meaning that, if ${}^{T}/_{H} \cong \mathbb{Z}\hat{\phi}/\mathbb{Z}\phi$, then $\beta \in \mathbb{Z}\phi$, which contradicts our assumptions.

Note 2.9.1.7. Let $J \subseteq I$, $T_J = \langle T_i \mid i \in J \rangle \leq T$ and $\pi_J \colon T \to T_J$ the canonical projection. Let L_J be the stabilizer in G of R_J and the residue opposite R_J in Σ, and put

$$H_J := L_J \cap T_J.$$

Then $(\pi_J \circ \omega_i)_{i \in J}$ is a cocharacter base of $(T_J, (\chi_i \mid_{T_J})_{i \in J})$ and

$$H_J = \langle\, \pi_J \circ h_i(K^*) \mid i \in I \,\rangle.$$

2.9.2. Levi Subgroups

Let Δ be a spherical Moufang building of rank ≥ 3 and simply-laced type, and let $G = \mathsf{Aut}^\dagger(\Delta)$ be its little projective group, which is a simple adjoint Chevalley group over some field $\mathbb{K}$.

Let Σ be an apartment of Δ and C a chamber in Σ, which we will consider to be the fundamental chamber. Let F be a face of C. A parabolic subgroup of $\mathsf{Aut}^\dagger(\Delta)$ is

$$G_F := \mathsf{Aut}^\dagger(\Delta)_F := \langle \{U_\alpha \mid F \subseteq \alpha\} \rangle.$$

Definition 2.9.2.8 (**Levi subgroup**)**.** A *Levi subgroup* of the parabolic subgroup G_F of $\mathsf{Aut}^\dagger(\Delta)$ is a subgroup L_F such that G_F is the semi-direct product of L_F and a unipotent subgroup (cf. [**Bor69**, p. 158, Definition 11.22]).

It is known that the parabolic subgroups opposite G_F correspond bijectively to the Levi subgroups of G_F (cf. [**Bor69**, p. 199, Proposition 14.21]). Hence, a Levi

subgroup fixes a simplex and a unique opposite simplex pointwise, and it acts as a group of automorphisms on the residue of each of these simplices.

Lemma 2.9.2.9. *Let F and F' be opposite simplices in Σ with $F \subseteq C$. We define the following subgroups.*

$$G_{C,[F]} := \langle \{U_\alpha \mid C \subseteq \alpha, F \subseteq \alpha \setminus \partial\alpha\}\rangle,$$
$$G_{C,]F[} := \langle \{U_\alpha \mid C \subseteq \alpha, F \subseteq \partial\alpha\}\rangle.$$

The intersection of the stabilizers G_F and $G_{F'}$ defines a Levi subgroup of G_C and we can write the parabolic subgroup G_C of the Chevalley group $\mathsf{Aut}^\dagger(\Delta)$ as the semidirect product $G_{C,[F]}G_{C,]F[}$.

Proof. We can write

$$\begin{aligned} G_F \cap G_{F'} &= \langle \{U_\alpha \mid F \subseteq \alpha\}\rangle \cap \langle \{U_\alpha \mid F' \subseteq \alpha\}\rangle \\ &= \langle \{U_\alpha \mid F, F' \subseteq \alpha\}\rangle = \langle \{U_\alpha \mid F \subseteq \partial\alpha\}\rangle, \end{aligned}$$

since F and F' are opposite. But we also have

$$\begin{aligned} G_F \cap G_{F'} &= \langle \{U_\alpha \mid F \subseteq \partial\alpha\}\rangle \\ &= \langle \{U_\alpha \mid C \subseteq \alpha, F \subseteq \partial\alpha\}\rangle = G_{C,]F[}, \end{aligned}$$

because we considered C as the fundamental chamber. With that,

$$G_C = G_{C,[F]}G_{C,]F[}.$$

The group $G_{C,[F]}$ is exactly the unipotent subgroup. □

Example 2.9.2.10. Let Δ be a building of simply-laced type and rank at least 3, and let F be a simplex of type J. By Tits's classification of thick irreducible spherical buildings of rank at least 3 (cf. [**Tit74**]), Δ admits an automorphism group $\mathsf{Aut}^\dagger(\Delta)$, which is a Chevalley group G, or — in case Δ corresponds to a projective space of dimension d defined over a non-commutative skew field $\mathbb{L}$ — a group G, which is the group $\mathsf{PGL}_{d+1}(\mathbb{L})$. Suppose $\mathsf{Res}_\Delta(F)$ is irreducible. Let P_F be the parabolic subgroup of G, which is the stabilizer of F. Let U_F be the unipotent subgroup and let L_F be the Levi subgroup such that $P_F = U_F L_F$. We now provide an explicit description of P_F, U_F and L_F for $G = \mathsf{PGL}_{d+1}(\mathbb{L})$.

First, we choose a basis of $\mathsf{PGL}_{d+1}(\mathbb{L})$ such that each subspace of F of dimension i is generated by the first $i+1$ base points. The flag F (cf. Note 2.1.0.8) consists of i-dimensional subspaces with $0 \leq i \leq d_1 - 1$ and $d - d_3 \leq i \leq d - 1$, where $|F| = d_1 + d_3$. Set $d_2 := d + 1 - d_1 - d_3$. Note that $J = \{1, \ldots, d_1, d - d_3 + 1, \ldots, d\}$. Then a generic element of P_F looks as follows,

$$\begin{pmatrix} T_{d_1} & M_{d_1\times d_2} & M_{d_1\times d_3} \\ O_{d_2\times d_1} & M_{d_2\times d_2} & M_{d_2\times d_3} \\ O_{d_3\times d_1} & O_{d_3\times d_2} & T_{d_3} \end{pmatrix},$$

where T_{d_i}, $i = 1, 3$, is an arbitrary invertible upper triangular matrix over F (T_{d_1} and T_{d_3} are independent of each other; even if $d_1 = d_3$ they are considered different), $M_{d_i\times d_j}$ is an arbitrary $d_i \times d_j$ matrix, $i \in \{1, 2\}$ and $j \in \{2, 3\}$ (with a similar remark as for the T_{d_i}), and $O_{d_i\times d_j}$ is the $d_i \times d_j$ zero matrix, $i \in \{2, 3\}$, $j \in \{1, 2\}$. With similar notation, and on top with U_{d_i}, $i \in \{1, 3\}$, an arbitrary unipotent upper triangular $d_i \times d_i$ matrix, D_{d_i}, $i \in \{1, 3\}$ an arbitrary invertible diagonal $d_i \times d_i$

matrix, and I_{d_2} the $d_2 \times d_2$ identity matrix, generic elements of U_F and L_F look as follows (where blanks replace zero matrices):

$$\begin{pmatrix} U_{d_1} & M_{d_1\times d_2} & M_{d_1\times d_3} \\ & I_{d_2} & M_{d_2\times d_3} \\ & & U_{d_3} \end{pmatrix} \text{ and } \begin{pmatrix} D_{d_1} & & \\ & M_{d_2\times d_2} & \\ & & D_{d_3} \end{pmatrix},$$

respectively. One indeed checks that $P_F = U_F L_F$ and $U_F \cap L_F = \{I_{d+1}\}$.

2.10. The Connection between Levi Subgroups & Little Projective Groups

In this section, we present some new results for spherical Moufang buildings, which are part of [**BSV25**] and [**BV25a**], which will be used later, and which explore the connection between the little projective groups and the special projectivity groups of Moufang spherical buildings.

When Δ has rank 2 — that is, when Δ is the building of a generalized polygon — any projectivity group $\Pi(F)$ of a simplex F is necessarily the projectivity group of a single vertex, which can be regarded either as a point (type 1) or a line (type 2) of the polygon. Knarr [**Kna88**] proved that if Δ is Moufang, then for every point or line x of Δ, the group $\Pi^+(x)$ coincides with the stabilizer of x in the little projective group of Δ.

We generalize this result to arbitrary simplices in arbitrary Moufang spherical buildings in Theorem 2.10.0.3. The proof follows the same strategy as in the rank-2 case, but it additionally relies on the fact that the unipotent radical of a parabolic subgroup in a Moufang spherical building fixes the corresponding residue pointwise and acts transitively on the simplices opposite that residue. A proof of this fact is provided in Lemma 2.10.0.4. Since this lemma admits a much shorter proof in the simply-laced case, we include Lemma 2.10.0.1, and present said shorter proof.

Lemma 2.10.0.1. *Let Δ be a spherical Moufang building of simply-laced type and let F be a simplex of Δ of type J. Let P_F be the stabilizer of F in $\mathsf{Aut}^\dagger(\Delta)$. Then the unipotent radical $U_F \leq P_F$ acts sharply transitively on the set $F^\equiv$ of simplices opposite F, and pointwise fixes $\mathsf{Res}_\Delta(F)$.*

Proof. From the definition of U_F in Section 8.5 of [**Car72**], it is readily deduced that U_F pointwise fixes $\mathsf{Res}_\Delta(F)$, because it is generated by root groups whose corresponding roots contain F, but not in their boundary. Furthermore, a Levi subgroup L_F is just the stabilizer in G of F and an opposite simplex F', as described in the introduction. Since P_F acts transitively on $F^\equiv$ (by the BN-pair property of Chevalley groups), we find that U_F acts transitively on $F^\equiv$. Since $U_F \cap L_F$ is just the identity (cf. Theorem 8.5.2 of [**Car72**]), the lemma follows. □

For Proposition 2.10.0.2, we will need a result that will be proved later in 5.6.5.5, where it makes more sense to do so. We state it here for convenience.

Lemma 5.6.5.5. *Let α be a root of an irreducible spherical Coxeter complex $\Sigma(W,S)$. Let F be the center of α (cf. 2.3.0.8). Let v be a vertex joinable to a vertex u that is joinable to F. Then v lies in α.*

Proposition 2.10.0.2. *Let F be a simplex of a Moufang spherical building Δ of simply-laced type. Let $\mathsf{Aut}^\dagger(\Delta)$ be the automorphism group of Δ generated by the root groups. Then the special projectivity group of F coincides with the faithful permutation group induced by the stabilizer $\mathsf{Aut}^\dagger(\Delta)_F$ of F in $\mathsf{Aut}^\dagger(\Delta)$ on the residue $\mathsf{Res}_\Delta(F)$ of F in Δ.*

Proof. (I) First, we want to show that every even self-projectivity of $\mathsf{Res}(F)$ is induced by a product of elations that stabilizes F. In fact, we are going to show that any even projectivity

$$\theta\colon \mathsf{Res}(F) \to \mathsf{Res}(T),$$

that maps F to a simplex T, is induced by an elation. Since self-projectivities are products of projectivities, it then follows that every even self-projectivity is induced by a product of elations that stabilizes F.

So let $\theta\colon \mathsf{Res}(F) \to \mathsf{Res}(T)$ be an even projectivity that maps F to a simplex T. It suffices to prove the assertion for the case that θ is a product of two perspectivities. Then there exists a simplex R opposite both F and T such that $\theta = \mathsf{proj}_T^R \circ \mathsf{proj}_R^F$. Since Δ is Moufang, it follows from Lemma 2.10.0.1 that there exists an elation g which maps F to T and fixes R pointwise. For an element f in $\mathsf{Res}(F)$, f^g is exactly the projection of $\mathsf{proj}_R(f)$ onto T, since elations preserve incidence. That means $g_{|\mathsf{Res}(F)} = \mathsf{proj}_T^R \circ \mathsf{proj}_R^F$.

(II) Now let $g\colon \Delta \to \Delta$ be a central elation. Let c be the center of any root corresponding to g. Let T be a simplex either containing c or joinable to it. Then, by Lemma 5.6.5.5, every vertex u of Δ joinable to T is contained in a root with center c (consider an apartment containing $\{c\} \cup T$ and $\{u\} \cup T$). It follows that g fixes $\mathsf{Res}(T)$ pointwise and moves a simplex F of the same type to a simplex F^g. First, we claim that the restriction $g_{|\mathsf{Res}(F)}$ is an even projectivity from $\mathsf{Res}(F)$ to $\mathsf{Res}(F^g)$.

Since Δ is Moufang, Δ is thick, and therefore there exists a simplex R in Δ opposite both F and T. Since elations preserve incidence, the image R^g is opposite both F^g and $T^g = T$.

Now for every $f \in \mathsf{Res}(F)$ we have:

$$f^g = \mathsf{proj}_{F^g}^{R^g} \circ \mathsf{proj}_{R^g}^T \circ \mathsf{proj}_T^R \circ \mathsf{proj}_R^F(f),$$

proving the claim. For an element h of the little projective group that stabilizes F, h is a product of central elations, and every such central elation gives rise to an even projectivity like above. □

The goal of the following is to extend this result to spherical Moufang buildings that are not necessarily of simply-laced type. To do so, it suffices to show that the unipotent radical of a parabolic subgroup in a Moufang spherical building fixes the corresponding residue pointwise and acts transitively on the simplices opposite that residue. Once this is established, the same arguments used in the proof of Proposition 2.10.0.2 apply verbatim.

We will first state the theorem in full generality and then prove the analogue of Lemma 2.10.0.1 for arbitrary spherical Moufang buildings in Lemma 2.10.0.4. We remark that one could instead appeal directly to [**MPW15**, Proposition 24.21], but we include a self-contained proof of Lemma 2.10.0.4.

Theorem 2.10.0.3. *Let F be a simplex of a Moufang spherical building Δ. Let $\mathsf{Aut}^\dagger(\Delta)$ be the automorphism group of Δ generated by the root groups. Then $\Pi^+(F)$ is permutation equivalent to the action of the stabilizer $\mathsf{Aut}^\dagger(\Delta)_F$ of F in $\mathsf{Aut}^\dagger(\Delta)$ on the residue $\mathsf{Res}_\Delta(F)$ of F in Δ.*

Lemma 2.10.0.4. *Let Δ be a spherical Moufang building and let F be a simplex of Δ. Let G_F be the stabilizer of F in $\mathsf{Aut}^\dagger(\Delta)$. Then there exists a subgroup $U_F \leq G_F$ which acts sharply transitively on the set $F^\equiv$ of simplices opposite F, and which pointwise fixes $\mathsf{Res}_\Delta(F)$.*

Proof. Let F' be an opposite simplex to F in Δ. Choose an apartment Σ containing F and F' and a chamber C in Σ containing F. For a root α of Σ, let U_α be the corresponding root group, that is, the group of automorphisms g of Δ such that g fixes all chambers having a panel in the interior of α.

- Let $G_F = \langle \{U_\alpha \mid \alpha \text{ is a root that contains } F\} \rangle$ be the group generated by all root groups of Σ for roots containing F. Similarly, we define $G_{F'}$ and G_C.
- Let $G_{C,[F]}$ be the group generated by all root groups U_α corresponding to roots α such that α contains C, and F is in α, but not in the boundary $\partial\alpha$ of α.
- Let $G_{C,]F[}$ be the group generated by all root groups U_α corresponding to roots α such that α contains C, and F is in the boundary $\partial\alpha$ of α.

We know that $G_{C,[F]}$ fixes $\mathsf{Res}(F)$ in Δ pointwise. This is equivalent to the fact that every one of the generating root groups of $G_{C,[F]}$ fixes $\mathsf{Res}(F)$ in Δ pointwise. That follows from [**AB08**, Lemma 7.33] and [**AB08**, Corollary 7.34]. We also know that G_C acts sharply transitively on the chambers opposite C in Δ. We want to show that $G_{C,[F]}$ acts sharply transitively on the simplices opposite F in Δ.

Every root α in Σ divides Σ into two half-apartments α and $-\alpha$ with $\partial\alpha = \alpha \cap -\alpha$. Notice that $G_{C,[F]}$ and $G_{C,]F[}$ generate G_C, since F is in C. We want to show that G_C is the product of group subsets $G_{C,[F]}$ and $G_{C,]F[}$. For that we have to see that $G_{C,[F]}$ is normal in G_C. Let β be a root such that β contains C and F, but F is not contained in $\partial\beta$, and let γ be a root such that γ contains C and F is contained in $\partial\gamma$. That means $U_\beta \in G_{C,[F]}$ and $U_\gamma \in G_{C,]F[}$.

First we claim that the open interval

$$(\beta, \gamma) = \{\text{Roots } \delta \text{ of } \Sigma \mid \delta \supseteq \beta \cap \gamma\} - \{\beta, \gamma\}$$

is completely contained in the set of roots that contain C and that contain F, but not in their boundary.

Since Σ is an apartment of a spherical building, Σ is a triangulated sphere. For every root δ of Σ, $-\delta$ is an opposite root and δ and $-\delta$ are two hemispheres such that $\Sigma = (\delta \cup -\delta)$.

The intersection $\beta \cap \gamma$ is the intersection of two different hemispheres that contain the chamber C. Let $Z := \partial\beta \cap \partial\gamma$ be the intersection of the boundaries. Every root in (β, γ) contains Z per definition. Since F is not in $\partial\beta$, F is not in Z.

For a root $\delta \in (\beta, \gamma)$, $(\beta \cap \gamma) \subseteq \delta$ and therefore $(-\beta \cap -\gamma) \subseteq -\delta$ and $(\partial\beta \cap \partial\gamma) \subseteq \partial\delta$. Hence, the boundaries of all roots in (β, γ) intersect in Z. The root γ is the unique hemisphere containing $\beta \cap \gamma$ that has a boundary that contains Z and F. Therefore all roots in (β, γ) contain F, but not in their boundary.

With [**AB08**, Proposition 7.72], we know that

$$[U_\beta, U_\gamma] \leq U_{(\beta,\gamma)} := \langle U_\delta \mid \delta \in (\beta, \gamma) \rangle.$$

Hence, we can now write $G_C = G_{C,[F]} G_{C,]F[}$.

Since F is opposite F' and F is in the boundary of roots corresponding to root groups generating $G_{C,]F[}$, F' must also be in the boundary of these roots. That means $G_{C,]F[}$ is contained in $G_{F'}$.

In summary, we know that G_C acts sharply transitively on the chambers opposite C in Δ, $G_C = G_{C,[F]} G_{C,]F[}$, and $G_{C,]F[} \subseteq G_{F'}$.

Now let F'' be an arbitrary simplex opposite F and C'' be an arbitrary chamber opposite C containing F''. Let C' be a chamber opposite C containing F'. Then there exist $a, b \in G_{C,]F[} G_{C,[F]}$, with $a \in G_{C,]F[}$ and $b \in G_{C,[F]}$, such that ab maps C' to C'' and hence maps F' to F''. So $F'^{ab} = F''$, and since $F'^{a} = F'$, we obtain $F'^{b} = F''$.

Therefore $G_{C,[F]}$ acts sharply transitively on simplices opposite F in Δ. By putting $U_F = G_{C,[F]}$, we arrive exactly at the statement of the lemma. □

Thus, we have established that, for a Moufang building Δ and any simplex F in Δ, every self-projectivity in the special projectivity group of F arises as a product of elations that pointwise fix F. Conversely, any element of the little projective group of Δ that fixes F pointwise can be expressed as a self-projectivity of F of even length. In particular, this yields the desired correspondence between the pointwise stabilizers of simplices in the little projective group and the special projectivity groups of those simplices.

2.11. Determining Special Projectivity Groups Algebraically

The results in this section are based on [**BSV25**]. We use the exact same notation as in Section 2.9.1. Furthermore, let $\alpha_i \subseteq \Sigma$ be the root such that the face of C_0, that does not contain v_i, is on the boundary of α_i.

We note that all groups T_i normalize each other and all H_{α_i} normalize each other. For each $i \in I$ the group T_i acts regularly on $R_i \setminus \Sigma$, which yields an isomorphism

$$\theta_i \colon \mathbb{K}^* \to T_i$$
$$\text{given by } \lambda \mapsto \theta_i(\lambda).$$

These isomorphisms provide a canonical isomorphism

$$T \cong (\mathbb{K}^*)^{|I|}.$$

Also, for each $i \in I$ there is a canonical homomorphism

$$h_i \colon \mathbb{K}^* \to T$$
$$\text{given by } \lambda \mapsto h_i(\lambda)$$

whose image is H_{α_i}. Note that $H \leq T$. The group H corresponds to the root lattice of G and, since $\hat{G}$ is the full linear type-preserving group, the group T corresponds to the weight lattice of G.

Let R'_J denote the unique residue of Σ that is opposite R_J, and let L_J be the corresponding Levi subgroup, that is, L_J is the stabiliser in G of $R_J \cup R'_J$.

By Theorem 2.10.0.3, $\Pi^+(R_J)$ is the image of L_J in $\mathsf{Aut}^\circ(R_J)$. Setting

$$T_J := L_J \cap T,$$

we find

$$L_J = \mathsf{Aut}^\dagger(R_J) T_J,$$

where $\mathsf{Aut}^\dagger(R_J)$ denotes the automorphism group of R_J generated by the root elations of R_J.

So, to obtain $\Pi^+(R_J)$, we have to determine T_J. This boils down to understanding the action of $h_i(\mathbb{K}^*)$ on R_J, for all $i \in I \setminus J$ (for $i \in J$, this action is already in $\mathsf{Aut}^\dagger(R_J)$).

If $i \in I \setminus J$ is not connected to J in the Coxeter diagram, then the action of H_{α_i} on R_J is trivial, as both U_{α_i} and $U_{-\alpha_i}$ pointwise fix R_J, and H_i is inside $L_{\{i\}}$. If $i \in I \setminus J$ is connected to a vertex $j \in J$ (and we write $i \sim j$), then j is uniquely determined. We claim that the action induced by H_{α_i} on R_J is the action of T_j.

Indeed, since by uniqueness of $j \sim i$, both H_i and T_j pointwise fix $R_{J\setminus\{j\}}$, since T_j acts faithfully on $R_{\{j\}}$, and $H_i \leq L_{\{i\}}$, it suffices to look in $R_{\{i,j\}}$, which is a projective plane.

We may choose coordinates such that $R_{\{j\}}$ corresponds to line $(*,*,0)$ and $R_{\{i\}}$ to the line pencil with vertex the point $(1,0,0)$ (thus, $C \cap R_{\{i,j\}} = \{(1,0,0),(*,*,0)\}$). Then

$$H_i/R_{\{i,j\}} = \left\{ \begin{pmatrix} 1 & & \\ & k & \\ & & k^{-1} \end{pmatrix} \mid k \in \mathbb{K}^\times \right\} \text{ and } T_j/R_{\{i,j\}} = \left\{ \begin{pmatrix} \ell & & \\ & 1 & \\ & & 1 \end{pmatrix} \mid \ell \in \mathbb{K}^\times \right\}.$$

Restricting to the first two coordinates and putting $\ell = k^{-1}$, the claim follows.

Hence $\Pi^+(R_J)$ is generated by $\mathsf{Aut}^\dagger(R_J)$ and all T_j, $j \in J$ such that there exists $i \in I \setminus J$ with $i \sim j$. It remains to determine $\mathsf{Aut}^\dagger(R_J)T_j$ for these values of $j \in J$ (and the groups they generate for suitable different j). These follow from straight forward computations in the corresponding Chevalley groups. We will explain the main examples and for that, we first introduce some notation.

Notation 2.11.0.1. Let

$$\mathsf{PSL}_n(\mathbb{K}, a) := \{M \in \mathsf{GL}_n(\mathbb{K}) \mid \det M = k^a, k \in \mathbb{K}\}.\mathsf{Sc}_n(\mathbb{K})/\mathsf{Sc}_n(\mathbb{K}),$$

where $\mathsf{Sc}_n(\mathbb{K})$ is the group of all scalar matrices over $\mathbb{K}$. We get $\mathsf{PGL}_n(\mathbb{K})$ by putting $a = 1$ and $\mathsf{PSL}_n(\mathbb{K})$ by putting $a = n$. Note that we can always choose a as a divisor of n since $\mathsf{PSL}_n(\mathbb{K}, a) = \mathsf{PSL}_n(\mathbb{K}, g)$, with $g = \gcd(a, n)$.

(1) Let R_J be the building of a projective space $\mathsf{PG}(d,\mathbb{K})$. We relabel
$$J = \{1, 2, \ldots, d+1\}$$
according to Bourbaki [**Bou68**]. Then, with respect to the standard chamber in $\mathsf{PG}(d,\mathbb{K})$, we have
$$T_j = \left\{ \begin{pmatrix} kI_j & \\ & I_{d-j+1} \end{pmatrix} \mid k \in \mathbb{K}^\times \right\},$$
where I_ℓ is the $\ell \times \ell$ identity matrix. It follows that
$$\mathsf{PSL}_{d+1}(\mathbb{K})T_j = \mathsf{PSL}_{d+1}(\mathbb{K}, j) = \mathsf{PSL}_{d+1}(\mathbb{K}, g), \text{ with } g = \gcd(d+1, j).$$
This has the following consequences:
   (*i*) If some endpoint of the subdiagram of the diagram of Δ corresponding to R_J has a neighbour outside J, then $\Pi^+(R_J)$ is $\mathsf{PGL}_{d+1}(\mathbb{K}) = \hat{G}$.
   (*ii*) If $j \in J$ corresponds to a Bourbaki label that is relatively prime to $d+1$, and it is connected to a vertex outside J, then, again, we have
   $$\Pi^+(R_J) = \mathsf{PGL}_{d+1}(\mathbb{K}) = \hat{G}.$$
   (Note (*i*) is a special case of this.)
   (*iii*) There are precisely three cases which are not applicable to either (*i*) or (*ii*). The first one is $J = \{1, 2, \ldots, n-1\}$ in a building of type D_n, with n even. Here, $j = n-2$ and we get $\mathsf{PSL}_n(\mathbb{K}, 2)$. The second one is $J = \{1, 3, 4, 5, 6\}$ in $\mathsf{E}_6(\mathbb{K})$. Here, the node 4 is joint to a vertex outside J, namely 2. The former node has in Bourbaki labelling for A_5 label 3 and hence $\Pi^+(R_J) = \mathsf{PSL}_6(\mathbb{K}, 3)$. The third situation is when $J = \{2, 4, 5, 6, 7\}$ in $\mathsf{E}_7(\mathbb{K})$. Here, the second node is joint to an outside node, leading to $\Pi^+(R_J) = \mathsf{PSL}_6(\mathbb{K}, 2)$.

(2) Let R_J be the building of type D_n over $\mathbb{K}$. Due to the existence of a branching vertex of valency 3 in the Dynkin diagram, only the end vertices can be joint to vertices outside this diagram when viewed as a subdiagram of the Dynkin diagram of Δ. So, we are only interested in GT_1, GT_{n-1}, GT_n and the groups generated by any combination of those. Referring forward to Notation 5.1.2.7 and Notation 5.7.1.2, it follows from Lemma 5.7.1.3 that
$$GT_1 = \mathsf{PGO}^\circ_{2n}(\mathbb{K}), \quad GT_nT_{n-1} = \overline{\mathsf{PGO}}^\circ_{2n}(\mathbb{K}) = \hat{G}$$
(but $T_n = T_{n-1}$ if n is odd), and $GT_n = GT_{n-1} = \overline{\mathsf{P\Omega}}_{2r}(\mathbb{K})$, if n is even. Lemma 5.7.1.3(*iii*) now implies $GT_1T_n = \overline{\mathsf{PGO}}^\circ_{2n}(\mathbb{K}) = \hat{G}$. All this implies that there are exactly two cases where $\Pi^+(R_{\{j\}})$ is not the full linear type-preserving group $\hat{G}$.
   (*i*) Δ is the building of type D_{n+1} over $\mathbb{K}$ and $I \setminus \{J\} = \{1\}$.
   Then the above implies $\Pi^+(R_J) = \mathsf{PGO}^\circ_{2n}(\mathbb{K})$.
   (*ii*) Δ is the building of type E_7 over $\mathbb{K}$ and $J = \{2, 3, 4, 5, 6, 7\}$.
   It follows from the above immediately that $\Pi^+(R_J) = \overline{\mathsf{P\Omega}}_{12}(\mathbb{K})$.

(3) Let R_J be the building of type E_6 over $\mathbb{K}$. We are only interested in $G^\dagger T_1$, which equals $\widehat{G}$ by Lemma 5.7.2.7. Another way to see this is by observing that the fundamental weight ω_1 does not belong to the root lattice (and the index $[\hat{G} : G] = 3$ is prime). Note that, in this case, the geometric approach

obtains $\Pi^+(R_J) = \widehat{G}$ in quite a different way, namely using Lemma 5.7.2.6 and symplectic polarities.

(4) Let Δ be the building of type E_8 over $\mathbb{K}$. Then $\hat{G} = G$, as the weight lattice and root lattice coincide. Hence for every $J \subseteq I$ we have $\Pi^+(R_J)$ is the full linear type-preserving group.

3. Point-Line Geometries

Spherical buildings can be viewed as certain *point-line geometries*, and we will proceed by giving a few basic definitions in that context. For instance, buildings of type A_n are *flag complexes* of *projective spaces*, while those of type C_n are flag complexes of *polar spaces*. For some buildings of exceptional type, we need to consider *parapolar spaces*.

After having introduced these geometries in general, we will talk about how to translate from a building to a geometry and vice versa, and discuss more specific properties in Chapter 4. We will already introduce the notions of opposition, projections, residues, and apartments for projective spaces and polar spaces here, since they are incredibly helpful. For parapolar spaces, however, the situation is considerably more intricate, and we will restrict our discussion of these notions to particular cases in Chapter 4.

The proofs of our main results later on will all be carried out within the corresponding geometries.

3.1. General Facts & Definitions

3.1.1. Point-Line Geometries, Incidence Systems, Duals & Collineations

Definition 3.1.1.1 (**(Pre-)Point-line geometries**)**.** A *pre-point-line geometry* is a tuple $\Gamma = (\mathscr{P}, \mathscr{L})$, where

(i) $\mathscr{P}$ is a set of points, and
(ii) $\mathscr{L}$ is a non-empty multiset of subsets of $\mathscr{P}$ of cardinality at least two, whose elements we will call *lines of the pre-point-line geometry*.

A *point-line geometry* is a pre-point-line geometry $\Gamma = (\mathscr{P}, \mathscr{L})$, where

(ii') $\mathscr{L}$ is a non-empty set of subsets of $\mathscr{P}$ of cardinality at least two.

If a point p is contained in a line L, we will say that *the point p is on the line L*, *p lives on L*, *p and L are incident*, or use other phrasings similar to that. In words, what (ii') tells us is that in a point-line geometry, every line has at least two points, and there are no repeating lines (cf. [**Van24**, Definition 2.2.1]). If we

just talk about a *geometry* in the following, we will mean a pre-point-line geometry without the property that $\mathscr{L}$ is necessarily non-empty.[1]

Definition 3.1.1.2 (**Subgeometries of (pre-)point-line geometries**)**.** For a (pre-)point-line geometry $\Gamma = (\mathscr{P}, \mathscr{L})$, a *subgeometry* is a (pre-)point-line geometry $\Gamma' = (\mathscr{P}', \mathscr{L}')$ such that $\mathscr{P}' \subseteq \mathscr{P}$ and $\mathscr{L}' \subseteq \mathscr{L}$.

Definition 3.1.1.3 (**Collinear points**)**.** We call two points $x, y \in \mathscr{P}$ *collinear* if there exists a line $L \in \mathscr{L}$ such that $x \in L$ and $y \in L$. Instead of saying that x and y are collinear, we sometimes just write $x \perp y$. Otherwise, we say x and y are *non-collinear* and write $x \not\perp y$. If x and y are two collinear points in a point-line geometry, then we refer to the line containing both by writing xy, if this line is unique.

Definition 3.1.1.4 (**Thick lines**)**.** We call a line in a pre-point-line geometry *thick* if it contains at least three points.

Definition 3.1.1.5 (**Incidence systems**)**.** Let I be a set. A triple $\Gamma = (X, *, \tau)$ is called an *incidence system over* I if

(i) X is a set,
(ii) $*$ is a symmetric and reflexive relation on X,
(iii) τ is a surjective map from X to I such that distinct elements $x, y \in X$ with $x * y$ satisfy $\tau(x) \neq \tau(y)$.

The relation $*$ is also called the *incidence relation of* Γ. If $x * y$, then we also say that x and y are *incident*. We call I the *type set* of Γ and its elements *types*. The map τ is called the *type map of* Γ. The cardinality of I is called the *rank of* Γ (cf. [**BC13**, Definition 1.2.2 and 1.2.3]).

Note 3.1.1.6. Any point-line geometry can be viewed as an incidence system (cf. [**BC13**, Definition 2.5.12]). Given a point-line geometry, we define

$$p * L :\Longleftrightarrow p \in L, \quad \text{for } p \in \mathscr{P},\ L \in \mathscr{L}.$$

The set X has as elements the elements of $\mathscr{P}$ and the elements of $\mathscr{L}$. The types are *points* and *lines*. Depending on the context, one view is sometimes more advantageous than the other. Partly because of that, one finds many different ways of defining a point-line geometry in the literature.

Definition 3.1.1.7 (**Collineations**)**.** An *automorphism* or a *collineation* of a point-line geometry is a permutation of the point set inducing a permutation of the line set, that is, the set of points on any line is mapped onto the set of points of another line, and each line occurs as such an image. In particular collineations preserve collinearity. The set of all collineations forms a group called the *collineation group* (cf. [**Van24**, §7]).

[1]There is not one consistent definition for point-line geometries in the literature. Sometimes, lines are not even defined as sets of points (cf. [**Shu10**, §§3.1.2]). Furthermore, sometimes other words are used to describe what we defined here, as for example in [**BC13**, Definition 2.5.8], where the expression *line space* is used. It is rather uncommon to require that the set of lines is non-empty, as we did here. However, the author believes that it would be strange to talk about a point-line geometry without lines. The expression *point-geometry* would appear more sensible in that instance.

Definition 3.1.1.8 (**Isomorphisms**)**.** An *isomorphism* between two point-line geometries is a bijection between the sets of points that induces a bijection between the sets of lines.

3.1.2. Collinearity Graphs, Incidence Graphs, Distances

Definition 3.1.2.9 (**Collinearity graphs** or **point graphs**)**.** Given a point-line geometry, we can construct a graph by drawing a vertex for every point $p \in \mathscr{P}$ and by drawing an edge between two points p and q if p and q are collinear. This graph is called the *point graph* or *collinearity graph* (cf. [**Van24**, Definition 9.2.1]).

Definition 3.1.2.10 (**Distances between points**)**.** Let Δ be a point-line geometry. The *distance between two points* p and q in Δ is the graph-distance between the corresponding vertices in the collinearity graph.

Definition 3.1.2.11 (**Diameters**)**.** Let Δ be a point-line geometry. We define the *diameter of* Δ to be the maximal distance that two points in Δ can have, and ∞ if such a maximum does not exist.

Since we will only be concerned with point-line geometries corresponding to spherical buildings, such a maximum will always exist.

Some authors prefer to use the *incidence graph* to define distance and diameter. We want to mention them here in contrast to collinearity graphs, for clarity and completeness. We will also use them a little bit later.

Definition 3.1.2.12 (**Incidence graphs**)**.** The *incidence graph* of a (pre)-point-line geometry $(\mathscr{P}, \mathscr{L})$ is the bipartite graph with vertex set $V = \mathscr{P} \cup \mathscr{L}$ and edge set $E = \{(p, L) \mid p \in L\}$.

Definition 3.1.2.13 (**Girths**, **gonalities**)**.** For a point-line geometry Δ, we define the *girth of* Δ as the girth of its incidence graph Γ. In particular, if there is no minimal circuit in the incidence graph of Γ, then the girth of Δ is ∞.

The *gonality* of Δ (or of Γ) is half of its girth.

3.1.3. Subspaces & Hyperplanes

Definition 3.1.3.14 (**Subspaces**, **convex**, **proper**, **maximal**, **singular**)**.**
Let $\Gamma = (\mathscr{P}, \mathscr{L})$ be a point-line geometry.

- A *subspace* Γ' *of* Γ is a point-line geometry with a point set $\mathscr{P}' \subseteq \mathscr{P}$ and a line set $\mathscr{L}' \subseteq \mathscr{L}$ such that, if for two points $p, q \in \mathscr{P}'$ there exists a line $L \in \mathscr{L}$ carrying both, then L is also contained in $\mathscr{L}'$ (and all points of L are contained in $\mathscr{P}'$).
- A subspace Γ' is called *convex* if all vertices of any shortest path in the collinearity graph of Γ between any two elements of Γ' are contained in Γ' (cf. [**Van24**, Note 1.2.3 and Definition 9.2.1]).

- A *proper* subspace is a subspace that does not coincide with Γ itself.
- A proper subspace of Γ is a *maximal subspace* of Γ if it is not contained in any other proper subspace of Γ.
- We call a subspace *singular* if any two points are collinear.
- A *maximal singular subspace* of Γ is a singular subspace that is not a proper subspace of any other singular subspace of Γ.

Definition 3.1.3.15 (**Line pencils**, **point sets**)**.** Let $\Gamma = (\mathscr{P}, \mathscr{L})$ be a point-line geometry and $p \in \mathscr{P}$. We define $\mathscr{L}_p$ as the set of all lines in Γ through p and call $\mathscr{L}_p$ a *line pencil through* p. For any singular subspace U, we define $\mathscr{P}\big|_U := \{x \in \mathscr{P} \mid x \in U\}$ and call it the *point set of* U.

Definition 3.1.3.16 (**Spanned subspaces**, **generating sets**)**.** Let Γ be a point-line geometry and X a set of points of Γ. We define $\langle X \rangle$ as the intersection of all subspaces of Γ that contain X. Since the intersection of subspaces is again a subspace (cf. [**Bue95**, Ch. 12]), $\langle X \rangle$ is a subspace and we call it the *subspace generated by* X. If $X = \{p_0, p_1, \ldots, p_n\}$, we write $\langle p_0, p_1, \ldots, p_n \rangle$ instead of $\langle \{p_0, p_1, \ldots, p_n\} \rangle$. As mentioned in Definition 3.1.1.3, if X only consists of two collinear points p and q, contained in a unique common line, then we usually write pq instead of $\langle p, q \rangle$. If we denote by U the subspace $\langle X \rangle$, then we also say that U is *spanned by* X or that X is a *generating set for* U. If U and V are two subspaces of Γ, then, instead of writing $\langle U \cup V \rangle$, we write $\langle U, V \rangle$.

Definition 3.1.3.17 (**Planes**)**.** A subspace generated by three mutually collinear points not on a common line is called a *plane*.

Note that, in general, a plane is not necessarily a singular subspace. However, in all the cases we will deal with — except for *icosahedral geometries* (cf. Definition 4.10.1.1), which we will only discuss in a much later chapter — subspaces generated by pairwise collinear points are singular; in particular, planes will be singular subspaces. (We will also prove that all planes of *certain* icosahedral geometries are always singular in Corollary 4.10.1.8.)

Definition 3.1.3.18 (**Radicals**, **collinear subspaces**)**.**
Let Γ be a point-line geometry and X a subspace of Γ. We set

$$X^\perp := \bigcap_{x \in X} x^\perp.$$

The *radical* of a subspace X is the set $\mathsf{Rad}(X) := X \cap X^\perp$. We say that two subspaces X and Y are *collinear* if every point of X is collinear to every point of Y.

Lemma 3.1.3.19. *Let Γ be a point-line geometry, and let X and Y be two subspaces of Γ. Then the following hold.*

- *If $X \subseteq Y$, then $Y^\perp \subseteq X^\perp$.*
- *$(X \cup Y)^\perp = X^\perp \cap Y^\perp$.*
- *$X \subseteq (X^\perp)^\perp$.*
- *$X^\perp = ((X^\perp)^\perp)^\perp$.*
- *X is singular if and only if $X \subseteq X^\perp$.*
- *If X is singular, then $\mathsf{Rad}(X^\perp) = (X^\perp)^\perp$.*

REFERENCE. This is [**Bue95**, Ch. 12, Closure Lemma 1.1]. □

Definition 3.1.3.20 (**Chains**)**.** Let $\Gamma = (\mathscr{P}, \mathscr{L})$ be a point-line geometry and

$$\Gamma_0 = (\mathscr{P}_0, \mathscr{L}_0),\ \Gamma_1 = (\mathscr{P}_1, \mathscr{L}_1),\ \Gamma_2 = (\mathscr{P}_2, \mathscr{L}_2), \dots$$

subspaces of Γ such that

$$\mathscr{P}_0 \subsetneq \mathscr{P}_1 \subsetneq \mathscr{P}_2 \subsetneq \dots \ .$$

Then we call $\Gamma_0 \subsetneq \Gamma_1 \subsetneq \Gamma_2, \dots$ a *chain in* Γ.

Definition 3.1.3.21 (**Geometric hyperplanes**)**.** Let Δ be a point-line geometry. A *geometric hyperplane* H of Δ is a proper subspace of Δ such that every line of Δ contains at least one point of H. We also say: each line of Δ *intersects* or *meets* H. Sometimes we omit the "geometric" and just say *hyperplane*.

3.1.4. Gamma Spaces

We will be interested in three different kinds of point-line geometries: *projective spaces*, *polar spaces*, and *parapolar spaces*. These are all gamma spaces (cf. [**Hig83**]).

Definition 3.1.4.22 (**Gamma spaces**)**.** A point-line geometry $\Gamma = (\mathscr{P}, \mathscr{L})$ is called a *gamma space* if, for every point $p \in \mathscr{P}$ and every line $L \in \mathscr{L}$, we have that p is either collinear to none, exactly one, or all points of L.

Lemma 3.1.4.23. *Let $\Gamma = (\mathscr{P}, \mathscr{L})$ be a gamma space. Then the following assertions hold.*

- (*i*) *For any subset X of $\mathscr{P}$, the subset $X^\perp$ is a subspace of Γ and coincides with $\langle X \rangle^\perp$.*
- (*ii*) *If X is a singular subspace, then $(X^\perp)^\perp$ is a singular subspace of Γ containing X.*
- (*iii*) *Every maximal singular subspace X satisfies $X = X^\perp = (X^\perp)^\perp$, and every singular subspace is contained in a maximal singular subspace.*
- (*iv*) *If X is a singular subspace, then $(X^\perp)^\perp$ is the intersection of all maximal singular subspaces containing X.*
- (*v*) *If x and y are two distinct collinear points, then $(\{x, y\}^\perp)^\perp$ is a singular subspace containing every line through x and y.*

REFERENCE. This is [**Bue95**, Ch. 12, Gamma Space Lemma 1.2]. □

Heuristically, we can view a polar space as glued-together projective spaces, and a parapolar space as glued-together polar spaces. Hence, we will first talk about projective spaces.

3.2. Projective Spaces

An important example of a projective space is a *projective plane*.

Definition 3.2.0.1 (**Projective planes**). A *projective plane* is a point-line geometry $(\mathscr{P}, \mathscr{L})$ such that (cf. [**Bue95**, Ch. 2, p. 34])

(i) for $p, q \in \mathscr{P}$, there always exists a unique $L \in \mathscr{L}$ such that $p, q \in L$. (Any two points are collinear.)
(ii) for $L, J \in \mathscr{L}$, there always exists a unique $p \in \mathscr{P}$ such that $p \in L \cap J$. (Any two lines intersect.)
(iii) there exist at least 4 points of which any 3 points are non-collinear. (There exists a quadrangle.)

Example 3.2.0.2. The smallest example of a projective plane is the so-called *Fano plane*.[2] It has 7 points and 7 lines. In Fig. 1 we have labeled the points with the

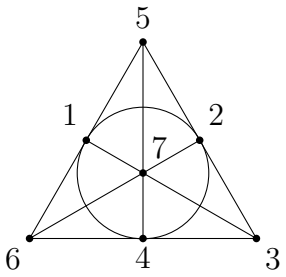


Fig. 1. The Fano plane is an example of a projective plane.

numbers $1, 2, \ldots, 7$. The "circle" in the middle is one line. It is the set of points $\{1, 2, 4\}$. For example, one can see a quadrangle by looking at the points labeled 2, 3, 4 and 7.

Suppose we wanted to find a smaller example of a projective plane. Since we need to be able to see a quadrangle, we need at least 4 lines. If we start with a quadrangle, take one line and call it L_0, and then go around clockwise and call the following lines L_1, L_2 and L_3. Then we can see that L_0 and L_2, and also L_1 and L_3, do not intersect. We have to add a point in which L_0 and L_2 intersect. Then also L_1 and L_3 still have to intersect. The intersection point $L_1 \cap L_3$ cannot be the same as the intersection point $L_0 \cap L_2$ because, in that case, L_0 and L_3 would intersect in two points. That means we have to add another point in which L_1 and L_3 intersect. Furthermore, we need to add lines between $L_0 \cap L_1$ and $L_2 \cap L_3$, and also between $L_0 \cap L_3$ and $L_1 \cap L_2$. These new lines, again, would have to intersect in a point n. This new point n needs to be collinear to all other points. We also still need a line between $L_0 \cap L_2$ and $L_1 \cap L_3$. We add another line carrying $L_0 \cap L_2$, $L_1 \cap L_3$ and n, and arrive at 7 points and 7 lines.

[2]Named after Gino Fano, who first found this minimal example and described it in the article *On foundations of projective geometry* in 1892. Interestingly enough, the Fano plane does not satisfy Fano's axiom (cf. [**BR04**, p. 86]).

Definition 3.2.0.3 (**Projective spaces**)**.** Let $\mathscr{P}$ be a set of points and $\mathscr{L}$ a set of subsets of $\mathscr{P}$, whose elements we call lines again. Then the point set $\mathscr{P}$, equipped with all its subspaces, is called a *projective space* if we are in one of the following scenarios:

(1) $\mathscr{P}$ is the empty set.
(2) $\mathscr{P}$ consists of one point.
(3) $(\mathscr{P}, \mathscr{L})$ consists of one line with at least 3 points.
(4) $(\mathscr{P}, \mathscr{L})$ is a projective plane.
(5) $(\mathscr{P}, \mathscr{L})$ has the following properties:
   (i) For $p, q \in \mathscr{P}$, there always exists a unique $L \in \mathscr{L}$ such that $p, q \in L$.
   (ii) If a, b, c, d are different points and ab and cd intersect, then ad and bc intersect.
   (iii) There exist at least 4 points of which any 3 points are non-collinear.
   (iv) Every line has at least three points.
   (v) Any chain of subspaces has finite length.

The characterization of (5) is due to Veblen and Young. Axiom (ii) of (5) is often called the *Veblen-Young axiom* or *Pasch's axiom*.

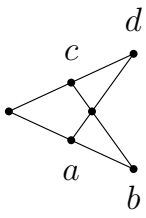


FIG. 2. A visualization of the Veblen-Young axiom.

Note 3.2.0.4. Notice that a projective space $(\mathscr{P}, \mathscr{L})$ is a point-line geometry as soon as $\mathscr{L}$ is not empty. We will call a projective space *trivial* if we are in one of the scenarios (1) – (3).

Definition 3.2.0.5 ($\mathsf{PG}(\boldsymbol{d}, \mathbb{L})$, $\mathsf{PG}(\boldsymbol{V})$)**.** Let V be a vector space of dimension $d+1$ at least 3. Let $\mathscr{P}$ be the set of 1-dimensional subspaces of V, and let $\mathscr{L}$ be the set of those subsets of $\mathscr{P}$ that are contained in a given 2-dimensional subspace of V. Then the point-line geometry $(\mathscr{P}, \mathscr{L})$ is denoted by $\mathsf{PG}(d, \mathbb{L})$, or $\mathsf{PG}(V)$, and we call it the *projective space on* V (cf. [**BC13**, Definition 5.2.1]).

Example 3.2.0.6. It is difficult to draw a projective space that is neither trivial nor just a projective plane. However, we provide an illustration for the smallest projective space of dimension 3 in Fig. 3.[3] It has 15 points, 35 lines and 15 planes. Every line has 3 points, and every line is contained in 3 planes. Every plane is a Fano plane with 7 points and 7 lines.

[3]As a model, we used an illustration drawn by Frans Marcelis, which is available on his webpage: `https://fgmarcelis.wordpress.com/pg32/pg32-1/`.

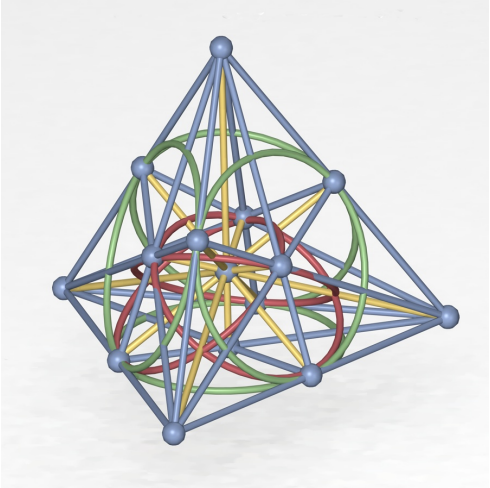


FIG. 3. An example of a projective space of dimension 3.

Despite the difficulty of actually drawing projective spaces that are not trivial or projective planes, we have the following theorem.

Theorem. *Let $\Gamma = (\mathscr{P}, \mathscr{L})$ be a non-trivial projective space. Then the point set, equipped with all its lines, is either a projective plane or $\mathsf{PG}(d, \mathbb{L})$ for some integer $d > 3$ and some skew field $\mathbb{L}$.*

REFERENCE. For a sketch of proof, we refer to [**Bue95**, Ch. 2, pp. 36–37]. □

Definition 3.2.0.7 (**Dimension of a projective space**)**.** Let $\Gamma = (\mathscr{P}, \mathscr{L})$ be a projective space. Then the *dimension* $\dim(\Gamma)$ *of* Γ is defined as follows.

$$\dim(\Gamma) := \begin{cases} -1, & \text{if } \mathscr{P} \text{ is the empty set.} \\ 0, & \text{if } \mathscr{P} \text{ consists of one point.} \\ 1, & \text{if } (\mathscr{P}, \mathscr{L}) \text{ consists of one line with at least 3 points.} \\ 2, & \text{if } (\mathscr{P}, \mathscr{L}) \text{ is a projective plane.} \\ d, & \text{if } (\mathscr{P}, \mathscr{L}) \text{ is } \mathsf{PG}(d, \mathbb{L}) \text{ for some integer } d > 3, \\ & \text{and some skew field } \mathbb{L}. \end{cases}$$

We will refer to projective spaces of dimension d as *d-spaces*, to projective planes as just *planes*, and, if we want to emphasize that we view a line as a projective space of dimension 1, then we will also sometimes refer to them as *projective lines*.

Definition 3.2.0.8. Let S be a set of points in a projective space Γ. The unique projective subspace of smallest dimension in which each point of S and each line between points of S is contained will be denoted by $\langle S \rangle$, which is compatible with 3.1.3.16. If U is a singular subspace and $S = \{p_0, p_1, \ldots, p_n\}$ is a set of points such that $\langle S \rangle = U$, but $\langle S \setminus \{p_i\} \rangle \neq U$ for all $i \in \{0, 1, \ldots, n\}$, then we call S a *minimal generating set of* U.

Note 3.2.0.9 (**Hyperplanes in projective spaces**)**.** The geometric hyperplanes (cf. 3.1.3.21) of a projective space Γ of dimension n are just the projective subspaces of dimension $n-1$. Each geometric hyperplane of Γ is a maximal subspace of Γ (cf. [**BC13**, Lemma 5.2.8]).

Definition 3.2.0.10 (**Dual projective spaces**, **dualities**)**.** The hyperplanes of a projective space can be considered as the points of the projective space corresponding to the dual vector space. We call this projective space the *dual projective space*. An isomorphism from a projective space to its dual is called a *duality*.

We also use the term *duality* for an isomorphism of a projective space to the dual of another projective space.

Definition 3.2.0.11 (**Elations**, **homologies**, **axes**, **centers**)**.** Let Γ be a projective space, let H be a hyperplane of Γ, and let c be a point of Γ. We say a collineation of Γ has *axis* H if it fixes all points of H, and we call such collineations *axial collineations*. We say a collineation of Γ has *center* c if it maps every line containing c onto itself. We then call it a *central collineation*. If a collineation α has an axis H and a center c such that $c \in H$, then we also call α an *elation*, and if $c \notin H$, then we call α a *homology*.

Lemma 3.2.0.12. *Let Γ be a projective space, and let H be a hyperplane of Γ. Then*

(i) each collineation of Γ fixing each point of H, as well as a point outside H, is a collineation with axis H and center c.

(ii) each non-trivial translation of Γ has a unique center and a unique axis.

(iii) if o, a, and b are collinear points with $a, b \notin H \cup \{o\}$, then the collineation group of Γ has at most one collineation α with axis H and center o such that $\alpha(a) = b$.

Reference. This is [**BC13**, Lemma 6.1.3]. □

Theorem 3.2.0.13 (Desargues' Theorem)**.** *Let Γ be a projective space of dimension at least three. Let $\{p_1, p_2, p_3\}$ and $\{q_1, q_2, q_3\}$ be two sets that each contain three distinct, mutually collinear points of Γ. Then the intersection points*

$$p_1p_2 \cap q_1q_2, \quad p_2p_3 \cap q_2q_3, \quad \text{and} \quad p_3p_1 \cap q_3q_1$$

are three distinct points that are all contained in a common line if and only if the lines p_rq_r, for $r \in \{1, 2, 3\}$, all intersect in the same point.

Reference. See [**Bue95**, Ch. 2]. □

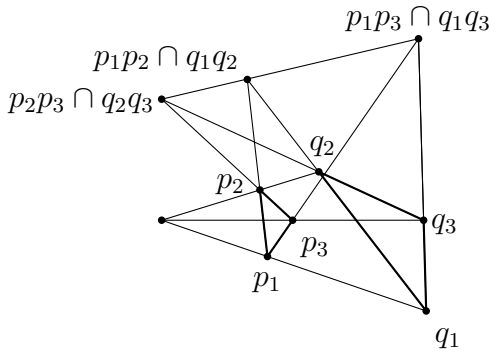


Fig. 4. A visualization of the Desargues' Theorem.

Proposition 3.2.0.14. *A projective plane is isomorphic to* $\mathsf{PG}(2, \mathbb{L})$ *for some skew field* $\mathbb{L}$ *if and only if it satisfies Desargues' Theorem.*

REFERENCE. This is [**Bue95**, Theorem 2.1]. □

Note 3.2.0.15. Let $\Gamma = (\mathscr{P}, \mathscr{L})$ be a projective space of dimension at least 1. If we view Γ as a point-line geometry, then the subspaces of Γ (as defined in 3.1.3.14) are again projective spaces.

Definition 3.2.0.16 (**Order of a finite projective space**)**.** A *finite projective space* Γ is a projective space where $\mathscr{P}$ is a finite set of points. In any finite projective space, each line contains the same number of points, and the *order of* Γ is defined as one less than this common number.

3.3. Polar Spaces

The main reference regarding polar spaces will be the book *Polar Spaces* by Hendrik Van Maldeghem [**Van24**]. We use the definition for *polar spaces* that is introduced in Chapter 2.

Definition 3.3.0.1 (**Polar Spaces**)**.**
A *polar space* is a point-line geometry $\Gamma = (\mathscr{P}, \mathscr{L})$ such that

(*i*) for every point p and every line L that does not contain p, either there exists exactly one point q on L that is collinear to p, or all points on L are collinear to p,
(*ii*) every line contains at least three points,
(*iii*) any chain of singular subspaces has finite length,
(*iv*) no point is collinear to all other points.

Axiom (i) is also called the *one-or-all axiom*.

Definition 3.3.0.2 (**Possibly degenerate**, **degenerate**, **non-degenerate**)**.** A *possibly degenerate polar space* Δ is a point-line geometry satisfying (i)-(iii), but possibly not (iv). If there exists a point in Δ collinear to all other points, then we call Δ *degenerate*. The point-line geometries satisfying (i)-(iv) are also sometimes called *non-degenerate polar spaces*.

Note 3.3.0.3. Recall from Definition 3.1.3.14 that a subspace of a point-line geometry is singular if any two points are collinear, and that in a projective space any two points are collinear. One can prove that the singular subspaces of a polar space are projective spaces. See [**Van24**, Proposition 2.3.6]. An important observation is that two collinear points determine a unique line.

Definition 3.3.0.4 (**Rank of a polar space**)**.** Let Γ be a polar space. The *rank of* Γ is one more than the highest dimension of any projective space contained in Γ.

Note 3.3.0.5. Since we defined polar spaces as point-line geometries, every polar space automatically has rank at least 2 now, because we required the set $\mathscr{L}$ to be non-empty in Definition 3.1.1.1. For some very specific purposes, and by slight abuse of language, we will define the following.

Definition 3.3.0.6 (**Rank-0** & **rank-1 polar spaces**)**.** A *rank*-1 *polar space* is a set of pairwise non-collinear points of cardinality at least 2. A *rank*-0 *polar space* is a set of just one point.

Some authors have less strict general definitions for polar spaces and point-line geometries that immediately include rank-1 and rank-0 polar spaces, see for example [**Shu10**, Definition 3.1.2, 7.3.1 and p. 175]. In this thesis, when we talk about a polar space, we always mean a polar space of rank ≥ 2, unless stated otherwise or unless the polar space occurs out of some construction in a higher-dimensional polar space, where we want to allow the rank to drop below 2.[4]

Example 3.3.0.7. If a polar space Γ contains projective planes but no 3-spaces, then the rank of Γ is 3. If a polar space Γ' contains projective lines but no projective planes, then the rank of Γ' is 2. (Notice that a polar space that contains $d-1$-spaces but no d-spaces cannot contain any projective subspaces of dimension greater than d, since we can always find projective subspaces of all lower dimensions in a higher-dimensional projective space.)

Note 3.3.0.8. Rank 2 polar spaces are also called *generalized quadrangles*. If we think of an *ordinary quadrangle* (a proper definition for that is given in Definition 3.5.0.1), we can see that every point is either incident with a given line or collinear to exactly one point of the line. (Furthermore, there is exactly one unique point that a given point is not collinear to. One could say that this point is *opposite* the given point. The concept of opposition will play an important role later.) However, if we think of an ordinary quadrangle, then we think of something that consists of four lines and four points. Our definition of a polar space requires that every line has at least three points. If we glue together several ordinary quadrangles, we can obtain a generalized quadrangle and thus a polar space (cf. Example 3.3.0.16 and Example 3.3.0.17).

Note 3.3.0.9. Let Γ be a polar space. Then the only convex proper subspaces of Γ (as defined in Definition 3.1.3.14) are the singular subspaces of Γ. For a proof see [**Van24**, Proposition 9.2.2].

Note 3.3.0.10. Let S be a set of mutually collinear points in a polar space Δ. As in Definition 3.2.0.8, the unique projective subspace of smallest dimension in which each point of S and each line between points of S is contained will be denoted by $\langle S \rangle$, and the existence and uniqueness are proved in [**Van24**, §1.3]. In completely the same way as in Definition 3.1.3.16 and Definition 3.2.0.8, we define (minimal) generating sets and what it means for a subspace to be spanned by a set.

[4]Even though this way of including rank-1 and rank-0 polar spaces might not seem pretty, the author chose to do it this way for two reasons: (1) The author would dislike calling something a *point-**line** geometry* when it does not contain any lines. (2) Projective spaces and polar spaces should be examples of point-line geometries.

Definition 3.3.0.11 (**Geometric hyperplanes of a polar space**)**.** A *geometric hyperplane* of a polar space Γ is a proper subspace of Γ that is not disjoint from any line in Γ (see also Definition 3.1.3.21).

Lemma 3.3.0.12. *Let Γ be a polar space, p a point in Γ, and U a singular subspace of Γ not containing p. If p is collinear with a generating set S of points of U, then $\langle \{p\} \cup S \rangle$ is a singular subspace of Γ containing U as a hyperplane. In particular, p is collinear with all points of $U = \langle S \rangle$.*

Reference. This is [**Van24**, Corollary 1.3.5]. □

Lemma 3.3.0.13. *Let Γ be a polar space. If U is a projective subspace and p a point not contained in U, then $p^\perp \cap U$ is either equal to U or a hyperplane of U. Furthermore,* $\dim(\langle p, p^\perp \cap U \rangle) = \dim(p^\perp \cap U) + 1$.

Reference. This is [**Van24**, Proposition 1.4.1]. □

Definition 3.3.0.14 (**Thick** & **top-thin polar spaces**)**.** A polar space Γ of rank n is called *thick* (*top-thin*) if each projective subspace of dimension $n-2$ is contained in at least 3 (exactly 2) projective subspaces of dimension $n-1$.

Example 3.3.0.15. Let Γ be a polar space of rank 2. Then Γ is top-thin if each point is on exactly 2 lines. Let Γ' be a polar space of rank 3. Then Γ' is thick if each line is contained in at least 3 planes.

Example 3.3.0.16. An example of a thick generalized quadrangle has 15 lines and 15 points, see Fig. 5. The space in the middle is white to indicate that none of the lines intersect there. If all lines intersected in the middle, then there would be a point collinear to all the others, which contradicts axiom (ii).

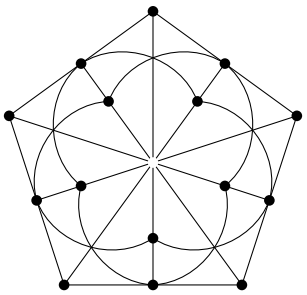


Fig. 5. An example of a thick rank 2 polar space.

Example 3.3.0.17. An example of a top-thin generalized quadrangle is a *grid*. See Figure 6.

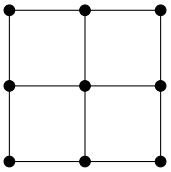


Fig. 6. A (3×3) grid is an example of a top-thin polar space of rank 2.

We point out that in a polar space of rank at least three the projective subspaces of dimension 2 are projective planes.

Definition 3.3.0.18 (**Submaximal singular subspaces**, **codimensions**)**.** Let Γ be a polar space of rank n.

- We will call a projective subspace of dimension $n-2$ a *submaximal singular subspace*.
- The *codimension* of a singular subspace U is $(n-1)-\dim(U)$.

Note that every singular subspace of Γ is contained in a maximal singular subspace.

Definition 3.3.0.19 (**Finite polar spaces**)**.** A polar space $\Gamma = (\mathscr{P}, \mathscr{L})$ is called *finite* if it has a finite number of points (i.e., $|\mathscr{P}| < \infty$).

Definition 3.3.0.20 (**Order of a finite polar space**)**.** A polar space Γ has *order* (s,t) if every line in Γ has exactly $s+1$ points, and each submaximal singular subspace of Γ is contained in exactly $t+1$ maximal singular subspaces.

Note 3.3.0.21. We note the following.

- Every polar space Δ of rank > 2 and every polar space of rank 2 that is not top-thin has an order (cf. [**Van24**, Corollary 1.7.5]).
- The top-thin polar spaces of rank 2 are grids like the one in Fig. 6 — just possibly with more points and lines (cf. [**Van24**, Note 1.7.6]).

We say more about finite polar spaces of certain orders in Fact 5.1.5.12.

Definition 3.3.0.22 (**Polar spaces of infinite rank**)**.** A *polar space of infinite rank* is a point-line geometry in which every line contains at least three points, for every point x, the set $x^\perp$ is a geometric hyperplane, and there exist chains of singular subspaces of infinite length.

Throughout this thesis, we will only consider polar spaces of *finite rank* (that is, their maximal singular subspaces are generated by a finite number of points), with the exception of Section 6.5.

Definition 3.3.0.23 (**Duals of thick polar spaces**)**.** Let Δ be a thick polar space of rank n. We define the *dual* Δ^{dual} of Δ as follows: the points of Δ^{dual} are the maximal singular subspaces of Δ, and a line of Δ^{dual} is formed by all maximal singular subspaces that intersect in a common submaximal singular subspace in Δ. That means two points of Δ^{dual} are collinear if the corresponding maximal singular subspaces in Δ intersect in a submaximal singular subspace. The dual Δ^{dual} is a point-line geometry (more precisely, a *parapolar space of symplectic rank* 2, cf. Section 3.4).

Definition 3.3.0.24 (**Elations**, **central** and **axial collineations**)**.** Let Δ be an arbitrary polar space. Let c be any point of Δ. Then a *central collineation* (with *center* c) is a collineation of Δ fixing each point of Δ that is collinear to c (cf. [**Van24**, Definition 7.2.1]).

Let L be any line of Δ. Then an *axial collineation* (with *axis* L) is a collineation of Δ preserving each line of Δ that intersects L in precisely one point (cf. [**Van24**, Definition 7.3.1]).

If a collineation of Δ is either central or axial, then we call it an *elation* of Δ.

3.3.1. Oriflamme Geometries

Top-thin polar spaces admit a description as so-called *oriflamme geometries* (cf. [**Van24**, Ch. 6]), which we will introduce in this subsection.

In the introduction, we mentioned that every Moufang building is a *flag complex* of some *Lie incidence geometry*. For instance, a thick building of type B_n is the standard flag complex of a thick polar space. Similarly, buildings of type D_n arise as flag complexes of *oriflamme geometries* constructed from top-thin polar spaces.[5] This does not contradict the first statement: oriflamme geometries have the same points and lines as top-thin polar spaces, but they come with a natural way for defining the flags that produce buildings of type D_n. Once the oriflamme geometry is defined, its standard flag complex is a building of type D_n.

Definition 3.3.1.25 (**Codimension of a singular subspace**)**.** Let Δ be a polar space of rank n. The *codimension of a singular subspace* U is

$$\mathsf{codim}(U) := n - 1 - \dim(U).$$

Let Δ be a top-thin polar space of rank n. One can show that there are two distinct types of maximal singular subspaces: given any maximal singular subspace M in Δ, every other maximal singular subspace intersects M in either even or odd codimension. The maximal singular subspaces intersecting M in odd codimension form one class, while those intersecting M in even codimension, together with M itself, form the other class (cf. [**Van24**, §6.4]). With that, we can define the following.

Definition 3.3.1.26 (**Oriflamme geometry**)**.** Let Δ be a top-thin polar space of rank n. Let Φ and Φ' be the two natural classes of maximal singular subspaces. We define an incidence system $\widetilde{\Delta} = (X, *, \tau)$ as follows. Let I' be the set

$$\{1, 2, \dots, n-2, n\},$$

and let X be the set of m-dimensional singular subspaces of Δ, with $m + 1 \in I'$. Then X is the set of all singular subspaces of Δ that do not have dimension $n - 2$. We define

$$\tau \colon X \to I = I' \cup \{n'\}$$

to be the map assigning to each m-dimensional subspace the type $m+1$ (for $m+1$ in $I' \setminus \{n\}$), and assigning type n and n' to members of Φ and Φ', respectively. For two singular subspaces $U, V \in X$ with at most one maximal and neither of dimension $n-2$, we declare U and V incident, written $U * V$, if either $U \subseteq V$ or $V \subseteq U$ in Δ. If $U \in \Phi$ and $V \in \Phi'$, then U and V are incident, written $U * V$, whenever $U \cap V$ has dimension $n - 2$.

In the oriflamme geometry, two maximal singular subspaces are opposite if and only if they are disjoint. If n is even, opposite maximal singular subspaces lie in the same class; if n is odd, they lie in different classes.

The term *oriflamme geometry* was introduced by Jacques Tits in [**Tit74**]. Historically, the Oriflamme was a banner used by French royalty in the Middle Ages (cf. [**Shu10**, p. 179]). Unlike ordinary flags, the Oriflamme had two pointed ends.

[5] One also says that buildings of type D_n are *oriflamme complexes* of top-thin polar spaces.

3.4. Parapolar Spaces

Again, the main reference will be the book *Polar Spaces* by Hendrik Van Maldeghem [**Van24**]. We use the definition for *parapolar spaces* that is introduced in Ch. 9.

Definition 3.4.0.1 (**Parapolar spaces**)**.** A *parapolar space* Γ is a point-line geometry that has the following properties (cf. [**Van24**, Definition 9.2.3]).

(i) The point graph of Γ is connected. That is, each pair of points is a finite distance apart.

(ii) For each pair of non-collinear points x and y, one of the following possibilities holds.
   (a) $x^\perp \cap y^\perp = \varnothing$,
   (b) $x^\perp \cap y^\perp$ consists of exactly one point,
   (c) $\{x, y\}$ is contained in a convex subspace isomorphic to a point-line geometry of a polar space. Such a subspace is called a *symplecton* or a *symp* for short.

(iii) Every line lies in at least one symplecton.

A *proper* parapolar space is a parapolar space that is not a polar space. That means it contains at least two symplecta. If a parapolar space Γ contains another parapolar space γ as a proper convex subspace, then we also call γ a *para* of Γ.

If a non-collinear pair of points (x, y) satisfies (ii)(b), we call it a *special pair*, we write $x \bowtie y$, and we denote the unique point collinear to both x and y by $[x, y]$. If it satisfies (ii)(c), we call it a *symplectic* or *polar pair*, we write $x \perp\!\!\!\perp y$, and we denote the unique symp containing both x and y by $\xi(x, y)$. A parapolar space without special pairs is called *strong*. The *symplectic rank* of a parapolar space is r if each symplecton has rank r. A parapolar space does not necessarily have a symplectic rank.

We will now list some general properties of parapolar spaces without proof. For each statement a proof can be found in [**Van24**, §9.2].

- Let p, q be two collinear points of Γ. Then p and q are contained in exactly one line. Furthermore, each line contains at least three points.
- Let p be a point and let L be a line that does not contain p. Then $p^\perp \cap L$ is either empty, exactly one point, or the whole line L. We also call this the *zero-one-or-all property*. Point-line geometries satisfying the zero-one-or-all property are exactly the gamma spaces as defined in Definition 3.1.4.22 (cf. [**Shu10**, §3.5.2] or [**BC13**, Remark 6.7.3]).
- For each point p the set $p^\perp$ is a subspace.
- Every set of mutually collinear points is contained in a singular subspace.
- Let Σ be a symplecton and p be a point not contained in Σ. Then $p^\perp \cap \Sigma$ is a singular subspace of Σ.
- Let Σ_1 and Σ_2 be two distinct symplecta. Then $\Sigma_1 \cap \Sigma_2$ is a singular subspace.
- Let x, y be two non-collinear points of Γ contained in a common symplecton ξ. Then ξ is unique.

For proper parapolar spaces, it is difficult to create drawings of examples. We will see some drawings later, but for these, we will need more context.

3.5. Generalized n-gons

In this thesis, we will mostly be concerned with point-line geometries that contain at least 2-dimensional projective subspaces (planes). However, sometimes we will be interested in subgeometries that are contained in our "bigger" ones and could be "smaller" — in the sense that their highest dimensional projective subspaces may only be lines. That is the reason why we want to talk about generalized polygons next.

For this section, the main reference is the book *Generalized Polygons* by Hendrik Van Maldeghem (cf. [**Van98**]). We already saw two examples of generalized polygons: (1) The Fano plane in Fig. 1 is an example of a generalized triangle. Indeed, the generalized triangles are exactly the projective planes. (2) As mentioned before, rank-2 polar spaces are generalized quadrangles, and we saw an example in Fig. 5. We will now give the general definitions for *ordinary polygons* and *generalized polygons*.

Definition 3.5.0.1 (**Ordinary polygons**)**.** Let $n \in \mathbb{N}_{\geq 2}$. An *ordinary n-gon* is a unique pre-point-line geometry with n points and n lines such that every point is contained in exactly 2 lines, every line has exactly 2 points, and its incidence graph is connected. In particular, an ordinary 2-gon has 2 lines and 2 points, which are incident with each of the two lines. If $n \geq 3$, then the ordinary n-gon is a point-line geometry.

Definition 3.5.0.2 (**Weak** & **thick generalized polygons**)**.** Let $n \geq 2$ be a natural number. A *weak generalized n-gon* is a geometry $\Gamma = (\mathscr{P}, \mathscr{L})$ such that the following two axioms are satisfied:

(i) Γ contains no ordinary k-gon as a subgeometry, for $3 \leq k < n$.
(ii) Any two elements $x, y \in \mathscr{P} \cup \mathscr{L}$ are contained in some ordinary n-gon as a subgeometry in Γ.

A *thick generalized n-gon* is a weak generalized n-gon Γ that also satisfies the following axiom.

(iii) There exists an ordinary $(n+1)$-gon (as a subgeometry) in Γ.

An equivalent definition is the following (cf. [**Van98**, Lemma 1.3.6]).

Definition 3.5.0.3 (**Generalized polygons**, **thin**)**.** A pre-point-line geometry Γ is a *generalized n-gon* if the incidence graph of Γ is a connected graph of diameter n and gonality n (cf. 3.1.2.13) such that each vertex lies on at least two edges.

- If every vertex lies on at least 3 edges, then Γ is thick.
- If every vertex lies on exactly two edges, then Γ is called *thin*.

When we just talk about a *generalized n-gon* in the following, it may either be a thick or a thin generalized n-gon.

If we just talk about a *generalized polygon*, then we mean a generalized n-gon for some $n \in \mathbb{N}_{\geq 2}$.

Example 3.5.0.4. In Fig. 7, we show a visualization of a thick generalized hexagon that was first described in [**Sch99**].

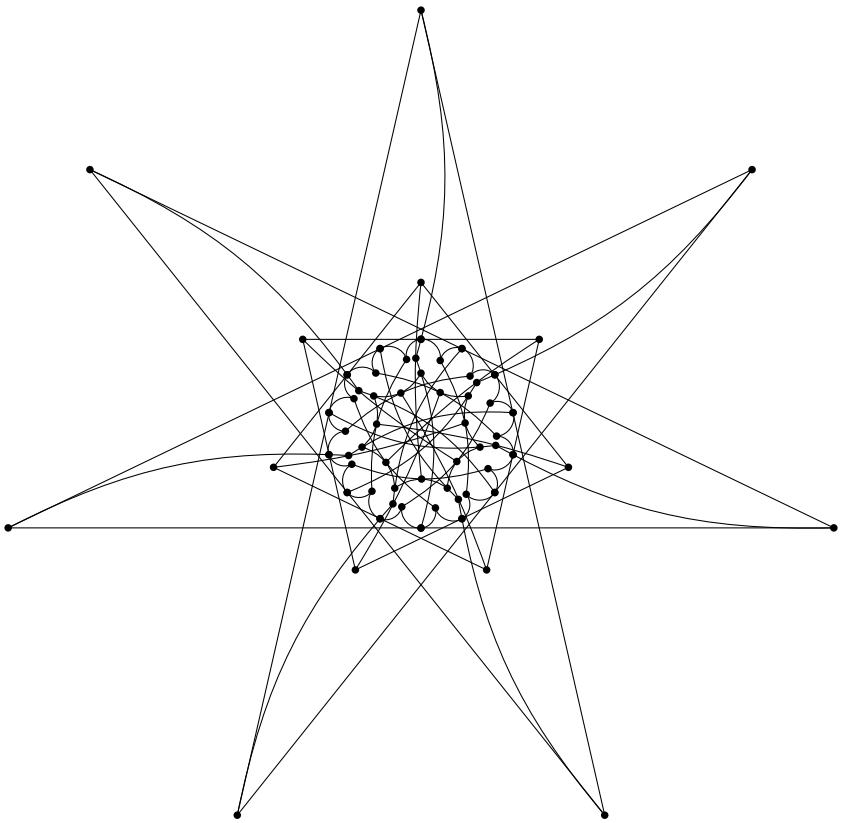


FIG. 7. A thick generalized hexagon.

Definition 3.5.0.5 (**Orders of generalized polygons**)**.** The *order* of a finite generalized n-gon (i.e., a generalized n-gon with a finite number of points) is the pair (s, t) such that each line contains precisely $s + 1$ points and each point is contained in precisely $t + 1$ lines.

Definition 3.5.0.6 (**Subpolygons**)**.** A *subpolygon* of a generalized polygon is a generalized polygon induced on a convex subgraph of the incidence graph of Γ.

Fact 3.5.0.7. An interesting fact is that finite generalized n-gons of order (s, t) with $s \geq 2$, $t \geq 2$ only exist for $n \in \{2, 3, 4, 6, 8\}$. This was shown in 1964 by Walter Feit and Graham Higman (cf. [**FH64**]).

3.6. Opposition, Projections & Residues

Now that we have introduced different kinds of (pre-)point-line geometries, we can discuss some important concepts that can be defined for all of them, but that look slightly different depending on the kind of (pre-)point-line geometry. In this section, we will only discuss opposition, projections, and residues in projective spaces and polar spaces, since for parapolar spaces it heavily depends on the kind of parapolar space what the aforementioned concepts look like. We will discuss this in Chapter 4.

3.6.1. In Projective Spaces

Definition 3.6.1.1 (**Opposite subspaces in projective spaces**)**.** Two subspaces A and B of a projective space are called *disjoint* if $A \cap B = \varnothing$.

Let Δ be a non-trivial projective space of dimension n, and let U and V be two subspaces of Δ such that $\dim(U) + \dim(V) = n - 1$. Then we call U and V *opposite* if $U \cap V = \varnothing$. If U and V are opposite, then every subspace of dimension $\dim(V)+1$ is not disjoint from U.

Example 3.6.1.2.

- In a projective plane, every opposite of a point is a line. Two lines are never disjoint.
- In a 3-dimensional projective space, every opposite of a point is a plane, and every opposite of a line is a line. A line and a plane are never disjoint.
- In a 4-dimensional projective space, every opposite of a point is a 3-space, and every opposite of a line is a plane.

Proposition 3.6.1.3. *Let α be a collineation of a non-trivial projective space Δ. Recall Definition 3.2.0.11 for axes and centers.*

(i) If α fixes all points of some hyperplane (i.e., has an axis), then α has a center.

(ii) If α fixes all lines through a point c (i.e., has a center), then α has an axis.

Reference. This is [**BC13**, Proposition 6.1.4]. □

Definition 3.6.1.4 (**Perspectivities of projective planes**)**.** Let Δ be a projective plane, p a point, and L a line in Δ such that $p \notin L$. We define the bijection

$$\begin{aligned} \mathsf{proj}_p^L : \mathscr{P}\big|_L &\to \mathscr{L}_p, \\ x &\mapsto xp, \end{aligned}$$

where $\mathscr{P}\big|_L$ is the point set of L and $\mathscr{L}_p$ is the line pencil through p as defined in Definition 3.1.3.15, and xp is the unique line in Δ containing both points x and p. We call proj_p^L the *projection from L onto p*. Dually, we define the bijection

$$\begin{aligned} \mathsf{proj}_L^p : \mathscr{L}_p &\to \mathscr{P}\big|_L, \\ K &\mapsto K \cap L, \end{aligned}$$

where K is a line through p and $K \cap L$ is its intersection point with L, and call this the *projection from p onto L*. In general, we call these kinds of bijections *perspectivities*.

Definition 3.6.1.5 (**Projectivities of projective planes**)**.** If Δ is a projective plane, $p_0, p_1, \ldots, p_{n-1}$ are n different points in Δ, and $L_0, L_1, \ldots, L_n$ are $n+1$ different lines in Δ such that $p_i \notin L_i, L_{i+1}$ for $i \in \{0, 1, \ldots, n-1\}$, then

$$\mathsf{proj}_{L_n}^{p_{n-1}} \circ \cdots \circ \mathsf{proj}_{L_1}^{p_0} \circ \mathsf{proj}_{p_0}^{L_0}$$

is called a *projectivity* of Δ, and we also write:

$$L_0 \barwedge p_0 \barwedge L_1 \barwedge p_1 \barwedge \ldots \barwedge L_{n-1} \barwedge p_{n-1} \barwedge L_n.$$

If $L_0 = L_n$, then we call this projectivity a *self-projectivity*. For a projectivity $L \barwedge \ldots \barwedge M$, we say that $L \barwedge \ldots \barwedge M$ is a *projectivity from L to M*.

The set of all self-projectivities of a line L of a projective plane Δ forms a group; we call it the *projectivity group of L in Δ* and denote it by Π_L. If M is another line of Δ and π is a projectivity from L to M, then $\Pi_M = \pi^{-1}\Pi_L\pi$. That means that the projectivity groups of two lines in Δ are isomorphic as permutation groups. This is well-known, and we used the notation that is common in the literature (cf. [**Kna88**, §1], [**Shu10**, §§7.5.3], or [**Van98**, §1.5]). We now want to define perspectivities and projectivities for higher-dimensional projective spaces.

Definition 3.6.1.6 (**pencil$_U$**)**.** Let $\Delta = (\mathscr{P}, \mathscr{L})$ be a projective space of dimension n, and let U be a subspace of Δ of dimension $m \leq n-1$. We define pencil_U as the set of all $m+1$-dimensional subspaces of Δ that contain U. The set pencil_U will also be called the *full pencil through U*.

Definition 3.6.1.7 (**Perspectivities** & **projectivities of projective spaces**)**.** Let Δ be a projective space of dimension n, and let U and V be two opposite subspaces of Δ. We define the following maps:

$$\begin{aligned} \pi \colon \mathscr{P}\big|_U &\to \{Y \in \mathrm{pencil}_V \mid \dim(Y \cap U) = 0\}, \\ x &\mapsto \langle x, V \rangle, \\ \pi' \colon \{Y \in \mathrm{pencil}_V \mid \dim(Y \cap U) = 0\} &\to \mathscr{P}\big|_U, \\ Y &\mapsto Y \cap V. \end{aligned}$$

Then π' is the inverse of π. For π we also write proj_V^U and call it the *projection from U onto V*. Again, both bijections π and π' will be called *perspectivities*. A product of perspectivities will be called a *projectivity*.

Definition 3.6.1.8 ($\mathscr{P}_{\boldsymbol{U}}^{\boldsymbol{\Delta}}, \mathscr{L}_{\boldsymbol{U}}^{\boldsymbol{\Delta}}$)**.** Let $\Delta = (\mathscr{P}, \mathscr{L})$ be a non-trivial projective space of dimension n, and let U be a subspace of Δ of dimension $n-2$. We define the sets

- $\mathscr{P}_U^\Delta$ as the set of all subspaces of Δ of dimension $1 + \dim(U)$ that contain U,
- $\mathscr{L}_U^\Delta$ as the set of all subspaces of Δ of dimension $2 + \dim(U)$ that contain U,

where we identify each member of $\mathscr{L}_U^\Delta$ with the members of $\mathscr{P}_U^\Delta$ it contains. We call $(\mathscr{P}_U^\Delta, \mathscr{L}_U^\Delta)$ the *residue of U in Δ* and denote it by $\mathsf{Res}_\Delta(U)$, or by $\mathsf{Res}(U)$ for short, if that is clear enough in a given context. The residue of a subspace in a projective space is always again a projective space of dimension $n - 1 - \dim(U)$.

Note 3.6.1.9. Perspectivities are not always collineations. They can also be dualities (cf. Definition 3.2.0.10).

3.6.2. In Polar Spaces

Notation 3.6.2.10 (**Perp of a subspace**)**.** Let Δ be a (para)polar space. Recall from Definition 3.1.3.18 that for a projective subspace V, we denote by $V^\perp$ the set of points collinear to each point of V and call it the *perp of* V. (This always includes V itself, see Note 3.3.0.5.)

Lemma 3.6.2.11. *Let $\Delta = (\mathscr{P}, \mathscr{L})$ be a polar space. If p and b are two opposite points in Δ, then $p^\perp \cap b^\perp$ is a subspace of Δ. If we denote by $\mathscr{L}_{p,b}$ the set of lines completely contained in $p^\perp \cap b^\perp$, then $\Delta_{p,b} = (p^\perp \cap b^\perp, \mathscr{L}_{p,b})$ is a polar space as soon as $\mathscr{L}_{p,b}$ is non-empty.*

Proof. This is [**Van24**, Lemma 2.3.1]. □

Proposition 3.6.2.12. *If U is a singular subspace of Δ, then $p^\perp \cap U$ is either equal to U or a hyperplane of U, and* $\dim(\langle p, p^\perp \cap U\rangle) = \dim(p^\perp \cap U) + 1$.

Proof. This is [**Van24**, Proposition 1.4.1]. □

Definition 3.6.2.13 (**Projections** & **Opposite subspaces in polar spaces**)**.** Let Δ be a polar space. Let U and V be two singular subspaces. We denote the set of points $V^\perp \cap U$ by $\mathsf{proj}_U(V)$ and call it the *projection of V onto U*. We call U and V *opposite* if both $\mathsf{proj}_U(V)$ and $\mathsf{proj}_V(U)$ are empty. The set of elements opposite a given singular subspace U is called the *opposition of U* and denoted by U^{op}.

One can show the following properties:

- $\mathsf{proj}_U(V)$ is a subspace of U ([**Van24**, Note 1.4.4]).
- Opposite subspaces are always disjoint; for two maximal singular subspaces, being disjoint is sufficient to be opposite ([**Van24**, Note 1.4.4]). Two singular subspaces U and V are opposite if and only if $\dim(U) = \dim(V)$ and no point of U is collinear with all points of V ([**Van24**, Corollary 1.4.7]).
- Let U and V be two singular subspaces. Then
$$\dim(V) - \dim(\mathsf{proj}_V(U)) = \dim(U) - \dim(\mathsf{proj}_U(V)).$$
In particular, if $\dim(U) \leq \dim(V)$, then $\dim(\mathsf{proj}_V(U)) \geq \dim(V) - \dim(U) - 1$, with equality if and only if $\mathsf{proj}_U(V)$ is empty ([**Van24**, Proposition 1.4.6]). Each singular subspace U admits an opposite in Δ ([**Van24**, Theorem 1.4.10]).
- Let S be a singular subspace disjoint from a maximal singular subspace M. Then $U := \langle S, \mathsf{proj}_M(S)\rangle$ is the unique maximal singular subspace containing S and intersecting M in a singular subspace of dimension $n - 2 - \dim(S)$. Moreover, U is the union of all singular subspaces containing S as a hyperplane and intersecting M in at least one point ([**Van24**, Corollary 1.4.8]).

For each singular subspace V, we consider the projection operator proj_V, which maps each singular subspace S onto $\mathsf{proj}_V(S)$, and we denote the restriction of the projection operator to a singular subspace $U \in V^{\mathsf{op}}$ by proj_V^U.

Proposition 3.6.2.14. *If U and V are two opposite singular subspaces, then the restricted projection operator proj_V^U defines a duality from the projective space defined on U to the projective space defined on V, with inverse proj_U^V.*

REFERENCE. This is [**Van24**, Theorem 1.4.11]. □

Note 3.6.2.15. Instead of proj_V^U, we also sometimes write $U \overline{\wedge} V$, and call this a *perspectivity of lower residues*. One can also define *perspectivities of upper residues*, and we will do so in Definition 3.6.2.22.

Example 3.6.2.16. In a rank 3 polar space, the following holds:

- Two points are opposite if they are not on a common line together.
- Two lines L and M are opposite if, for each point l on L, there exists exactly one point m on M such that l and m are on a common line together, and vice versa.
- Two projective planes are opposite if they do not intersect.
- If we project a point onto a plane that does not contain the point, we obtain a line.
- If we project a line onto a plane that has no intersection with the line, we obtain a point.

Definition 3.6.2.17 (**Residues in polar spaces**)**.** Let Δ be a polar space of rank n, and let U be a singular subspace of Δ of dimension at most $n-3$. We define the set

- $\mathscr{P}_U^\Delta$ as the set of all singular subspaces of Δ of dimension $1+\dim(U)$ that contain U.
- V/U^Δ as the set
$$\{W \in \mathscr{P}_U^\Delta \mid W \subseteq V\}$$
for all singular subspaces V of Δ that contain U.
- $\mathscr{L}_U^\Delta$ as the set
$$\{V/U \mid U \subseteq V, V \text{ singular subspace of } \Delta, \dim(V) = 2 + \dim(U)\}.$$
- Ω_U^Δ as the set $\{V/U \mid U \subseteq V, V \text{ singular subspace of } \Delta\}$.

Then we call the point-line geometry $(\mathscr{P}_U^\Delta, \mathscr{L}_U^\Delta)$ the *upper residue of U in Δ*, or briefly the residue of U in Δ, if $\mathscr{L}_U$ is not empty.

One can show that it is always a polar space of rank $n-1-\dim(U)$ (cf. [**Van24**, Theorem 1.6.4]). If $\mathscr{L}_U$ is empty, then we say that the residue of U in Δ is a rank-1 polar space. We denote the residue of U in Δ by $\mathsf{Res}_\Delta(U)$, or by $\mathsf{Res}(U)$ for short, if that is clear enough in a given context. The set Ω_U^Δ is the set of all singular subspaces of $\mathsf{Res}_\Delta(U)$.

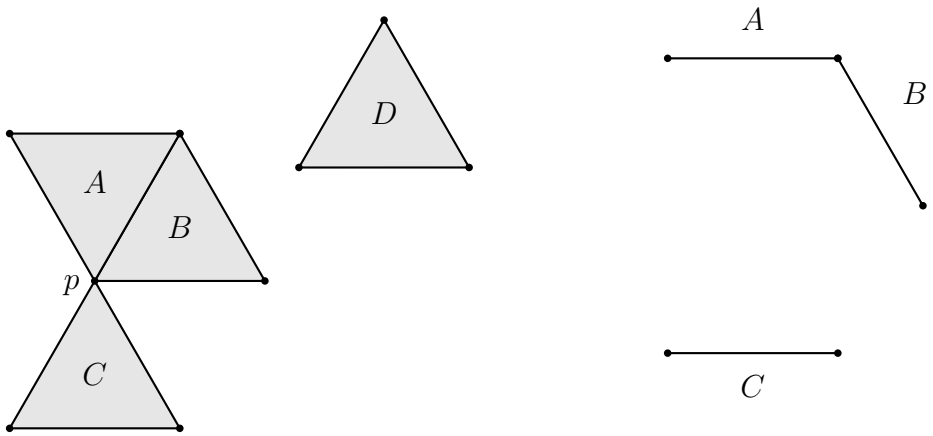
The *lower residue of U in Δ* is the singular subspace U itself, considered as a projective space.

Example 3.6.2.18. Let p be a point in a rank 3 polar space $\Delta = (\mathscr{P}_\Delta, \mathscr{L}_\Delta)$. Then the residue $\mathsf{Res}(p)$ of p in Δ is the point-line geometry with point set

$$\mathscr{P}_{\mathsf{Res}(p)} := \{L \in \mathscr{L}_\Delta \mid p \in L\}$$

and we draw a line between two points of $\mathscr{P}_{\mathsf{Res}(p)}$ if the corresponding lines in Δ are in a common plane together.

In Δ:	In $\mathsf{Res}_\Delta(p)$;



Proposition 3.6.2.19. *As mentioned before, the structure* $\mathsf{Res}_\Delta(U)$ *is a polar space of rank* $n - 1 - \dim(U)$*. Moreover, it is thick if and only if* Δ *is thick and top-thin if and only if* Δ *is top-thin. If* U *and* U' *are two opposite singular subspaces of dimension at most* $n-3$*, then the polar spaces* $\mathsf{Res}_\Delta(U)$ *and* $\mathsf{Res}_\Delta(U')$ *are isomorphic to each other.*

Proof. This is [**Van24**, Theorem 1.6.4] and [**Van24**, Corollary 1.6.8]. □

Since some of our proofs will be inductive, we must recognize opposite singular subspaces from *locally opposite* subspaces.

Definition 3.6.2.20 (**Local opposition**)**.** Two singular subspaces U, W of a polar space are called *locally opposite (at* $U \cap W$*)* if no point of $(U \cap W) \setminus (U \cap W)$ is collinear to all points of $U \cup W$. This is equivalent to U and W being opposite in $\mathsf{Res}_\Delta(U \cap W)$.

We now have the following local-to-global characterization.

Lemma 3.6.2.21. *Let* U *and* W *be two singular subspaces of some polar space* Δ*. Let* $A \subseteq U$ *be a subspace. Set* $B := A^\perp \cap W$*. Then the singular subspace* S*, spanned by* A *and* B*, is locally opposite* U *at* A*, and locally opposite* W *at* B *if and only if* U *and* W *are opposite in* Γ*.*

Proof. Suppose first that S is locally opposite U at A and locally opposite W at B. Then no point of $W \setminus B$ is collinear to all points of U, since no such point is collinear to all points of A. No point of $B \subseteq S$ is collinear to all points of U, since S is locally opposite U at A. No point of $U \setminus A$ is collinear to all points of B, as such a point would otherwise be collinear to all points of S (recalling that A and B

generate S), contradicting the local opposition of S and U at A. Finally, no point of A is collinear to all points of W, since W is locally opposite S at B. Hence U and W are opposite.

The converse is proved similarly. □

Definition 3.6.2.22 (**Lower** and **upper projectivities**)**.** Let U and V be two opposite singular subspaces. In Note 3.6.2.15, we already mentioned that the map $U \overline{\wedge} V$ mapping any subspace $S \subseteq U$ to $S^\perp \cap V$ defines an isomorphism from U to the dual of V and is called a *perspectivity (of lower residues)*.

- The composition of (a finite number of lower) perspectivities is a *projectivity (of lower residues)*.
- When such a projectivity has the same image as origin, then we call it a *self-projectivity (of that lower residues)*.
- The set of all self-projectivities of the lower residue of a subspace U forms a group denoted by $\Pi_{\leq}(U)$ and called the *general projectivity group of the lower residue of* U.
- The set of self-projectivities of the lower residue of a singular subspace, which are the composition of an even number of perspectivities of lower resdieus, forms a normal subgroup of $\Pi_{\leq}(U)$ of index at most 2, called the *special projectivity group of the lower residue of* U and denoted as $\Pi^+_{\leq}(U)$.
- If U is a maximal singular subspace, then the upper residue is empty, and so we denote the respective projectivity groups of the lower residue of U as $\Pi^+(U)$ and $\Pi(U)$.

We can now do the same for upper residues. Let U and V still be opposite singular subspaces. The map $U \overline{\wedge} V$ (it will always be clear from the context whether lower or upper residues is meant) mapping a subspace $U' \supseteq U$ of dimension $1 + \dim U$ to the unique subspace $V' \supseteq V$ of dimension $1 + \dim V$ intersecting U' in a point, and also denoted by $\mathsf{proj}_V(U')$, induces an isomorphism from $\mathsf{Res}(U)$ to $\mathsf{Res}(V)$, called a *perspectivity (of upper residues*.

- The composition of (a finite number of) perspectivities of upper residues is a *projectivity (of upper residues)*.
- When such a projectivity has the same image as origin, then we call it a *self-projectivity (of that upper residues)*.
- The set of all self-projectivities of the upper residue of a subspace U forms a group denoted by $\Pi_{\geq}(U)$ and called the *general projectivity group of the upper residue of* U.
- The set of self-projectivities of the upper residue of a subspace U, which are the composition of an even number of perspectivities of upper residues, forms a normal subgroup of $\Pi_{\geq}(U)$ of index at most 2, called the *special projectivity group of the upper residue of* U and denoted by $\Pi^+_{\geq}(U)$.
- If U is a point, then the lower residue is trivial, and so, since there is no confusion, we denote the respective projectivity groups of the upper residue of U as $\Pi^+(U)$ and $\Pi(U)$.

In Section 6.3.2, we will need a special way of viewing residues in rank-3 polar spaces, and a special way of projecting.

Note 3.6.2.23. Let Δ be a polar space of rank n. Let p and b be two opposite points in Δ. Then

$$\mathsf{Res}_\Delta(p) \cong \mathsf{Res}_\Delta(b) \cong p^\perp \cap b^\perp,$$

and $p^\perp \cap b^\perp$ defines a polar subspace in Δ of rank $n-1$ (cf. Lemma 3.6.2.11). Let u be a point in $p^\perp$. Then we can view u as a point of $\mathsf{Res}_\Delta(p)$ by identifying it with the line pu.

Definition 3.6.2.24 (**Projection from the residue of p to the residue of b**)**.** With the notation as in Note 3.6.2.23, we say that we *project pu to b* by projecting b onto pu in the usual way, obtaining a point u' collinear to b, and then setting bu' as the image of the *projection of pu onto b*. Like this, we define a *projection from the residue of p to the residue of b*, and we also write proj_b^p.

3.7. Apartments

Again, in this subsection, we will not discuss apartments of parapolar spaces, because for them, the description of an apartment becomes way more difficult, and we will only focus on that when we discuss different types of parapolar spaces in more detail in Chapter 4. For now, we will focus on projective spaces, polar spaces, and generalized polygons.

We will start with generalized polygons, for which defining an apartment can be done relatively quickly.

Definition 3.7.0.1 (**Apartments of generalized polygons**)**.** The *apartments* of a generalized n-gon Δ are the ordinary n-gons contained in Δ (cf. [**Van98**, Definitions 1.3.1]).

We move on to projective spaces, which require only slightly more attention.

Definition 3.7.0.2 (**Projective frames**)**.** Let Δ be a non-trivial projective space of dimension n. A *projective frame of* Δ is a set F of $n+1$ different points

$$\{x_1, x_2, \ldots, x_n, x_{n+1}\},$$

such that F is not contained in any proper subspace of Δ (cf. [**Tit74**, §6.2]).

Definition 3.7.0.3 (**Apartments of a projective space**)**.** An *apartment* of a non-trivial projective space Δ is the set of all singular subspaces spanned by the points of a projective frame.

The apartments of a projective plane are ordinary triangles, and the apartments of a projective space of dimension 3 can be visualized as tetrahedra. Apartments for projective spaces of dimension $n > 3$ can be viewed as hypertetrahedra or n-simplices.

Now we get to polar spaces, for which we not only need the definition of an apartment, but also some results in that context for later use.

Definition 3.7.0.4 (**Polar frames**)**.** Let Δ be a polar space of rank n. A *polar frame of* Δ is a set F of $2n$ different points

$$\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\},$$

such that x_i and y_i are opposite for all $i \in \{1, 2, \ldots, n\}$, and each element of F is collinear with exactly $2n-1$ points of F.

Proposition 3.7.0.5. *Let $F = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ be a polar frame. Consider two disjoint subsets J, K of $\{1, 2, \ldots, n\}$. Then the dimension of*

$$U = \langle \{x_j \mid j \in J\} \cup \{y_k \mid k \in K\} \rangle$$

is equal to $|J| + |K| - 1 = |J \cup K| - 1$.

Reference. This is [**Van24**, Proposition 1.5.2]. □

Construction 3.7.0.6. Given a polar space Δ of rank n, we can construct polar frames. This is described in [**Van24**, Construction 1.5.3], but we will also describe it here: Consider two disjoint maximal singular subspaces U_x and U_y. Consider a set of points $\{x_1, x_2, \ldots, x_n\}$ in U_x that generates U_x. Since $\mathsf{proj}_{U_y}^{U_x}$ is a duality between U_x and U_y, the hyperplane $H_i := \langle x_j | j \in \{1, \ldots, n\} \setminus \{i\} \rangle$, for each $i \in \{1, \ldots, n\}$, is collinear with a unique point y_i of U_y. The set $\{x_i, y_i \mid i \in \{1, \ldots, n\}\}$ is a polar frame.

Definition 3.7.0.7 (**Apartments of polar spaces**)**.** An *apartment* of a polar space Δ is the set of all singular subspaces spanned by the points of a polar frame.

For rank 2 polar spaces, the apartments are ordinary quadrangles, and for rank 3 polar spaces, the apartments can be visualized as octahedra. Apartments for polar spaces of higher rank can be viewed as hyperoctahedra (or cross polytopes).

Proposition 3.7.0.8. *In a polar space Δ, every two singular subspaces are contained in a common apartment.*

Reference. This is [**Van24**, Proposition 1.6.10]. □

Definition 3.7.0.9 (**Adjacent subspaces**)**.** We call two singular subspaces U and U' *adjacent* and write $U \sim U'$, if $U \cap U'$ is a hyperplane of U and a hyperplane of U'. One can verify that adjacent subspaces always have the same dimension.

Lemma 3.7.0.10. *Let U and U' be two singular subspaces of the same dimension. Then there exists a sequence of singular subspaces $(U = U_0, U_1, \ldots, U_m = U')$ such that $U_i \sim U_{i-1}$ for all $i \in \{1, 2, \ldots, m\}$.*

Reference. This is [**Van24**, Lemma 1.6.13]. □

4. The Connections between Diagrams, Geometries and Buildings

As mentioned earlier, our focus lies on certain geometries because some of them correspond to spherical buildings. Recall that every spherical building is associated with a Coxeter system (W, S), where W is a finite Coxeter group and S a generating set. For each Coxeter system we can draw a Coxeter diagram Π, and if Δ is a spherical building with Coxeter system (W, S) and diagram Π, then we say that Δ is a building of type Π. Thus, it is clear how we obtain a diagram from a building. Since finite Coxeter groups were classified by Harold Scott MacDonald Coxeter (cf. [**Cox35**]), we know precisely which Coxeter diagrams may occur.

On the other hand, given a Coxeter diagram, one can construct geometries corresponding to spherical buildings, and in this chapter we hope to shed some light on that. The above should explain why we restrict our attention to diagrams of finite Coxeter groups. Geometries attached to spherical buildings — and hence to groups of Lie type (cf. Section 2.9) — are called *Lie incidence geometries*, and all geometries under discussion here fall into this class. Later, we will see how such geometries, tied to spherical Coxeter diagrams Π, translate into buildings of type Π.

Thanks to Jacques Tits, all thick irreducible spherical buildings of rank at least 3 are classified (cf. [**Tit74**]), and thanks to Jacques Tits & Richard Weiss (cf. [**TW02**]) and Bernhard Mühlherr & Hendrik Van Maldeghem (cf. [**MV99**]), all thick irreducible spherical Moufang buildings of rank 2 are classified as well. Thanks to Francis Buekenhout and Arjeh Marcel Cohen, there exists a comprehensive book describing in detail how diagrams prescribe the local structure of geometries corresponding to spherical buildings (cf. [**BC13**]). Finally, thanks to Andries Evert Brouwer, Arjeh Marcel Cohen, and Arnold Neumaier, many further properties of the geometries we will encounter can also be found in book form (cf. [**BCN89**]).

We begin in Section 1 by explaining how to obtain the diagram for a thick polar space and for a projective space. In Section 2, we show how to translate a projective space into the corresponding building. Section 3 introduces the notion of a Lie incidence geometry, specifies precisely what we mean by this term, and presents a list indicating which types of buildings correspond to which kinds of Lie incidence geometries. In addition, we define Grassmannians, introduce notation, and record some first observations on ranks, distances, and opposition, and we state explicitly which types of simplices in a building can be identified with which elements in the corresponding Lie incidence geometry. Section 4 then gives a brief account of how to construct a geometry from a given spherical building. The subsequent sections are devoted to deriving specific properties of certain Lie incidence geometries that play a central role in the later parts of this thesis.

4.1. Diagrams of Projective Spaces and Thick Polar Spaces

We begin by collecting some additional definitions and results from [**Van24**, §6.2]. With these in place, we first define the diagram of a thick polar space, followed by that of a projective space.

Definition 4.1.0.1 (**Generalized digons**)**.** A *generalized digon* is a pre-point-line geometry Δ such that every point is incident to every line and Δ contains at least two lines.

Fact 4.1.0.2. A bipartite graph Γ is the incidence graph (as defined in Definition 3.1.2.12) of

- a generalized digon if and only if Γ is a complete graph with diameter 2 and girth 4.
- a projective plane if and only if Γ has diameter 3 and girth 6 and the valency of each vertex is at least 3.
- a generalized quadrangle if and only if Γ has diameter 4 and girth 8 and the valency of each vertex is at least 2.
- a generalized polygon if and only if Γ has diameter $n \geq 2$ and girth $2n$ and the valency of each vertex is at least 2.

Definition 4.1.0.3 (**Matrix of a generalized polygon**)**.** We define the *matrix of a generalized n-gon* as:

$$\begin{pmatrix} 1 & n \\ n & 1 \end{pmatrix}$$

Note that projective planes and generalized digons may be treated as thick generalized 3-gons and generalized 2-gons, respectively.

Definition 4.1.0.4 (**Chambers in finite polar spaces**)**.** A *chamber of a polar space* Δ of rank n is a set of nested singular subspaces, one of each dimension, ranging from dimension 0 through dimension $n-1$.

Proposition 4.1.0.5. *Consider a chamber C in a thick polar space Δ of rank n and remove from it two elements, say the singular subspaces U_i and U_j of respective dimensions $i-1$ and $j-1$, $i<j$ and $i,j \in \{1,2,\ldots,n\}$. Define the pre-point-line geometry $\Delta_{i,j}$ as follows. The points are the singular subspaces of dimension $i-1$, which are contained in a chamber together with $C \setminus \{U_i, U_j\}$, and the lines are the singular subspaces of dimension $j-1$ that are also contained in a common chamber together with $C \setminus \{U_i, U_j\}$ (now we explicitly allow repeating lines). Incidence is inclusion. Then $\Delta_{i,j}$ is a*

(i) generalized digon, if $i \leq j-2$,
(ii) projective plane, if $i+1=j<n$,
(iii) thick generalized quadrangle, if $i+1=j=n$.

Reference. This is [**Van24**, Proposition 6.3.2]. □

Definition 4.1.0.6 (**Matrix of a finite thick polar space**)**.** Let Δ be a thick polar space of rank n. We define the *matrix* $M(\Delta) = (m_{ij})_{1 \leq i,j \leq n}$ *of the polar space* Δ as the matrix with entries

- $m_{ii} = 1$, for all $i \in \{1, 2, \ldots, n+1\}$, and
- for $i \neq j$, $\Delta_{i,j}$ is a thick generalized m-gon for some $m \in \mathbb{N}_{\geq 2}$, and we set $m_{ij} = m$.

Example 4.1.0.7. The matrix of a thick rank 3 polar space is the following:

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \\ 2 & 4 & 1 \end{pmatrix}$$

Conclusion 4.1.0.8 (**Diagrams of finite thick polar spaces**)**.** Since all of these matrices are symmetric, have 1s on their diagonal, and their entries contain positive integers ≥ 2, we can view them as Coxeter matrices and draw the corresponding Coxeter diagrams. We then also call this the *diagram of the thick polar space* Δ. One can verify that diagrams of thick polar spaces are always of type B_n, meaning that the thick generalized m-gons $\Delta_{i,j}$ are always either generalized digons, projective planes, or thick generalized quadrangles.

For projective spaces, we can follow the same strategy, and it will only become easier. First, we define chambers exactly as we did before for polar spaces.

Definition 4.1.0.9 (**Chambers in finite projective spaces**)**.** A *chamber of a projective space* Δ of dimension n is a set of nested singular subspaces, one of each dimension, ranging from dimension 0 through dimension $n-1$.

Now, we can define pre-point-line geometries $\Delta_{i,j}$ also exactly as before, and we can see that, if Δ is a projective space, then the $\Delta_{i,j}$ are

(i) generalized digons, for $i \leq j - 2$,
(ii) projective planes, for $i + 1 = j = n$.

Hence, if we also define a matrix for a projective space in the exact same way as before, we obtain a symmetric matrix with 1s on the diagonal and all other entries equal to either 2 or 3.

Conclusion 4.1.0.10 (**Diagrams of finite projective spaces**)**.** Viewing the matrices for projective spaces as Coxeter matrices and attaching the corresponding Coxeter diagrams to them, we can see that the diagrams of projective spaces are all of type A_n.

One could ask why we only defined diagrams for thick polar spaces and not for top-thin ones or for parapolar spaces that correspond to spherical buildings. The reason is that it is far more difficult for these kinds of point-line geometries. A good description of how it is done for top-thin polar spaces can be found in [**Van24**, §6.4].

4.2. Projective Spaces as Buildings

In this subsection, we describe how a building structure can be seen in a projective space. We begin with the example of a projective plane to illustrate the general idea, noting that an accessible account of this can be found in [**Hau92**].

Example 4.2.0.1. Let $\Gamma_{\text{geo}} = (\mathscr{P}, \mathscr{L})$ be a projective plane. Let $\mathscr{C} \subseteq \mathscr{P} \times \mathscr{L}$ be the set of all pairs (p, L), where p is a point and L is a line such that p lies on L. Then $\mathscr{C}$ is the set of all chains of maximal length of Γ_{geo}. The elements of $\mathscr{C}$ are chambers in the corresponding building Γ_{build}. In the context of projective geometry, they are also called *flags*. As we have seen, for a building, we can always define a function that measures the distance between two chambers. We now want to see how to measure distances between flags in a projective plane. We say that two flags (p, L) and (q, J) are *g-adjacent* if $p = q$ and $L \neq J$ (and we write $(p, L) \sim_g (q, J)$), and that they are *h-adjacent* if $p \neq q$ and $L = J$ (and we write $(p, L) \sim_h (q, J)$).

Let $F = (p, L)$ and $F' = (p', L')$ be two different flags such that $p \notin L'$ and $p' \notin L$. We know that, in a projective plane, we can always find a connecting line between two points, and two lines always intersect. As usual, we denote the connecting line between p and p' as pp', and we will refer to the intersection point of L and L' as $L \cap L'$. Then $(L, L \cap L')$, $(L', L \cap L')$, (p, pp'), and (p', pp') are also elements of $\mathscr{C}$. We get the following two sequences:

$$F = (p, L) \sim_g (p, pp') \sim_h (p', pp') \sim_g (p', L') = F'$$
$$F = (p, L) \sim_h (L \cap L', L) \sim_g (L \cap L', L') \sim_h (p', L') = F'$$

Following this approach, we define a map $\delta \colon \mathscr{C} \times \mathscr{C} \to W$ that takes ordered pairs of flags and maps them to reduced words of a Coxeter group with generators g and h as depicted in Table 1.

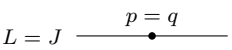
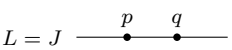
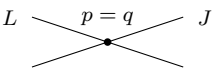
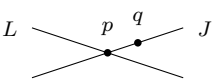
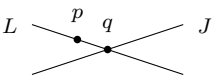
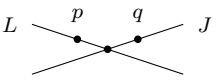
Mutual position of the flags	$\delta((p, L), (q, J))$
	id
	g
	h
	gh
	hg
	$ghg = hgh$

Table 1

We can see that the group has to be a Coxeter group of type A_2.

We now turn to projective spaces of higher dimension. The following is well-known and sometimes called *Structure Theorem*.

Proposition 4.2.0.2. *Projective spaces are either*

- *trivial (just one point, or just one line with ≥ 3 points),*
- *projective planes, or*
- *isomorphic to $\mathsf{PG}(V, \mathbb{K}) := (\mathscr{P}, \mathscr{L})$, where V is a vector space over the (skew) field $\mathbb{K}$, $\mathscr{P}$ is the set of one-dimensional subspaces of V, $\mathscr{L}$ is the set of two-dimensional subspaces of V, and an incidence relation is given by inclusion.*

Reference. See [**BC13**, §6.3]. □

Definition 4.2.0.3. Let Γ be a projective space isomorphic to $\mathsf{PG}(V, \mathbb{K})$, for V an $(n+1)$-dimensional vector space over a (skew) field $\mathbb{K}$. Then *flags in* $\mathsf{PG}(V, \mathbb{K})$ are increasing sequences of subspaces

$$0 \neq U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_k \neq V.$$

The *length of this flag* is k. The maximal length of a flag in $\mathsf{PG}(V, \mathbb{K})$ is n, and we call flags of length n *maximal flags*. Note that a maximal flag of $\mathsf{PG}(V, \mathbb{K})$ contains subspaces of every dimension from 1 to n.

Note 4.2.0.4. When we switch from talking about projective spaces from an axiomatic view to a linear algebraic view, we have to translate a little bit. The definition of *'flag'* is a direct translation of the definition of *'chain'*, given in Definition 3.1.3.20, for $\mathsf{PG}(V, \mathbb{K})$. We gave this definition of 'flag' here mostly because it is more commonly used in this context.

A projective k-space is a subspace whose vector space dimension is $k+1$. That means, for example, points, lines, and planes have vector space dimension 1, 2, and 3 and *geometric dimension* (viewed as subspaces of a point-line geometry) 0, 1, and 2, respectively (cf. [**Bue95**, Ch. 2, p. 30]). Note as well that we did not make a mistake in Example 4.2.0.1 by talking about flags: the case of lines already being maximal subspaces, and flags only consisting of a point and a line, was just relatively easy.

We state another well-known theorem.

Theorem 4.2.0.5. *Let V be an $(n+1)$-dimensional vector space over a (skew) field $\mathbb{K}$. Let Δ be the set of maximal flags of $\mathsf{PG}(V, \mathbb{K})$, and let $\mathscr{U}, \mathscr{W} \in \Delta$,*

$$\mathscr{U} = \{U_1, \ldots, U_n\}, \quad \mathscr{W} = \{W_1, \ldots, W_n\}.$$

Then there exists a basis $\mathscr{B} = \{b_1, \ldots, b_{n+1}\}$ of V and a permutation $\pi \in \mathsf{Sym}_{n+1}$ such that

$$U_i = \langle b_1, \ldots, b_i \rangle \quad \text{and} \quad W_i = \langle b_{\pi(1)}, \ldots, b_{\pi(i)} \rangle$$

for $0 \leq i \leq n+1$. The permutation π is independent of the choice of $\mathscr{B}$.[1]

Reference. This is described on page 84 of [**Bro89**]. □

[1] This permutation is sometimes called the *Jordan-Hölder permutation*.

Now, with notation as in Theorem 4.2.0.5, we can define

$$\delta\colon \Delta \times \Delta \to \mathsf{Sym}_{n+1}$$
$$\delta(\mathscr{U}, \mathscr{W}) := \pi.$$

With that, we obtain a distance map

$$\mathscr{C}(\Delta) \times \mathscr{C}(\Delta) \to W,$$
$$(C, C') \mapsto \pi(C, C'),$$

where W is a Coxeter group of type A_n.

4.2.1. Note on Polar Spaces as Buildings

For thick polar spaces, we can construct the building out of the geometry in a similar way. A description on how to obtain a Weyl-distance map in this case is given in [**DV18**, §2.2]. For top-thin polar spaces we have to consider the flag complex of the corresponding oriflamme geometry as mentioned in Section 3.3.1.

4.3. Constructing the Geometry from the Building

For this and the following subsections, our goal is not to present proofs or to rigorously explain why things are the way they are, but rather to use what is already known to give a simple description of how to extract information about the corresponding Lie incidence geometry from a given diagram. We will provide a set of rules, so to speak, that indicate how to read a diagram in this context.[2]

To begin, we clarify what exactly we mean by a *Lie incidence geometry*, and gather information about which types of spherical buildings correspond to which kinds of Lie incidence geometries. Some of these geometries have not yet been discussed, but we will introduce them in the following.

We refer to [**BCN89**, §10.2–10.3], [**BC13**], [**BP98**, Appendix A], [**Van24**, §9.3], [**Shu10**, §9.5] and to [**Tit74**] with regards to Table 2.

Construction & Definition 4.3.0.1 (**Lie incidence geometries**).
A *Lie incidence geometry* $\Delta = (\mathscr{P}, \mathscr{L})$ can be constructed from a spherical building Ω of rank $r \geq 2$ in the following way (cf. [**DSV24**]). Let $T = \{1, 2, \ldots, r\}$ be the type set of Ω (using Bourbaki labeling) and choose $J \subseteq T$. Then we obtain a Lie incidence geometry by

- letting the point set $\mathscr{P}$ be the set of simplices of type J, and
- letting the lines be the sets of simplices of type J completing flags of type $T \setminus \{j\}$ to a chamber, for $j \in J$.

[2]This might be dissatisfying, but there is a reason why other authors have written entire books on this topic, and it would not make sense to reproduce their contents here. Unfortunately, this thesis must adhere to a finite length.

Note that a chamber is a simplex of type T, and that different simplices of type $T \setminus \{j\}$ can give rise to the same line.

If Ω has Coxeter diagram X_r, then we say that Δ *has type* $\mathsf{X}_{r,J}$, omitting the braces when $J = \{j\}$ is a singleton.

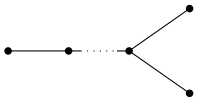
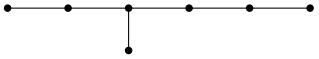
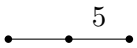
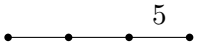
Type of Δ		Lie incidence geometries
A_n		Projective spaces
$\mathsf{B}_n, \mathsf{C}_n$		Thick polar spaces
D_n		Top-thin polar spaces
E_6		Parapolar spaces (of type E_6)
E_7		Parapolar spaces (of type E_7)
E_8		Parapolar spaces (of type E_8)
F_4		Parapolar spaces (of type F_4) / metasymplectic spaces
I_j		Generalized polygons
H_3		Generalized icosahedra
H_4		Generalized hypericosahedra (600-cell)

TABLE 2. Types of spherical buildings and their corresponding Lie incidence geometries.

If Ω has a simply laced diagram, then it is determined by its Coxeter diagram X_r and a field $\mathbb{K}$. If the field could play a role, or if we want to be exact, we also sometimes denote Δ *as* $\mathsf{X}_{r,J}(\mathbb{K})$.

If Ω is of non-simply laced type, then it is generally not uniquely determined by its Coxeter diagram (cf. [**Van24**, §6.3]). However, for buildings of type F_4, we have that they are uniquely determined by their Dynkin diagram, a field $\mathbb{K}$, and a quadratic alternative division algebra $\mathbb{A}$. In that case, we also denote Δ *as* $\mathsf{F}_{r,J}(\mathbb{K}, \mathbb{A})$ (cf. [**LV25**, §2.2]). If we consider $J = \{j\}$, we also write $\mathsf{X}_{r,j}$.

For a Lie incidence geometry Δ with diagram Π, constructed for some $J \subseteq T$,

we sometimes encircle all vertices, whose type is in J, to make clear, which subset of T we considered. For clarity:

- Lie incidence geometries are exactly the geometries associated to spherical buildings, and every spherical building corresponds to Lie incidence geometries constructed as above.
- Thick spherical buildings do not exist for every spherical Coxeter diagram (and we will talk more about that in Section 6.6).
- Every Lie incidence geometry is a gamma space, but not every gamma space is a Lie incidence geometry. For example, Proposition 4.3.0.4 below does not hold for general gamma spaces (cf. [**Van24**, Example 9.6.3]).
- We will only be interested in Lie incidence geometries constructed for $|J| = 1$ with the notation above; i.e., with one encircled vertex.

Among the Lie incidence geometries that can be constructed from a spherical building Ω, there is typically a canonical choice that is the most convenient to work with. For instance, strong parapolar spaces of small diameter are generally easier to handle than non-strong parapolar spaces of larger diameter. In this thesis, we will mostly focus on Lie incidence geometries for which $J = \{1\}$ if Ω is of classical type and $J = \{r\}$ if Ω is of exceptional type (using the same notation as in 4.3.0.1). Thus, we define the following for convenience.

Definition 4.3.0.2 (**Pick-me Lie incidence geometry**)**.** We call a Lie incidence geometry a *pick-me (Lie incidence) geometry* if it has type

- $\mathsf{X}_{r,1}$ if X is A, B, D, or H,
- $\mathsf{X}_{r,r}$ if X is E, or F.

Note that, because of the symmetry of the Dynkin diagrams,

$$\mathsf{A}_{r,1} = \mathsf{A}_{r,r}, \quad \text{and} \quad \mathsf{E}_{6,1} = \mathsf{E}_{6,6}.$$

In the case that we talk about a pick-me Lie incidence geometry, we sometimes omit the second index.

Definition & Fact 4.3.0.3. Let Δ be a pick-me Lie incidence geometry with diagram Π of rank n. In the fully labeled version of Π, let γ be a maximal subgraph of Π of type A_m, $m \leq n$. Then either

- $\gamma = \Pi$, Δ is a projective space, and the dimension of Δ is n,
- or m is either $n-1$ or $n-2$, we define *the rank of* Δ as $m+1$, and the maximal projective subspaces of highest dimension of Δ are m-spaces. We have $m = n-2$ if Δ is a metasymplectic space and Π is of type F_4.

This is compatible with how we defined the rank of a polar space in Definition 3.3.0.4.

Proposition 4.3.0.4. *If Δ is a Lie incidence geometry, then all its singular subspaces are projective spaces. More generally, the singular subspaces of parapolar spaces with symplectic rank ≥ 3, the singular subspaces of projective spaces, and the singular subspaces of polar spaces are projective spaces.*

Reference. This is [**Van24**, Proposition 9.6.2]. □

4.3.1. Grassmannians

We now give a brief introduction to *Grassmannians*, which provide a way to construct Lie incidence geometries from a given one. Again, a reference for this is [**Van24**].

In fact, a Grassmannian in a Lie incidence geometry is another Lie incidence geometry. It provides a way to understand the latter in terms of the former, without having to think about the underlying building. This is in particular helpful if the former is a well-understood geometry. Examples of these are projective spaces, polar spaces, and maybe the strong parapolar spaces. Here, we only consider projective and polar spaces, but we phrase the definitions in general.

Definition 4.3.1.5. Let $\Delta = (\mathscr{P}, \mathscr{L})$ be a Lie incidence geometry of rank n, let I be an ordinary index set, and let m be an integer such that $0 < m \leq n-1$. For a family $S := (U_i)_{i\in I}$ of different m-spaces such that $\bigcap_{i\in I} U_i$ is an $m-1$-space V, we say that S is an *m-space pencil through V*. If all U_i are contained in some $m+1$ space W, for $i \in I$, we call it an *m-space pencil through V in W*. We call an m-space pencil in W *filling* if every point of W is contained in some U_i, and in that case, we also write $\text{pencil}_V^W (U_i)_{i\in I}$ instead of S. For $m = 0$, a *pencil* will be a set of points such that all points are contained in some common line. A pencil of points S on the common line L will be called *filling* if S is exactly the point set of L.

Note 4.3.1.6. In a projective space in which every line has exactly $t+1$ points, a set S of m-spaces that form a pencil inside some $m+1$ space is filling if S has exactly $t+1$ elements. A pencil of projective subspaces of dimension ≥ 1 in a projective space is filling if it is full (cf. Definition 3.6.1.6).

This now allows to define $k+1$-Grassmannians (of type $\mathsf{A}_{n,k+1}$ and $\mathsf{B}_{n,k+1}, \mathsf{D}_{n,k+1}$) for projective and polar spaces.

Definition 4.3.1.7. Let Δ be a projective or polar space of dimension n. Let Γ be its corresponding diagram of type A_n, B_n or D_n, enriched with the Bourbaki labeling of the vertices. Encircle the vertex of Γ with label $k+1$ (for $0 \leq k \leq n-1$ in case A_n, for $0 \leq k \leq n-2$ in case B_n, and for $0 \leq k \leq n-3$ in case D_n), associated to the k-dimensional subspaces of Δ. We construct a new point-line geometry $\Delta_{n,k+1}$ as follows.

- The points of $\Delta_{n,k+1}$ are the k-dimensional subspaces of Δ.
- The lines of $\Delta_{n,k+1}$ are the filling k-space pencils of Δ.

We call $\Delta_{n,k+1}$ the *$k+1$-Grassmannian* of Δ and refer to the corresponding diagram with encircled vertex $k+1$ as the $\mathsf{A}_{n,k+1}$ diagram, the $\mathsf{B}_{n,k+1}$ diagram, or the $\mathsf{D}_{n,k+1}$ diagram, respectively.

Note 4.3.1.8. The previous construction also works for type D_n and $k = n-2$, but then we obtain a Lie incidence geometry of type $\mathsf{D}_{n,\{n,n-1\}}$. If, for thick polar spaces, one wants to do the same thing for $k = n-1$, then one has to consider as a filling pencil the set of all $(n-1)$-dimensional subspaces through a given $(n-2)$-dimensional subspace.

Notation 4.3.1.9. In the following, instead of saying: *"Lie incidence geometries of type* $\mathsf{X}_{n,1}$*"*, we will sometimes say: *"geometries of type* $\mathsf{X}_{n,1}$*"*. We will also sometimes shorten: *"in parapolar spaces of type* $\mathsf{X}_{n,1}$*"*, to: *"in* $\mathsf{X}_{n,1}$*"*. We will do the same for *"of"* that we do for *"in"*. For example, instead of saying: *"maximal singular subspaces of parapolar spaces of type* $\mathsf{X}_{n,1}$*"*, we will say *"maximal singular subspaces of* $\mathsf{X}_{n,1}$*"*.

Furthermore, if we talk of a Coxeter diagram of type $\mathsf{X}_{n,1}$, we mean the diagram of type X_n with encircled first vertex, according to the Bourbaki labeling of the vertices, and by the corresponding Lie incidence geometry we mean the geometry of type $\mathsf{X}_{n,1}$.

4.3.2. Distances

In Definition 3.1.3.14, we defined the distance between two points in a point-line geometry. The first thing we want to do is to describe how we can see the distances that points can have in a Lie incidence geometry by reading off the diagram. We already know that all points in a projective space are collinear, meaning they have at most distance 1. In a polar space, each point is on a line, and each point that is not on a certain line is collinear to at least one point of the line. That means points in a polar space have distance at most 2.

Fact 4.3.2.10. For all fully labeled spherical Coxeter diagrams Π of rank 2, the diameter of the incidence graph of the corresponding Lie incidence geometry coincides with the edge label j of Π, and the girth is $2j$ (cf. [**BC13**, §2.4] and [**Van98**, Lemma 1.3.6]). That means that, in the collinearity graph, points in the corresponding Lie incidence geometry can have at most distance $j/2$ if j is even, and $j-1/2$ if j is odd.

Observation 4.3.2.11. The girth of the incidence graph of any Lie incidence geometry of type $\mathsf{X}_{n,1}$ and rank $n \geq 3$ is always equal to 3, as they contain simplices of size at least 3. The girth of the collinearity graph is also equal to 3 because we have thick lines and every triple of points on a line defines a triangle in the collinearity graph.

Definition 4.3.2.12 (**Symmetrical(*) Coxeter diagrams**)**.** Let Π be a spherical Coxeter diagram. Then we call Π *symmetrical* if Π has either rank 2, or type A_n, E_6, F_4, or D_n. We call it *symmetrical** if Π is symmetrical, but not of type D_n for n even. So we only allow D_n if the rank n of Π is odd.

Fact 4.3.2.13 (version 1)**.** Let Π be a spherical Coxeter diagram of rank $n \geq 3$, associated to a pick-me Lie incidence geometry Γ.

- If the reductively labeled form of Π does not contain any vertex contained in 3 edges, and the fully labeled form does not contain any edge label > 3, then the diameter of Γ is 1.
  (This concerns A_n.)
- If the reductively labeled form of Π contains exactly one vertex contained in 3 edges, and that vertex is contained in at most 4 subdiagrams of type A_4, then Γ has diameter 2.
  (This concerns B_n, C_n, D_n, and E_6.)

- If Π has 3 edges, one of which has label 5, then the diameter of Γ is 5. (This concerns H_4.)
- In all other cases, the diameter of Γ is 3. (This concerns E_7, E_8, F_4, and H_3.)

It might not be clear why this fact should be true, but we will shed some more light on it later. A reference for this is [**BCN89**, Table 10.5][3]. Another way to phrase it is the following.

Fact 4.3.2.14 (version 2)**.** Projective spaces have diameter 1, polar spaces and parapolar spaces with a symmetrical and simply laced diagram of type $\mathsf{X}_{n,1}$ have diameter 2, and for all other Lie incidence geometries of type $\mathsf{X}_{n,1}$, corresponding to spherical buildings of rank ≥ 3, the diameter is 3, except for Lie incidence geometries of type $\mathsf{H}_{4,1}$, which have diameter 5.

4.3.3. Types

In Remark 1.2.0.5 we mentioned that we can view a Coxeter diagram as a vertex-colored pre-graph. In Definition 3.1.1.5, we defined incidence systems as triples $\Gamma = (X, *, \tau)$, and we described in Note 3.1.1.6 that we can always view a point-line geometry as an incidence system. Now we want to bring these things together for at least some of the Lie incidence geometries we mentioned before.

Observation 4.3.3.15. Let Π be a Coxeter diagram of type $\mathsf{X}_{n,1}$ without a vertex of valency 3 in its fully labeled version, I its set of vertex colors, n its rank, and let Δ be a Lie incidence geometry of type Π. Let X be the set of all singular subspaces of Δ, and let $*$ be the following incidence relation: a singular subspace x of dimension $0 < k \leq n-1$ is incident with a singular subspace y of dimension $< k$ if y is contained in x. We define a map τ as follows:

$$\tau : X \to I$$
$$\{k\text{-dimensional subspaces}\} \mapsto k+1, \text{ for all } k \in \{0, \ldots, n\}.$$

Then $(X, *, \tau)$ is an incidence system over I. This way, we can view certain Lie incidence geometries as incidence systems.

If the fully labeled Coxeter diagram of type $\mathsf{X}_{n,1}$ contains a vertex of valency 3, it becomes more complicated to define an incidence system like this, because we do not have a one-to-one correspondence between the vertices with label $k+1$ and the k-spaces of Δ for every $0 \leq k < n$. Why this is the case will hopefully become clear when we talk about how to translate between a building and a Lie incidence geometry. For now, we will just state which kinds of subgeometries have which type for $\mathsf{D}_{n,1}$, $\mathsf{E}_{6,1}$, and $\mathsf{E}_{7,1}$ without explanation.

[3]With regards to using the reference to obtain the desired information: we can find out about the maximal distance that two points in — for example — H_4 can have, by looking at the first number below the first vertex from the left of the H_4 diagram in [**BCN89**, Table 10.5], since that vertex contains information about the points, and the first number below from the left represents the diameter.

For $\mathsf{D}_{n,1}$:		For $\mathsf{E}_{6,1}$:		For $\mathsf{E}_{7,1}$:	
Elements	Types	Elements	Types	Elements	Types
Points	1	Points	1	Points	1
Lines	2	5-spaces	2	6-spaces	2
Planes	3	Lines	3	Lines	3
k-spaces for $3 < k \leq n-3$	$k+1$	Planes	4	Planes	4
Maximal subspaces of the one kind	$n-1$	4-spaces	5	4-spaces	5
Maximal subspaces of the other kind	n	Symps	6	Symps	6
				Paras	7

Definition 4.3.3.16 (**Chambers of Lie incidence geometry of type $\mathsf{X}_{n,1}$**)**.** Let Δ be a Lie incidence geometry of type $\mathsf{X}_{n,1}$ with diagram Π. A *chamber* C of Δ is an n-tuple whose entries are subgeometries of Δ such that C contains an element of every type of Π, and all two subgeometries have the maximal possible intersection that two elements of the respective types can have.

This matches how we defined chambers for projective spaces in Definition 4.1.0.9 and for thick polar spaces in Definition 4.1.0.4 (where the chambers are just the maximal flags and the maximal intersection is always incidence).

Example 4.3.3.17. In a top-thin polar space of type $\mathsf{D}_{5,1}$, for example, a chamber consists of a point p, a line L, a plane π, a 4-space V, and a $4'$-space V' such that

$$p \in L \subseteq \pi \subseteq V,\ \pi \subseteq V',\ \text{and } V \cap V' \text{ is a 3-space,}$$

where 4-spaces and $4'$-spaces denote the two classes of maximal singular subspaces as a top-thin polar space. In a parapolar space of type $\mathsf{E}_{7,1}$, a chamber consists of a point p, a line L, a plane π, a 4-space V, a 6-space A, a symp ξ, and a para Ξ such that

$$p \in L \subseteq \pi \subseteq V \subseteq \xi \subseteq \Xi,\ \pi \subseteq A, A \cap \xi \text{ is a } 4'\text{-space and } A \cap \Xi \text{ is a 5-space,}$$

where $4'$-spaces are 4-dimensional projective subspaces contained in 6-spaces, and 4-spaces are 4-dimensional projective spaces not contained in any higher-dimensional projective subspaces.[4] We will give more explanation and see many more examples for chambers in the following.

Definition 4.3.3.18 (**Duality**)**.** A *duality* of a Lie incidence geometry is an automorphism of the underlying building that acts as an involution on the type set.

4.3.4. Opposition

Depending on the type of the Lie incidence geometry, it varies what it means for two subspaces to be opposite. We will investigate possible mutual positions of

[4]Compare with [**NPVV25**].

subspaces in different Lie incidence geometries in the respective subsections. In this subsection, we only want to describe how we can see what types of elements are opposite what types of elements by looking at the diagram. We will use the opposition map introduced in Notation & Definition 2.3.0.7.

Fact 4.3.4.19. For this fact, we refer to [**BC13**, Remark 4.6.11]. Let Δ be a Lie incidence geometry with diagram Π of rank $n \geq 3$. Let σ be the opposition map. We view Π in its reductively labeled version and use the Bourbaki labeling of the vertices.

- If Π is not symmetrical*(4.3.2.12), or not simply-laced, then all types are self-opposite.
- If Π is of type A_n, then σ exchanges the types i and $n-i+1$ for $1 \leq i \leq n$.
- If Π is of type D_n and n is odd, then σ exchanges the vertices with label n and $n-1$ and fixes all others.[5]
- If Π is of type E_6, then σ is a reflection along the edge between the vertices with label 2 and 4, meaning that σ fixes elements of type 2 and 4 and exchanges elements of type 1 and 6, and elements of type 3 and 5.
- If Δ has rank 2, then the types are self-opposite if the label of the diagram is even, and σ exchanges the types if that label is odd.

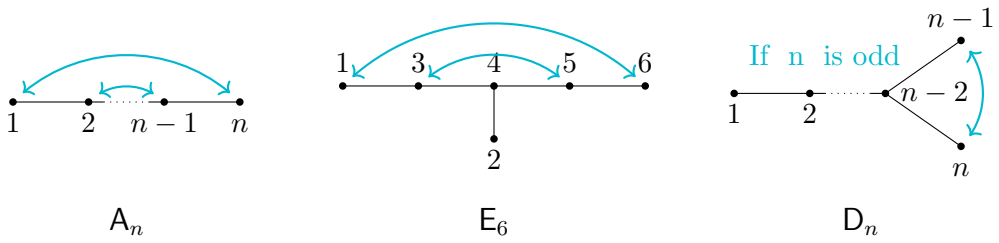


TABLE 3. The non-trivial opposition maps for spherical Coxeter diagrams.

Note 4.3.4.20. For polar spaces, we defined that points are opposite if they are non-collinear. Now we define that two subgeometries are opposite if their types are images of each other under the opposition map, and if each point of either is opposite some element incident with the other. This matches, since elements of type 1 are opposite elements of type 1 for Lie incidence geometries of type B_n or D_n.

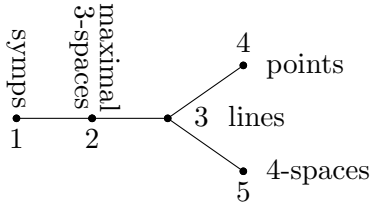
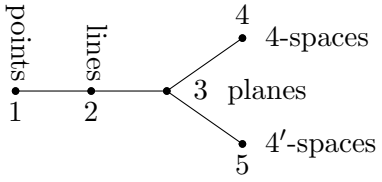
Notation 4.3.4.21. Let Δ be a Lie incidence geometry with diagram Π. Let γ be a (pre-)subgraph of Π and let Γ be a subgeometry of Δ with diagram γ. Let A and B be two subgeometries of Γ such that they are associated to vertices of γ of opposite types, and such that A and B are opposite in Γ. Then, instead of saying that A and B are subgeometries in Δ that are opposite inside the subgeometry Γ of Δ, we might sometimes call A and B Γ-*opposite* in Δ.

[5]For this you can also see [**Van24**, Exercise 8.2].

4.4. Parapolar Spaces of Type $\mathsf{D}_{5,5}$

Note 4.4.0.1. For the following subsections about the different kinds of parapolar spaces, the main reference will be the book *Projective Geometry over Arbitrary Fields* by Bart De Bruyn, Koen Thas, and Hendrik Van Maldeghem (cf. [**DTV25**]). We will always assume that elements of every type exist in a given Lie incidence geometry and that all possible mutual positions get realized. For detailed information about this, we refer to [**BV22**], [**DTV25**], and [**Tit57**].

We begin our more detailed description of some relevant parapolar spaces by considering the duals of polar spaces of type $\mathsf{D}_{5,1}$, which are of type $\mathsf{D}_{5,5}$.

$\mathsf{D}_{5,5}$		$\mathsf{D}_{5,1}$
		
points	$\leftrightarrow$	4-spaces
lines	$\leftrightarrow$	planes
maximal 3-spaces	$\leftrightarrow$	lines
4-spaces	$\leftrightarrow$	$4'$-spaces
symps	$\leftrightarrow$	points

We will differentiate between the two kinds of maximal singular subspaces in $\mathsf{D}_{5,1}$ by referring to them as 4-spaces or $4'$-spaces, as before. As noted in Section 3.3.1, two maximal singular subspaces of the same kind in a geometry of type $\mathsf{D}_{n,1}$ intersect in even codimension. That means, for geometries of type $\mathsf{D}_{5,1}$, two 4-spaces intersect either in a point or in a plane, and the same is true for the intersection of two $4'$-spaces. Conversely, a 4-space and a $4'$-space in $\mathsf{D}_{5,1}$ can either be opposite, or intersect in a line or a 3-space. Let Δ^* be a polar space of type $\mathsf{D}_{5,1}$. As described in Section 4.3.1, we obtain a geometry Δ of type $\mathsf{D}_{5,5}$ from Δ^* by taking maximal singular subspaces of one kind in Δ^* as points in Δ, and by drawing lines between two points $a, a' \in \Delta$ whenever there exists a chamber $C = (x_0, x_1, x_2, x_3, a)$ in Δ^* such that $C' = (x_0, x_1, x_2, x_3, a')$ is a chamber in Δ^* again. As a convention, the maximal singular subspaces corresponding to points in $\mathsf{D}_{5,5}$ will be the 4-spaces in the following (and not the $4'$-spaces). The following lemma will be useful to us.[6]

Lemma 4.4.0.2. *Two distinct symps in a Lie incidence geometry of type* $\mathsf{D}_{5,5}$ *are either disjoint or intersect in a* 3*-space.*

Lemma 4.4.0.2 is proved in [**DTV25**, Proposition 8.2.15]. For a short, heuristic description: symps in $\mathsf{D}_{5,5}$ correspond to points in $\mathsf{D}_{5,1}$. The "closest" distinct points

[6]It will be useful to us because Lie incidence geometries of type $\mathsf{D}_{5,5}$ appear as subgeometries in higher-rank Lie incidence geometries that we are interested in. Already in the next subsection, we will see an application of this lemma.

can be in $\mathsf{D}_{5,1}$, is, to be on a common line. The lines of $\mathsf{D}_{5,1}$ correspond to maximal 3-spaces of $\mathsf{D}_{5,5}$. If we see a line containing two points in $\mathsf{D}_{5,1}$, it corresponds to two symps intersecting in a 3-space in $\mathsf{D}_{5,5}$. As mentioned, this is not meant as a proof, but as a short, heuristic description, and a proper proof can be found in the cited reference.

4.5. Parapolar Spaces of Type $\mathsf{E}_{6,1}$

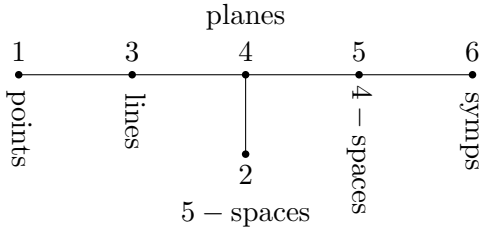
In this section, we will investigate parapolar spaces of type $\mathsf{E}_{6,1}$. That means that points will correspond to the vertex with label 1, as we have been used to before, and our previous definitions and observations hold. Looking at the $\mathsf{E}_{6,1}$ diagram, we can see that the biggest subdiagram of type $\mathsf{A}_{n,1}$ that we can find is $\mathsf{A}_{5,1}$. Hence, the rank of $\mathsf{E}_{6,1}$ is 5, and 5-spaces will be maximal projective subspaces.

We can see the $\mathsf{A}_{5,1}$ subdiagram if we cover the vertex with Bourbaki-label 2. Therefore (and this is introducing a new rule), 5-spaces will correspond to the vertex with label 2. Similarly, we can apply this new rule to all the other vertices:

- If we cover the vertex with label 6, we see a subdiagram of type $\mathsf{D}_{5,1}$. That means that symps in $\mathsf{E}_{6,1}$ will be top-thin polar spaces of rank 5 and correspond to the vertex with label 6 in the diagram.
- Covering the vertex with label 5 and looking at the left-hand side, we see a diagram of type $\mathsf{A}_{4,1}$, and hence 4-spaces correspond to the vertex with label 5.
- Covering the vertex with label 4 and looking at the left-hand side, we see a diagram of type $\mathsf{A}_{2,1}$, and hence planes correspond to the vertex with label 5, et cetera.

An interesting observation is the following: since maximal projective subspaces in $\mathsf{D}_{5,1}$ are 4-spaces, we can find two different kinds of 4-spaces in $\mathsf{E}_{6,1}$: those that are contained in symps, and those that are contained in 5-spaces. To differentiate between the two, we will call 4-spaces contained in 5-spaces $4'$-*spaces*. We will call all projective subspaces that are not contained in any higher-dimensional projective subspace *maximal*. Hence, both 4-spaces and 5-spaces are maximal projective subspaces in $\mathsf{E}_{6,1}$. We will also call $4'$-spaces *non-maximal* 4-dimensional projective subspaces.

$\mathsf{E}_{6,1}$



Now we cover the vertex with label 1. Looking at the right-hand side this time, we can see a diagram of type $\mathsf{D}_{5,5}$. That means (and this is introducing a new rule) that point residues in $\mathsf{E}_{6,1}$ are subgeometries of type $\mathsf{D}_{5,5}$. If we do the same for planes and look at the right-hand side (but not strictly), then we can see the vertex below and the two vertices connected by an edge to the right. Hence (and this belongs to the new rule), the residue of a plane in $\mathsf{E}_{6,1}$ is a direct product of a line and a plane. As a whole, the underlying rule is based on the upcoming Fact 4.9.0.1.

Lemma 4.5.0.1. *Let Δ be a parapolar space of type $\mathsf{E}_{6,1}$. Then two distinct, non-disjoint symps intersect in either a point or a maximal 4-space.*

PROOF. Consider two symps ξ_0 and ξ_1 and let x be a point in $\xi_0 \cap \xi_1$. The residue of x in ξ_i, for $i \in \{0, 1\}$, is a subgeometry of type $\mathsf{D}_{4,1}$ and contained in the residue of x in Δ. In $\mathsf{Res}_\Delta(x)$, both $\mathsf{Res}_{\xi_i}(x)$ form symps. Symps in parapolar spaces of type $\mathsf{D}_{5,5}$ are either disjoint or intersect in a maximal 3-space (see Lemma 4.4.0.2). Hence, in Δ, $x^\perp \cap \xi_1 \cap \xi_2$ has to be either just the point x or a maximal 4-space. □

Lemma 4.5.0.2. *Let Δ be a parapolar space of type $\mathsf{E}_{6,1}$. Then Δ is strong.*

PROOF. Since the point residues are subgeometries of type $\mathsf{D}_{5,5}$, the symps inside these point residues are of type $\mathsf{D}_{4,1}$. Consider two points p and q at distance 2 from each other in $\mathsf{E}_{6,1}$ and let o be a point collinear to both of them. We can consider the residue $\mathsf{Res}_\Delta(o)$ of the point o, in which we can see the lines op and oq as points. Because parapolar spaces of type $\mathsf{D}_{5,5}$ are strong, the points corresponding to op and oq are symplectic and hence contained in a symp ξ in $\mathsf{Res}_\Delta(o)$. If we consider the intersection of perps of the points corresponding to op and oq in ξ, then we obtain a top-thin polar space of rank 3 (Note 3.6.2.23), and every point in that intersection of perps gives rise to a path of length 2 from op to oq in the collinearity graph of $\mathsf{Res}_\Delta(o)$. Let π be a plane in the before-mentioned intersection of perps. Then in Δ, we can see π as a 3-space through o with $p \perp \pi$ and $q \perp \pi$. Therefore, $p^\perp \cap q^\perp$ is more than a unique point, and since p and q were two arbitrary points in Δ at distance 2, it follows that Δ is strong. □

Lemma 4.5.0.3. *Let Δ be a parapolar space of type $\mathsf{E}_{6,1}$. Then Δ has diameter 2.*

PROOF. Suppose there existed two points a and d at distance 3 from each other. Let b and c be two points such that $a \perp b \perp c \perp d$. Let $\xi(a, c)$ and $\xi(b, d)$ be the symps containing a and c, and b and d, respectively. Since symps are convex, $b \in \xi(a, c)$ and $c \in \xi(b, d)$. Since $b \perp c$, $\xi(a, c)$ and $\xi(b, d)$ already have to intersect in the line bc. But that means, according to Lemma 4.5.0.1, that $\xi(a, c) \cap \xi(b, d)$ is a 4-dimensional projective space V. Both a and d have to be collinear to a hyperplane of V, and these hyperplanes necessarily intersect in at least a plane, and every point p of that plane gives rise to a path of length 2 between a and d in the collinearity graph. □

Corollary 4.5.0.4. *Let Δ be a parapolar space of type $\mathsf{E}_{6,1}$. Then points are either collinear or symplectic.*

As mentioned before, since the diagram of $\mathsf{E}_{6,1}$ is symmetric, the opposition map fixes the vertices with label 2 and 4 and interchanges the vertices with label 3 and 5, and the vertices with label 1 and 6. So we have:

- Points and symps are opposite.
- Lines and 4-spaces are opposite.
- Planes are self-opposite.
- 5-spaces are self-opposite.

To understand what that really entails, we have to see which relations the various objects may have and define what *'opposite'* means in each case. We already know about point-point and symp-symp relations. Next we can investigate point-symp relations.

Lemma 4.5.0.5. *Suppose p is a point in a parapolar space Δ of type $\mathsf{E}_{6,1}$ and ξ is a symp of Δ not containing p. Then either p is not collinear to any point of ξ, or p is collinear to a $4'$-space contained in ξ.*

Proof. Suppose p were only collinear to one point q of ξ. Let u be a point in ξ collinear to q. Then p and q are symplectic and we denote the associated symp by $\xi(p,u)$. Since symps are convex, $\xi(p,u)$ contains q. That means that $\xi \cap \xi(p,u)$ already contains the line uq and hence, ξ and $\xi(p,u)$ have to intersect in a 4-space. But then p has to be collinear to a 3-space of that 4-space.

Now consider the residue of q in Δ. In $\mathsf{Res}_\Delta(q)$, we can see the line pq as a point, and we will refer to this point as $\mathsf{pt}(pq)$. If we translate $\mathsf{Res}_\Delta(q)$ into a geometry of type $\mathsf{D}_{5,1}$, $\mathsf{pt}(pq)$ can be seen as a 4-space. We will refer to this 4-space by PQ. In ξ, the point residue of q is a geometry of type $\mathsf{D}_{4,1}$ and we can see it as a symp χ in $\mathsf{Res}_\Delta(q)$. Again, if we translate this into $\mathsf{D}_{5,1}$, we see that symp as a point. We will refer to this point by x. So now we are in $\mathsf{D}_{5,1}$ and consider the point x and the 4-space PQ. Because $\mathsf{D}_{5,1}$ is a polar space, we know exactly what happens: x has to be collinear to a 3-space of PQ. Now we translate back to $\mathsf{D}_{5,5}$. A 3-space in $\mathsf{D}_{5,1}$ can be seen as the intersection of a 4-space and a $4'$-space. That means in $\mathsf{D}_{5,5}$, what corresponds to a 3-space is a point inside a symp. We can now see the situation in $\mathsf{D}_{5,5}$ as follows: We have the point $\mathsf{pt}(pq)$ and the symp χ, and the relation between the two has to be that $\mathsf{pt}(pq)$ is inside χ. The point residue of a point in χ is a subgeometry of type $\mathsf{D}_{3,2}$, which is the same as $\mathsf{A}_{3,2}$, so a 3-dimensional projective space. Therefore, we get that $\mathsf{pt}(pq)^\perp \cap \chi$ is a 3-space. Finally, we translate back to Δ and see that p has to be collinear to a 4-dimensional projective space V of ξ including q. This 4-dimensional space has to be a $4'$-space since $\langle p, V\rangle$ spans a 5-space in which V is contained. A point outside a given symp in $\mathsf{E}_{6,1}$ cannot be collinear to more than a maximal subspace of that symp.

The only thing left to see is that a point can be non-collinear to all points of a symp. Let ξ_0 and ξ_1 be two symps that only intersect in a point q. Let p be a point in ξ_0 not collinear to q. Suppose p were collinear to a $4'$-space W of ξ_1. Then q has to be collinear to a 3-space of W and $\langle q, q^\perp \cap W\rangle$ has to be contained in the symp spanned by p and q, which is ξ_0. But then $q^\perp \cap W$ is contained in $\xi_0 \cap \xi_1$, and it follows that ξ_0 and ξ_1 intersect in a 4-space. □

Definition 4.5.0.6. Let Δ be a parapolar space of type $\mathsf{E}_{6,1}$. We say that a point p and a symp ξ in Δ are *opposite* if p is not collinear to any point of ξ.

Corollary 4.5.0.7. *Let Δ be a parapolar space of type $\mathsf{E}_{6,1}$. The maximal intersection that a symp and a 5-space can have is a $4'$-space. A chamber in $\mathsf{E}_{6,1}$ consists of*

a point p, a line L, a plane π, a 4-space V, a 5-space A, and a symp ξ, such that

$$p \in L \subseteq \pi \subseteq V \subseteq \xi,\ \pi \subseteq A,\ \text{and } A \cap \xi \text{ is a } 4'\text{-space.}$$

Next, we will focus on the relations between maximal singular subspaces with each other, with symps, and with points.

Lemma 4.5.0.8. *Let Δ be a parapolar space of type $\mathsf{E}_{6,1}$. Then two non-disjoint 5-spaces either intersect in a point or a plane.*

PROOF. Let ξ be a symp in Δ and let A and B be two 5-spaces in Δ that both intersect ξ in respective $4'$-spaces U and V.[7] Symps in Δ are of type $\mathsf{D}_{5,1}$, and $4'$-spaces in $\mathsf{D}_{5,1}$ intersect either in a point or a plane.

If two distinct 5-spaces intersected in a $4'$-space, the point residue of a point in the intersection would contain two 4-spaces intersecting in a non-maximal 3-space. Translating that situation to $\mathsf{D}_{5,1}$, we would see two $4'$-spaces that do not intersect in even codimension, which is impossible.

If two distinct 5-spaces A and B intersected in a 3-space C, then we could consider two non-collinear points $a \in A$ and $b \in B$ and a symp containing both of them. Then that symp would also contain C and hence two $4'$-spaces, namely $\langle a, C\rangle$ and $\langle b, C\rangle$, intersecting in a 3-space. That is impossible because $4'$-spaces intersect in even codimension. □

We can also see this in a different way: Consider the residue of a plane π inside the 3-space C. The residue of a plane in Δ is a direct product of a line and a plane. In $\mathsf{Res}_\Delta(\pi)$ we would see A and B as planes and C as a point. So we would have two planes intersecting in a point in $\mathsf{Res}_\Delta(\pi)$, which is impossible because there are no planes intersecting in a point in a direct product of a line and a plane.

Lemma 4.5.0.9. *Let Δ be a parapolar space of type $\mathsf{E}_{6,1}$. Let A be a 5-space and p be a point of Δ not contained in A. Then p is either collinear to a unique point of A or p is collinear to a 3-space of A.*

PROOF. Let q be a point of A not collinear to p, let $\xi(p, q)$ be the symp spanned by p and q, and let U be a 3-space contained in $p^\perp \cap q^\perp$. In the residue $\mathsf{Res}_\Delta(q)$, we see A as a 4-space, $\xi(p, q)$ as a symp, and U as a 3-space. In $\mathsf{D}_{5,5}$, a 4-space and a symp intersect either in a point or in a 3-space. If the 4-space corresponding to A in $\mathsf{Res}_\Delta(q)$, and the symp corresponding to $\xi(p, q)$ in $\mathsf{Res}_\Delta(q)$ intersect in a 3-space, then that 3-space and the 3-space corresponding to $p^\perp \cap q^\perp$ intersect in a plane, and therefore A and $p^\perp \cap q^\perp$ intersect in a 3-space in Δ, implying that $p^\perp \cap A$ is a 3-space.

If the 4-space corresponding to A in $\mathsf{Res}_\Delta(q)$, and the symp corresponding to $\xi(p, q)$ in $\mathsf{Res}_\Delta(q)$ intersect in a point ℓ, then the line L corresponding to ℓ in Δ through q is contained in both A and $\xi(p, q)$, and since $\xi(p, q)$ is a polar space, p has to be collinear to a unique point of L. □

Definition 4.5.0.10. Let Δ be a parapolar space of type $\mathsf{E}_{6,1}$. Let A be a 5-space and p be a point of Δ not contained in A. We say that p is *close* to A if $p^\perp \cap A$ is a 3-space, and we say that p is *far* from A if $p^\perp \cap A$ is a unique point.

[7] Recall Note 4.4.0.1 with regards to the existence.

We have demonstrated how one can find out about the relations between different objects. Since all of these relations are known and the arguments can get quite long, we will continue to state the remaining relations without proof. For the rest of this subsection, $\Delta = (\mathscr{P}, \mathscr{L})$ will always be a parapolar space of type $\mathsf{E}_{6,1}$, and we will always assume to be inside a parapolar space of type $\mathsf{E}_{6,1}$.

Facts 4.5.0.11.

(i) Each singular subspace V of dimension 4 is contained in a unique maximal singular subspace. If the latter is a 5-space, then V is a $4'$-space.

(ii) For each symp ξ, its maximal singular subspaces contained in some other symp are 4-spaces and form one natural class of generators, and its maximal singular subspaces contained in a 5-space of Δ are $4'$-spaces and form the other class.

(iii) For each point $p \in \mathscr{P}$, there exists a symp ξ disjoint from $p^\perp$.

(iv) A singular 5-space that intersects a given symp in at least a plane intersects the symp in a $4'$-space. There exist 5-spaces intersecting a given symp in precisely a given line, and 5-spaces that have no intersection with a given symp. No 5-space intersects any symp in just a point.

(v) Each singular 3-space is contained in a unique maximal singular 4-space and a unique singular 5-space.

Proposition 4.5.0.12. *Let A and B be two 5-spaces. Then exactly one of the following holds.*

(0) $A = B$.

(1) $A \cap B$ *is a plane — and every point of* $A \setminus B$ *is close to* B.

(2) $A \cap B$ *is a point — and there exists a unique* $4'$*-space* H *of* A *with the property that a point* $x \in A \setminus B$ *is close to* B *if and only if* $x \in H$.

(2') $A \cap B = \varnothing$ *— and there exists a unique 5-space* W *such that* $\alpha := A \cap W$ *and* $\beta := W \cap B$ *are both planes; in that case, every point of* $A \setminus \alpha$ *is far from* B*, and each such point is collinear to a point of* β.

(3) $A \cap B = \varnothing$ *— and every point of* A *is collinear to a unique point of* B*; that means that all points of* A *are far from* B.

REFERENCE. This is [**NV25**, Lemma 2.25]. □

Definition 4.5.0.13. We use the terms that are commonly used in the literature and call two 5-spaces A and B

(0) *incident* if $A = B$.

(1) *adjacent* if their intersection is a plane.

(2) *symplectic* if their intersection is a point.

(2') *special* if they are disjoint and there exists another 5-space intersecting each of them in a plane.

(3) *opposite* if they are disjoint and all points of A are far from B.

Proposition 4.5.0.14. *Let $p \in \mathscr{P}$ and $x_1, x_2 \in \mathscr{P} \setminus p^\perp$. Then exactly one of the following holds.*

(*i*) $p^\perp \cap x_1^\perp \cap x_2^\perp$ *contains a singular* 3*-space and* $x_1 \perp x_2$.
(*ii*) $p^\perp \cap x_1^\perp \cap x_2^\perp$ *properly contains a singular plane but not a singular* 3*-space, and* $x_1 \not\perp x_2$.
(*iii*) $p^\perp \cap x_1^\perp \cap x_2^\perp$ *is a singular* 3*-space and* $x_1 \perp x_2$.
(*iv*) $p^\perp \cap x_1^\perp \cap x_2^\perp$ *is a singular plane and* $x_1 \not\perp x_2$.
(*v*) $p^\perp \cap x_1^\perp \cap x_2^\perp = \varnothing$ *and* $x_1 \not\perp x_2$.

PROOF. This is [**DTV25**, Corollary 9.1.6]. □

Lemma 4.5.0.15. *Let A be a singular* 5*-space and let π be a plane. If each point of π is collinear to a unique point of A, then there exists a unique singular* 5*-space B such that B is not opposite A and contains π. In that case, the* 5*-space B is special to A. Let $\alpha \subseteq A$ be the plane that is collinear to a plane of B. Then the point set of α coincides with the set of points of A that are collinear to some point of π.*

REFERENCE. This is [**NV25**, Lemma 2.26]. □

Lemma 4.5.0.16. *Given two opposite* 5*-spaces A and B, and any line L which intersects both A and B, then for any point p on L, there exists a* 5*-space C containing p such that C intersects each line connecting a point of A with a point of B.*

REFERENCE. This is [**NV25**, Lemma 2.27]. □

Definition & Fact 4.5.0.17. A line L and a 4-space V are *opposite* if no point of L is collinear with any point of V. Similarly, a plane α and a plane β are *opposite* if no point of α is collinear with any point of β.

Lemma 4.5.0.18. *Let L and π be a line and a plane, respectively, disjoint from a symp Σ, but all of whose points are close to Σ. Then there exist a unique plane π' and a unique line L' in Σ such that $L \perp \pi'$ and $\pi \perp L'$. Also, $\langle L, \pi' \rangle$ is a* 4*-space, whereas $\langle L', \pi \rangle$ is a* 4*′-space.*

PROOF. We first prove the lemma for the line L. Since all points of L are close to ξ, every point of L is collinear to a 4′-space of ξ. By Facts 4.5.0.11, two 4′-spaces in a symp intersect in either a point or a plane or they coincide. If two points of L were collinear to the same 4′-space of ξ, then every point of that 4′-space would be collinear to every point of L and L and that 4′-space would span a 6-space, which is impossible. Now, let x and y be two points of L, let V be the 4′-space that y is collinear to in ξ and let x' be some point of $\xi \setminus V$ that x is collinear to. Then x' has to be collinear to a 3-space $U \subseteq V$. The symp $\xi(y, x')$ contains U and x. With that, x has to be collinear to a plane π of U. That means $x^\perp \cap \xi$ and $y^\perp \cap \xi$ intersect in π and since every point of π is collinear to $x, y \in L$, and $\mathsf{E}_{6,1}(\mathbb{K})$ is a gamma space, every point of π has to be collinear to every point of L. With that, L is collinear to a unique plane of ξ. Then $W := \langle L, \pi \rangle$ is clearly a 4-space as it has a 3-space in common with the 4′-space V.

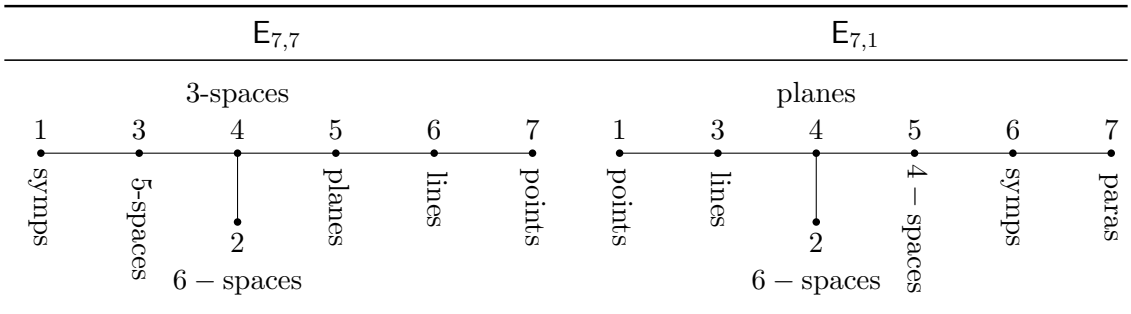
Now we prove the lemma for the plane π. We may assume without loss of generality that $L \subseteq \pi$. Let q be a point of π not on L. Let ζ be a symp containing W. If $q \in \zeta$, then $q^\perp \cap W$ is a 3-space, which intersects π' in a line L'. Hence π is collinear to a line L', and since $< \pi, L' >$ intersects W in a 3-space, the former is a $4'$-space. If $q \notin \zeta$, then it is collinear to a $4'$-space U_ζ of ζ and a $4'$-space U_ξ of ξ. Now, by the first part of this proof, U_ξ intersects V in a plane, and so q is collinear to some point of π'. Then U_ζ, which contains that point as well as L, intersects W in at least a plane, and hence in a 3-space. The rest is similar to the case $q \in \zeta$. □

Lemma 4.5.0.19. *Let Δ be a parapolar space of type $\mathsf{E}_{6,1}$. Let ξ be a symp of Δ and let π be a plane of Δ intersecting ξ in a unique point x. Then there exists a unique plane $\alpha \subseteq \xi$, all points of which are collinear to all points of π.*

Proof. Let L be a line in π not intersecting ξ. The lemma now follows from Lemma 4.5.0.18. □

4.6. Parapolar Spaces of Type $\mathsf{E}_{7,1}$ & $\mathsf{E}_{7,7}$

Regarding parapolar spaces associated with the group E_7, we are interested in both $\mathsf{E}_{7,1}$ and the dual $\mathsf{E}_{7,7}$. We want to find out about $\mathsf{E}_{7,1}$, because it is the so-called *long root (subgroup) geometry* related to buildings of type E_7 — we will explain later what that means — and we want to get to know $\mathsf{E}_{7,7}$, because it is, contrary to $\mathsf{E}_{7,1}$, a strong parapolar space, and it is generally easier to work in parapolar spaces without special pairs of points. We start by presenting the associated diagrams.



By looking at the diagram, we can see, just as before, that both $\mathsf{E}_{7,1}$ and $\mathsf{E}_{7,7}$ have rank 6. According to Fact 4.3.2.14 and Fact 4.3.4.19, $\mathsf{E}_{7,1}$ has diameter 3 and a trivial opposition map. The same is true for $\mathsf{E}_{7,7}$ for the same heuristic reasons: The diagram is not symmetrical, has a vertex of valency 3, but no edge labels. Certainly, it requires a proper proof, and it is possible to prove these facts similarly to how we did in the last subsection. For a reference, we point to [**DTV25**, Lemma 9.2.2]. In the following, we will first state some more facts about $\mathsf{E}_{7,1}$ and then about $\mathsf{E}_{7,7}$.

4.6.1. Parapolar Spaces of Type $\mathsf{E}_{7,1}$

Facts 4.6.1.1. Let Δ be a parapolar space of type $\mathsf{E}_{7,1}$. Then the following assertions hold.

(i) Δ has rank 6, diameter 3, a trivial opposition map, and Δ is non-strong.

(ii) Paras in Δ are isomorphic to parapolar spaces of type $\mathsf{E}_{6,1}$.
(Covering the vertex with label 7 and looking at the left-hand side yields an $\mathsf{E}_{6,1}$ diagram.)

(iii) Symps in Δ are isomorphic to polar spaces of type $\mathsf{D}_{5,1}$.
(Covering the vertices with label 6 and looking at the left-hand side yields a $\mathsf{D}_{5,1}$ diagram.)

(iv) Point residues in Δ are isomorphic to parapolar spaces of type $\mathsf{D}_{6,6}$.
(Covering the vertex with label 1 and looking at the right-hand side yields a $\mathsf{D}_{6,6}$ diagram.)

(v) The maximal singular subspaces of highest dimension in Δ are projective spaces of dimension 6. We call 4-dimensional projective subspaces contained in 6-spaces $4'$-spaces. These are non-maximal 4-dimensional projective subspaces. There are 4-dimensional projective subspaces in Δ that are not contained in any 6-space, and we will refer to them as maximal 4-spaces. These maximal 4-spaces are contained in symps.

(vi) Each 5-space occurs as the intersection of a unique para and a unique 6-space, and hence corresponds to a flag of type $\{2, 7\}$.

Fact 4.6.1.2 (Point-symp relations)**.** Let Δ be a parapolar space of type $\mathsf{E}_{7,1}$. If p is a point and ξ a symp of Δ with $p \notin \xi$, then precisely one of the following occurs.

(i) p is collinear to a unique $4'$-space U contained in ξ, p and ξ are contained in a unique para, and p is symplectic to the points of ξ that are not contained in U.

(ii) p is collinear to a unique line L of ξ, p is symplectic to the points $\xi \cap (L^\perp \setminus L)$, and p is special to the points $\xi \setminus L^\perp$.

(iii) p is symplectic to all points of ξ, and p and ξ are contained in a unique para, in whose subgeometry they are opposite.

(iv) p is symplectic to all points of a 4-space U contained in ξ, and p is special to the points of ξ that are not contained in U.

(v) p is symplectic to a unique point $q \in \xi$, p is special to all points of $\xi \cap (q^\perp \setminus \{q\})$, and opposite to all points of $\xi \setminus q^\perp$.

Fact 4.6.1.3 (Symp-symp relations)**.** Let Δ be a parapolar space of type $\mathsf{E}_{7,1}$, and let ξ and ξ' be two distinct symps of Δ. Then precisely one of the following occurs.

(i) $\xi \cap \xi'$ is a unique 4-space.

(ii) $\xi \cap \xi'$ is a unique plane.

(iii) $\xi \cap \xi'$ is a unique point.

(iv) $\xi \cap \xi' = \varnothing$.

Fact 4.6.1.4 (Point-para relations)**.** Let Δ be a parapolar space of type $\mathsf{E}_{7,1}$. If p is a point and Ξ a para of Δ with $p \notin \Xi$, then precisely one of the following occurs.

(i) p is collinear to a unique 5-space A contained in Ξ, and either symplectic or special to the points of $\Xi \setminus A$. The point p is special to a point $q \in \Xi$ if q is collinear to a unique point of A. In this case, we call p *close* to Ξ.

(ii) p is not collinear to any point of Ξ, but it is contained in a unique para Ξ' that intersects Ξ in a symp ξ. In this case, p and ξ are opposite in the subgeometry Ξ' of Δ, and we call p *far* from Ξ.

Fact 4.6.1.5 (Para-para relations)**.** Let Δ be a parapolar space of type $\mathsf{E}_{7,1}$. If Ξ and Ξ' are two distinct paras, then precisely one of the following occurs.

(i) $\Xi \cap \Xi'$ is a symp.

(ii) $\Xi \cap \Xi'$ is a point.

(iii) $\Xi \cap \Xi' = \varnothing$, and each point $x \in \Xi$ is far from Ξ'. Let ξ_x be the unique symp of Ξ' contained in a para with x. Then each point of $\Xi' \setminus \xi_x$, collinear to a point of ξ_x, is special to x. Each point in Ξ' that is opposite ξ_x in the subgeometry given by Ξ', is at distance 3 from x. The correspondence $\Xi \to \Xi'$, $x \mapsto \xi_x$ induces an isomorphism from Ξ onto the dual of Ξ'.

Fact 4.6.1.6. Let Δ be a parapolar space of type $\mathsf{E}_{7,1}$. Let Ξ be a para of Δ and let A, A' be two singular 5-spaces of Ξ, which are opposite in the subgeometry given by Ξ. Then every point $q \in A$ is collinear to a unique point $\theta(q)$ of A'. Let G and G' be the unique singular 6-spaces containing A and A', respectively. Let $p \in G$ and $p' \in G'$. Then

(i) $p \perp p'$ if and only if $p \in A$, and $p' = \theta(p)$.

(ii) $p \perp\!\!\!\perp p'$ if and only if $p \in A$, $p' \in A'$, and $p' \neq \theta(p)$.

(iii) p and p' are special if and only if either $p \in G \setminus A$, and $p' \in A'$, or $p \in A$, and $p' \in G' \setminus A'$.

(iv) p is opposite p' if and only if $p \in G \setminus A$, and $p' \in G' \setminus A'$.

Conversely, let B and B' be two singular 5-spaces such that each point of B is collinear to a unique point of B'. Then B and B' are contained in a unique para Π, where they are Π-opposite.

Notation 4.6.1.7. In the following, we will refer to the maximal singular subspaces of highest dimension as *maxhighs* for short. In the case of $\mathsf{E}_{7,1}$, the maxhighs are precisely the 6-spaces.

Fact 4.6.1.8 (Point-maxhigh relations)**.** Let Δ be a parapolar space of type $\mathsf{E}_{7,1}$, let G be a 6-space, and let p be a point in Δ. Then precisely one of the following occurs.

(i) $p \in G$.

(ii) $p^\perp \cap G$ is a unique 3-space.

(iii) $p^\perp \cap G$ is a unique point.

(iv) $p^\perp \cap G = \varnothing$.

Fact 4.6.1.9 (Maxhigh-maxhigh relations)**.** Let Δ be a parapolar space of type $\mathsf{E}_{7,1}$, and let G and G' be two distinct 6-spaces in Δ. Then precisely one of the following occurs.

(i) $G \cap G'$ is a unique plane.
(ii) $G \cap G'$ is a unique point.
(iii) $G \cap G' = \varnothing$.

Fact 4.6.1.10 (Maxhigh-symp relations)**.** Let Δ be a parapolar space of type $\mathsf{E}_{7,1}$, let G be a 6-space, and let ξ be a symp in Δ. Then precisely one of the following occurs.

(i) $G \cap \xi$ is a unique $4'$-space.
(ii) $G \cap \xi$ is a unique line.
(iii) $G \cap \xi = \varnothing$.

4.6.2. Parapolar Spaces of Type $\mathsf{E}_{7,7}$

Facts 4.6.2.11. Let Δ be a parapolar space of type $\mathsf{E}_{7,7}$. Then the following assertions hold.

(i) Δ has rank 6, diameter 3, a trivial opposition map, and Δ is strong.
(ii) Symps in Δ are isomorphic to polar spaces of type $\mathsf{D}_{6,1}$.
(The maximal subdiagram associated to a polar space that we can see is of type $\mathsf{D}_{6,1}$.)
(iii) Point residues in Δ are isomorphic to parapolar spaces of type $\mathsf{E}_{6,1}$.
(Covering the vertex with label 7 and looking at the left-hand side yields an $\mathsf{E}_{6,1}$ diagram.)
(iv) The maximal singular subspaces of highest dimension in Δ are projective spaces of dimension 6. We call 5-dimensional projective subspaces contained in 6-spaces $5'$-spaces, and those contained in symps (maximal) 5-spaces.
(v) Maximal 5-spaces occur as the intersection of two symps. On the other hand, $5'$-spaces occur as the intersection of a 6-space and a symp and correspond to flags of type $\{1, 2\}$.
(vi) For each symp ξ, its 5-spaces contained in some other symp form one natural class of generators. Its $5'$-spaces contained in a 6-space of Δ form the other class.

Fact 4.6.2.12 (Point-symp relations)**.** Let Δ be a parapolar space of type $\mathsf{E}_{7,7}$. If p is a point, and ξ a symp of Δ with $p \notin \xi$, then precisely one of the following occurs.

(i) p is collinear to a $5'$-space A of ξ, p is symplectic to the points $\xi \setminus A$, and we say that p is *close* to ξ.
(ii) p is collinear to a unique point $q \in \xi$, p is symplectic to the points $\xi \cap (q^\perp \setminus \{q\})$, and p is opposite to the points $\xi \setminus q^\perp$. We say that p is *far* from ξ.

This fact implies that, on each line L, there is at least one point symplectic to a given point p (unique when L contains at least one point opposite p).

Fact 4.6.2.13 (Symp-symp relations)**.** Let Δ be a parapolar space of type $\mathsf{E}_{7,7}$, and let ξ and ξ' be two distinct symps of Δ. Then precisely one of the following occurs.

(i) $\xi \cap \xi'$ is a 5-space, and we call ξ and ξ' *adjacent*.

(ii) $\xi \cap \xi'$ is a line L. Then points $x \in \xi \setminus L$ and $x' \in \xi' \setminus L$ are never collinear, and $\mathsf{dist}(x, x') = 3$ if and only if $x^\perp \cap L$ is disjoint from $x'^\perp \cap L$. We call $\{\xi, \xi'\}$ *symplectic*.

(iii) $\xi \cap \xi' = \varnothing$, and there is a unique symp ξ'' intersecting both ξ and ξ' in respective 5-spaces A and A', which are opposite in ξ''. All points of $\xi \setminus A$ are far from ξ', and each point of A is close to ξ'. Each line containing a point of ξ and a point of ξ' contains a point of $A \cup A'$. We call $\{\xi, \xi'\}$ *special*.

(iv) $\xi \cap \xi' = \varnothing$, and every point of ξ is collinear to a unique point of ξ'. In this situation, each point of ξ is far from ξ', and ξ and ξ' are called *opposite*.

Note 4.6.2.14. The point-symp and symp-symp relations in $\mathsf{E}_{7,7}$ are similar to the point-maxhigh and maxhigh-maxhigh relations of $\mathsf{E}_{6,1}$ (cf. Lemma 4.5.0.9 and Proposition 4.5.0.12).

Fact 4.6.2.15. Let Δ be a parapolar space of type $\mathsf{E}_{7,7}$, and let ξ_1 and ξ_2 be two opposite symps of Δ. Let $\mathscr{L}$ be the set of all lines that contain a point of ξ_1 and a point of ξ_2. Then, for each point p that is contained in a line of $\mathscr{L}$, there exists a symp ξ_p that intersects each line $L \in \mathscr{L}$.

Fact 4.6.2.16 (Point-maxhigh relations)**.** Let Δ be a parapolar space of type $\mathsf{E}_{7,7}$, let G be a 6-space, and let p be a point of Δ. Then precisely one of the following occurs.

(i) $p \in G$.

(ii) $p^\perp \cap G$ is a unique 4-space.

(iii) $p^\perp \cap G$ is a unique line.

(iv) $p^\perp \cap G = \varnothing$.

Fact 4.6.2.17 (Maxhigh-maxhigh relations)**.** Let Δ be a parapolar space of type $\mathsf{E}_{7,7}$, and let G and G' be two distinct 6-spaces in Δ. Then precisely one of the following occurs.

(i) $G \cap G'$ is a unique 3-space.

(ii) $G \cap G'$ is a unique line.

(iii) $G \cap G'$ is a unique point.

(iv) $G \cap G' = \varnothing$.

Fact 4.6.2.18 (Symp-maxhigh relations)**.** Let Δ be a parapolar space of type $\mathsf{E}_{7,7}$, let G be a 6-space, and let ξ be a symp of Δ. Then precisely one of the following occurs.

(i) $G \cap \xi$ is a unique 5′-space.

(ii) $G \cap \xi$ is a unique plane.

(iii) $G \cap \xi$ is a unique point.

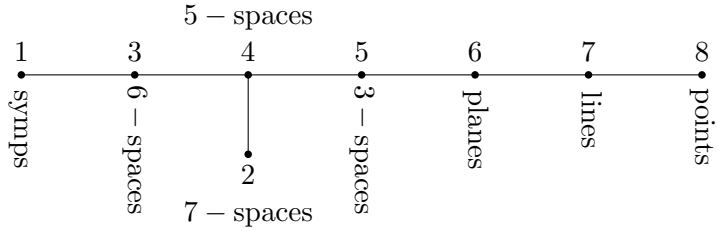
(iv) $G \cap \xi = \varnothing$.

Fact 4.6.2.19. Let Δ be a parapolar space of type $\mathsf{E}_{7,7}$. Let M be a maximal 5-space and let ξ and ξ' be two distinct symps containing M. Let $p \in \xi \setminus M$ and $p' \in p^\perp \cap (\xi' \setminus M)$. Then every point on the line $\langle p, p' \rangle$ is contained in a (unique) symp which contains M.

4.7. Parapolar Spaces of Type $\mathsf{E}_{8,8}$

For E_8, we will only consider parapolar spaces of type $\mathsf{E}_{8,8}$. Parapolar spaces of type $\mathsf{E}_{8,1}$ are complicated, not much is known about them, and we will actually neither work with them nor need them in this thesis. For E_8, parapolar spaces of type $\mathsf{E}_{8,8}$ correspond to the *long root (subgroup) geometry.*

$\mathsf{E}_{8,8}$



Facts 4.7.0.1. Let Δ be a parapolar space of type $\mathsf{E}_{8,8}$. Then the following assertions hold.

(i) Δ has rank 7, diameter 3, a trivial opposition map, and is non-strong.

(ii) Symps in Δ are isomorphic to polar spaces of type $\mathsf{D}_{7,1}$.
(The maximal subdiagram associated to a polar space that we can see is of type $\mathsf{D}_{7,1}$, and we see it if we cover the vertex with label 1. Hence, that vertex has to correspond to symps.)

(iii) Point residues in Δ are isomorphic to parapolar spaces of type $\mathsf{E}_{7,7}$.
(Covering the vertex with label 8 and looking at the left-hand side yields an $\mathsf{E}_{7,7}$ diagram. It is not $\mathsf{E}_{7,1}$, as one might expect, because of the way we had to deal with the case of the symps.)

(iv) The maximal singular subspaces of highest dimension in Δ are projective spaces of dimension 7. Again, we call 6-dimensional projective subspaces contained in 7-spaces 6′-spaces and those contained in symps (maximal) 6-spaces.

(v) Each 6′-space occurs as the intersection of a unique symp and a unique 7-space, and hence corresponds to a flag of type $\{1, 2\}$.

(vi) For each symp ξ, its 6-spaces contained in some other symp form one natural class of generators. Its 6′-spaces contained in a 7-space of Δ form the other class.

Note 4.7.0.2. These general facts about $\mathsf{E}_{8,8}$ are similar to the general facts about $\mathsf{E}_{7,1}$ (cf. Facts 4.6.1.1).

Fact 4.7.0.3 (Point-symp relations)**.** Let Δ be a parapolar space of type $\mathsf{E}_{8,8}$, let p be a point, and let ξ be a symp of Δ with $p \notin \xi$. Then precisely one of the following occurs.

(i) p is collinear to a 6′-space G' of ξ, p is symplectic to all points in $\xi \setminus G'$, and we say that p is *max-collinear* to ξ.

(ii) p is collinear to a unique line L of ξ, p is symplectic to all points in $\xi \cap L^\perp$, p is special to all points in $\xi \setminus L^\perp$, and we say that p is *line-collinear* to ξ.

(iii) p is symplectic to all points of a 6-space G of ξ, p is special to all points in $\xi \setminus G$, and we say that p is *max-symplectic* to ξ.

(iv) p is symplectic to a unique point q of ξ, p is special to all points in

$$\xi \cap q^\perp \setminus \{q\},$$

and p is opposite to all points in $\xi \setminus q^\perp$. In this case, we say that p is *point-symplectic* to ξ.

Note 4.7.0.4. Some point-symp relations in $\mathsf{E}_{8,8}$ are similar to the point-symp relations in $\mathsf{E}_{7,1}$ (cf. Fact 4.6.1.2).

Fact 4.7.0.5 (Symp-symp relations)**.** Let Δ be a parapolar space of type $\mathsf{E}_{8,8}$ and let ξ_1 and ξ_2 be two distinct symps in Δ. Then $\xi_1 \cap \xi_2$ is either empty, a point, a plane, or a 6-space.

Firstly, if $\xi_1 \cap \xi_2 = \varnothing$, then one of the following occurs.

(i) Each point of ξ_1 is point-symplectic to ξ_2, and "being symplectic" induces a duality between ξ_1 and ξ_2 (an isomorphism which switches 6- and 6′-spaces). We say that ξ_1 and ξ_2 are *opposite*.

(ii) There is a unique point p that is max-collinear to both ξ_1 and ξ_2. In this case, with $\{i, j\} = \{1, 2\}$, each point $q \in p^\perp \cap \xi_i$ is max-symplectic to ξ_j, each point $q \in \xi_i \setminus p^\perp$ is point-symplectic to ξ_j, and the unique point of ξ_j symplectic to q lies in $p^\perp \cap \xi_j$. We say that ξ_1 and ξ_2 are *point-disjoint*.

(iii) There is a unique symp ξ intersecting ξ_1 and ξ_2 in respective planes α_1 and α_2, which are opposite each other in ξ. With $\{i, j\} = \{1, 2\}$, the points in α_i are line-collinear to ξ_j. We say that ξ_1 and ξ_2 are *symp-disjoint*.

(iv) There is a unique line intersecting ξ_1 and ξ_2 in points p_1, p_2 that are max-collinear to ξ_2, ξ_1, respectively. We say that ξ_1 and ξ_2 are *line-disjoint*.

(v) There is a unique 7-space intersecting ξ_1 and ξ_2 in respective 3-spaces V_1, V_2. Each point in V_i is collinear to a 6′-space containing V_j for $\{i, j\} = \{1, 2\}$, and ξ_i contains no points point-symplectic to ξ_j. We say that ξ_1 and ξ_2 are *max-disjoint*.

Secondly, if $\xi_1 \cap \xi_2$ is a single point p, then one of the following occurs.

(i) ξ_1 and ξ_2 correspond to opposite symps in $\mathsf{Res}_\Delta(p)$. This means that each line L_i in ξ_i through p is collinear to a unique line L_j through p in ξ_j for $\{i, j\} = \{1, 2\}$. Moreover, each point of $\xi_i \cap p^\perp \setminus \{p\}$ is line-collinear to ξ_j. A point p_1 of ξ_1 is opposite a point p_2 of ξ_2 if and only if p_i is ξ-opposite p for $i = 1, 2$. We say that ξ_1 and ξ_2 are *locally opposite*.

(ii) There is a unique symp ξ intersecting ξ_1 and ξ_2 in respective 6-spaces G_1 and G_2. The points in G_i are max-collinear to ξ_j for $\{i,j\} = \{1,2\}$. No point of ξ_1 is opposite any point of ξ_2.

Finally, if $\xi_1 \cap \xi_2$ is at least a line, then one of the following occurs.

(i) Each point of $\xi_1 \setminus \xi_2$ is max-collinear to ξ_2, In this case, $\xi_1 \cap \xi_2$ is a 6-space.

(ii) Some point of ξ_1 is line-collinear to ξ_2, but no point of ξ_1 is max-symplectic to ξ_2. In this case, $\xi_1 \cap \xi_2$ is a plane.

(iii) $\xi_1 = \xi_2$.

Fact 4.7.0.6 (Point-maxhigh relations)**.** Let Δ be a parapolar space of type $\mathsf{E}_{8,8}$, let G be a 7-space, and let p be a point in Δ. Then precisely one of the following occurs.

(i) $p \in G$.

(ii) $p^\perp \cap G$ is a unique 5-space.

(iii) $p^\perp \cap G$ is a unique plane.

(iv) $p^\perp \cap G$ is a unique point.

(v) $p^\perp \cap G = \varnothing$.

Fact 4.7.0.7 (Maxhigh-maxhigh relations)**.** Let Δ be a parapolar space of type $\mathsf{E}_{8,8}$, and let G and G' be two distinct 7-spaces in Δ. Then precisely one of the following occurs.

(i) $G \cap G'$ is a unique 4-space.

(ii) $G \cap G'$ is a unique plane.

(iii) $G \cap G'$ is a unique line.

(iv) $G \cap G'$ is a unique point.

(v) $G \cap G' = \varnothing$.

Fact 4.7.0.8 (Maxhigh-symp relations)**.** Let Δ be a parapolar space of type $\mathsf{E}_{8,8}$, let G be a 7-space, and let ξ be a symp in Δ. Then precisely one of the following occurs.

(i) $G \cap \xi$ is a unique $6'$-space.

(ii) $G \cap \xi$ is a unique 3-space.

(iii) $G \cap \xi$ is a unique line.

(iv) $G \cap \xi$ is a unique point.

(v) $G \cap \xi = \varnothing$.

Fact 4.7.0.9. Let Δ be a parapolar space of type $\mathsf{E}_{8,8}$. Let p be a point, π a plane, and $L \subseteq \pi$ a line. Suppose p is collinear to some point $q \in \pi \setminus L$ and special to at least two points of L. Then it is symplectic to precisely one point of L.

The following properties can be found in [**CI07**].

Lemma 4.7.0.10. *Let Δ be a parapolar space of type $\mathsf{E}_{8,8}$. Let p, q be two points and ξ a symp.*

(i) The points p and q are opposite if and only if there exist points x, y with $p \perp x \perp y \perp q$ such that $\{p, y\}$ and $\{q, x\}$ are special.

(ii) Each line contains at least one point not opposite p. If it contains at least two such points, then no point on the line is opposite p.

(iii) If ξ contains a point opposite p, then ξ contains a unique point x symplectic with p; all points of $\xi \setminus \{x\}$ collinear to x are special to p, and all other points of ξ are opposite p.

A point and a symp containing an opposite point will be called *far from each other*.

4.8. Parapolar Spaces of Type $\mathsf{F}_{4,1}$ & $\mathsf{F}_{4,4}$

When Jacques Tits classified all thick irreducible spherical buildings of rank at least 3, he proved that almost all such buildings correspond to simple algebraic groups defined over arbitrary fields — as we have talked about in Chapter 2 and Appendix C. We have to say 'almost' because: (1) There are classical groups that are not algebraic groups, but still give rise to buildings, and (2) for each pair of fields $(\mathbb{K}, \mathbb{K}')$ of characteristic 2, with $\mathbb{K}^2 \subseteq \mathbb{K}' \subseteq \mathbb{K}$ (where $\mathbb{K}^2$ denotes the field of all squares of $\mathbb{K}$), there is a family of buildings of type F_4, associated to so-called *mixed groups* (cf. [**AB08**, §9.1]). Hence, we sometimes differentiate between *classical buildings*, *algebraic buildings*, and *mixed buildings*, which are associated to classical groups, algebraic groups, and mixed groups, respectively. Since many algebraic groups are classical, the first two classes of buildings have a big overlap.[8]

What is important for us right now is that buildings of type F_4 are a very special case. They come in 5 different flavors, which are completely determined by pairs $(\mathbb{K}, \mathbb{A})$, where $\mathbb{K}$ is a field and $\mathbb{A}$ is a quadratic alternative division algebra over $\mathbb{K}$:

(K) $\mathbb{K} = \mathbb{A}$ and $\mathsf{char}\,\mathbb{K} \neq 2$,

(L) $\mathbb{A}$ is a *separable quadratic extension* of $\mathbb{K}$,

(H) $\mathbb{A}$ is a *quaternion division algebra* over $\mathbb{K}$,

(O) $\mathbb{A}$ is a *Cayley division algebra*, or also called *octonion division algebra*, over $\mathbb{K}$,

(M) $\mathsf{char}\,\mathbb{K} = 2$ and $\mathbb{A}$ is an extension of $\mathbb{K}$, such that $\mathbb{A}^2 \subseteq \mathbb{K}$.

As a reference, we point to [**Tit74**, §10.2].

[8] Another way to divide buildings into three different classes would be to consider the buildings associated to classical groups as before, but then gather all buildings associated to non-classical algebraic groups in the second class, and the buildings associated to mixed groups in the last one. Some authors then call the buildings of the last two classes *exceptional buildings*. However, we will not do that, because in this thesis, we generally focus more on the associated Coxeter group of a building and usually use the adjective 'exceptional' in that context. There are buildings associated to non-classical algebraic groups that are not of exceptional type. Again, we refer to [**AB08**, §9.1].

Note 4.8.0.1. We will give a little bit more context now that is not important for the rest of this thesis, but could still be interesting. The buildings in cases (K), (L), (H) and (O) are algebraic. The buildings in case (M) are algebraic if and only if either $\mathbb{K} = \mathsf{A}$ or $\mathbb{K} = \mathsf{A}^2$; otherwise they are mixed. Case (K) corresponds to the Chevalley group of type F_4 over $\mathbb{A}$, and case (L) corresponds to the *twisted Chevalley group* of type ${}^2\mathsf{E}_6$. In cases (H), (O) and (M), the associated buildings are never finite except if $\mathbb{K} = \mathsf{A}$ in case (M) (cf. [**AB08**, §9.7] and [**Tit74**, §10.4]).

Notation & Definition 4.8.0.2. For this entire section, $\mathbb{K}$ always denotes a field. In the following, we use the notation that we introduced in Table 4 for the associated quadratic alternative division algebras over $\mathbb{K}$. We will speak of the *separable case* if we are given a quadratic alternative division algebra associated to one of the classes (K), (L), (H), or (O), and we will speak of the *inseparable case* if we are given a quadratic alternative division algebra associated to the class (M).

	symbol	$\dim_{\mathbb{K}}(\mathbb{A})$	$\mathsf{char}(\mathbb{K})$	class	properties
separable	$\mathbb{K}$	1	$\neq 2$	(K)	commutative, associative
	$\mathbb{L}$	2		(L)	commutative, associative
	$\mathbb{H}$	4		(H)	non-commutative, associative
	$\mathbb{O}$	8		(O)	non-commutative, non-associative, alternative
inseparable	$\mathbb{K}'$	$2^h, \infty$	$= 2$	(M)	commutative, associative

TABLE 4. Quadratic alternative division algebras over $\mathbb{K}$.

The last case in the table is exactly the inseparable case. This class includes the case where $\mathbb{K} = \mathbb{A}$ and $\mathsf{char}\,\mathbb{K} = 2$. We refer to the case $\mathbb{K} = \mathbb{A}$ in either characteristic as the *split case*. The other cases are called *non-split*.

Recall that a projective plane has a diagram of type A_2; i.e., a diagram with two vertices that are connected by a single edge. If we see the diagram of a projective plane with $\mathbb{K}$ written underneath both vertices, then it describes a projective plane which is coordinatized over the field $\mathbb{K}$. If we see the diagram of a projective plane with $\mathbb{A}$ written underneath both vertices, then it describes a projective plane which is coordinatized over the quadratic alternative division algebra $\mathbb{A}$.

A projective plane over $\mathbb{K}$.

A projective plane over $\mathbb{A}$.

Note that we can see exactly two subdiagrams of type A_2 in a diagram of type F_4. When we enrich the Coxeter diagram associated to F_4 with labels $\mathbb{K}$ and $\mathbb{A}$, it

becomes clear why we will have to differentiate between $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ and $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$ in the following.

$\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$				$\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$			
symps	planes	lines	points	points	lines	planes	symps
1	2	3	4	1	2	3	4
$\mathbb{K}$	$\mathbb{K}$	$\mathbb{A}$	$\mathbb{A}$	$\mathbb{K}$	$\mathbb{K}$	$\mathbb{A}$	$\mathbb{A}$

Definition 4.8.0.3 (**Cayley plane**)**.** The projective plane defined over an octonion division algebra is called a *Cayley plane*. Thus, it has the diagram $\overset{\mathbb{O}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\bullet\overset{\mathbb{O}}{}$.

With this background, we will now state some facts about parapolar spaces of type $\mathsf{F}_{4,1}$ and $\mathsf{F}_{4,4}$.

Facts 4.8.0.4. Let Δ be a parapolar space of type $\mathsf{F}_{4,1}$ or $\mathsf{F}_{4,4}$. Then the following assertions hold.

(i) Δ has rank 3, diameter 3, a trivial opposition map, and Δ is non-strong.

(ii) Symps and point residues in Δ are thick polar spaces of rank 3.

(iii) The maximal singular subspaces in Δ are projective planes.

(iv) (Symp-symp relations). The intersection of two symps is either empty, a point, or a plane.

Note 4.8.0.5. Parapolar spaces of type $\mathsf{F}_{4,1}$ are again *long root subgroup geometries*.

Fact 4.8.0.6 (Point-symp relations)**.** Let Δ be a parapolar space of type $\mathsf{F}_{4,1}$ or $\mathsf{F}_{4,4}$, let ξ be a symp, and let p be a point in Δ with $p \notin \xi$. Then precisely one of the following occurs.

(i) $p^\perp \cap \xi$ is a line L, p is symplectic to all points in $\xi \cap (L^\perp \setminus L)$, and p is special to all points in $\xi \setminus L^\perp$. We say that p and ξ are *close*.

(ii) $p^\perp \cap \xi = \varnothing$, but there is a unique point q of ξ symplectic to p. Then p is special to all points of $\xi \cap (q^\perp \setminus \{q\})$, and p is opposite to all points of $\xi \setminus q^\perp$. We say that p and ξ are *far*.

Fact 4.8.0.7 (Point-line relations)**.** Let Δ be a parapolar space of type $\mathsf{F}_{4,1}$ or $\mathsf{F}_{4,4}$, let L be a line, and let x be a point in Δ with $x \notin L$. Then precisely one of the following occurs.

(i) $x \perp L$.

(ii) $x \perp p \in L$ for exactly one point p, and $x \perp\!\!\!\perp q$ for all $q \in L \setminus \{p\}$.

(iii) $x \perp p \in L$ for exactly one point p, and $x \bowtie q$ for all $q \in L \setminus \{p\}$, with $[x, q] = p$.

(iv) $x \perp\!\!\!\perp p \in L$ for exactly one point p, and $x \bowtie q$ for all $q \in L \setminus \{p\}$, with $[x, q] = a$, where a is independent of q and collinear to all points of L.

(v) $x \bowtie L$, and the set $\{[x, p] \mid p \in L\}$ forms a line.

(vi) $x \bowtie p \in L$ for exactly one point p, and x is opposite q for all $q \in L \setminus \{p\}$.

Fact 4.8.0.8. Let Δ be a parapolar space of type $\mathsf{F}_{4,1}$ or $\mathsf{F}_{4,4}$. Let ξ be a symp of Δ, and let p, q be two points close to ξ. Then p and q are opposite if and only if the lines $p^\perp \cap \xi$ and $q^\perp \cap \xi$ are opposite in ξ.

4.9. Summary & Overview

In this section, we will summarize the most basic facts about some parapolar spaces that will be relevant to us in table form, and list some references for further reading.

Basic Properties

Fact 4.9.0.1. Let Δ be a Lie incidence geometry of type Π. The type of the residue of an element of type i can be seen in the diagram by neglecting every vertex that is strictly on the left-hand side of the vertex of type i and the vertex of type i itself, and considering the remaining diagram.

type of Δ	diameter	strength	max. singular subspaces	symps	point residues
$\mathsf{E}_{6,1}(\mathbb{K})$	2	strong	4 & 5	$\mathsf{D}_{5,1}(\mathbb{K})$	$\mathsf{D}_{5,5}(\mathbb{K})$
$\mathsf{E}_{7,1}(\mathbb{K})$	3	non-strong	4 & 6	$\mathsf{D}_{5,1}(\mathbb{K})$	$\mathsf{D}_{6,6}(\mathbb{K})$
$\mathsf{E}_{7,7}(\mathbb{K})$	3	strong	5 & 6	$\mathsf{D}_{6,1}(\mathbb{K})$	$\mathsf{E}_{6,1}(\mathbb{K})$
$\mathsf{E}_{8,8}(\mathbb{K})$	3	non-strong	6 & 7	$\mathsf{D}_{7,1}(\mathbb{K})$	$\mathsf{E}_{7,7}(\mathbb{K})$
$\mathsf{D}_{5,5}(\mathbb{K})$	2	strong	3 & 4	$\mathsf{D}_{4,1}(\mathbb{K})$	$\mathsf{A}_{4,3}(\mathbb{K})$
$\mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$, $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$	3	non-strong	2	$\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$, $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$	$\mathsf{C}_{3,3}(\mathbb{A},\mathbb{K})$, $\mathsf{B}_{3,3}(\mathbb{K},\mathbb{A})$

Notation 4.9.0.2. In the following, p is an arbitrary point in Δ, and we use the following notation.

symbol	meaning	abbreviation
$\in$	contained in	cont.
$\perp$	collinear	coll.
$\perp\!\!\!\perp$	symplectic	sympl.
$\bowtie$	special	spec.
$\mathsf{dist}(3)$	distance 3	–
$\equiv$	opposite	opp.
–	intersect	inters.
–	disjoint	disj.

Some Relations

$\mathsf{typ}(\Delta)$	point-point	point-symp	symp-symp
$\mathsf{E}_{6,1}(\mathbb{K})$	$\perp$ or $\perp\!\!\!\perp$	contained, coll. to 4-space, or opposite	intersect in point or 4-space
$\mathsf{E}_{7,1}(\mathbb{K})$	$\perp$, $\perp\!\!\!\perp$, $\bowtie$ or $\mathsf{dist}(3)$	contained, p coll. to $4'$-space, or p sympl. to entire symp, 4-space or point	disjoint, or intersect in point, plane or 4-space
$\mathsf{E}_{7,7}(\mathbb{K})$	$\perp$, $\perp\!\!\!\perp$ or $\mathsf{dist}(3)$	contained, or coll. to point or max. 5-space	disjoint, or intersect in line or max. 5-space
$\mathsf{E}_{8,8}(\mathbb{K})$	$\perp$, $\perp\!\!\!\perp$, $\bowtie$ or $\mathsf{dist}(3)$	contained, p coll. to $6'$-space or line, or p either sympl. to 6-space or point.	disjoint (opp., point-disj., symp-disj., line-disj. or max-disj.), or inters. in point, plane or 6-space
$\mathsf{D}_{5,5}(\mathbb{K})$	$\perp$ or $\perp\!\!\!\perp$	contained, or p coll. to 3-space	disj. or inters. in 3-space
$\mathsf{F}_{4,i}(\mathbb{K},\mathbb{A})$	$\perp$, $\perp\!\!\!\perp$, $\bowtie$ or $\mathsf{dist}(3)$	contained, coll. to line, or sympl. to point	disj. or inters. in point or plane

In the following, let $p \in \Delta$ be an arbitrary point, ξ a symp, and let A and B be two maximal singular subspaces not contained in a symp. As we did before, we abbreviate the latter by "maxhigh".

type of Δ	maxhigh-maxhigh	point-maxhigh	symp-maxhigh
$\mathsf{F}_{4,i}(\mathbb{K},\mathbb{A})$	$A \cap B$ is either empty, a point, or a line	$p \in A$, or $p^\perp \cap A$ is empty, a point, or a line	$A \subseteq \xi$, or $A \cap \xi$ is either empty, a point, or a line
$\mathsf{D}_{5,5}(\mathbb{K})$	$A \cap B$ is either empty, or a line	$p \in A$, or $p^\perp \cap A$ is empty, or a plane	$A \cap \xi$ is either a point, or a 3-space
$\mathsf{E}_{6,1}(\mathbb{K})$	$A \cap B$ is either empty, a point, or a plane.	$p \in A$, or $p^\perp \cap A$ is a point, or a 3-space	$A \cap \xi$ is either empty, a line, or a $4'$-space
$\mathsf{E}_{7,1}(\mathbb{K})$	$A \cap B$ is either empty, a point, or a plane	$p \in A$, or $p^\perp \cap A$ is empty, a point, or a 3-space	$A \cap \xi$ is either empty, a line, or a $4'$-space
$\mathsf{E}_{7,7}(\mathbb{K})$	$A \cap B$ is either empty, a point, a line, or a 3-space	$p \in A$, or $p^\perp \cap A$ is empty, a line, or a 4-space	$A \cap \xi$ is either empty, a point, a plane, or a $5'$-space
$\mathsf{E}_{8,8}(\mathbb{K})$	$A \cap B$ is either empty, a point, a line, a plane, or a 4-space	$p \in A$, or $p^\perp \cap A$ is empty, a point, a plane, or a 5-space	$A \cap \xi$ is either empty, a point, a line, a 3-space, or a $6'$-space

For further reading about properties of parapolar spaces corresponding to thick spherical buildings, we refer to [**BV22**], [**DTV25**], [**JSV25**], [**LV25**], [**SV**], [**DSV22**], and [**Tit57**].

To end this chapter, we will introduce a bit more notation.

Notation 4.9.0.3. Let Δ be a projective space, a polar space, or a parapolar space.

- We call points and/or lines in Δ *coplanar* if there exists a projective plane in Δ that contains them all.
- Let y be a point in Δ. Suppose points are opposite points. Then we denote by $y^{\not\perp}$ the set of all points that are not opposite y.

4.10. Lie Incidence Geometries of Type H_3

As mentioned, one goal of this thesis is to present a new elementary proof for the fact that thick buildings of types H_3 and H_4 do not exist. For that, we want to work in the associated Lie incidence geometries as well. The non-existence of thick buildings of type H_4 will follow directly from the non-existence of thick buildings of type H_3. The goals of this section are

- to introduce *icosahedral geometries* as point-line geometries satisfying certain axioms (cf. Definition 4.10.1.1),
- to show that their maximal singular subspaces are projective planes in the case that every line has at least three points (cf. Corollary 4.10.1.8),
- to define their apartments (as icosahedra, cf. Definition 4.10.2.15),
- to prove that any two chambers of an icosahedral geometry lie in a common apartment (cf. Proposition 4.10.2.24),
- and to observe that Lie incidence geometries of type H_3 are icosahedral geometries (cf. Section 4.10.3).

This gives us an elementary axiom system to work with, which might make our proofs and results more accessible and potentially more broadly applicable.

4.10.1. Icosahedral Geometries

We denote by $p^{\perp\!\!\!\perp}$ the set of all points of Δ that are at distance 2 from p.

Definition 4.10.1.1 (**Icosahedral geometries**)**.** An *icosahedral geometry* Δ is a point-line geometry such that the following axioms are satisfied.

(I1) Δ has diameter 3, and two distinct points p and q can have one of the following relations.

(i) p and q are collinear, and there is a unique line containing p and q, which is denoted by pq,
(ii) p and q have distance 2, and $p^\perp \cap q^\perp$ is a line,
(iii) p and q are opposite, and $p^\perp \cap q^{\perp\perp}$ and $p^{\perp\perp} \cap q^\perp$ are generalized pentagons.

(I2) For every point p, there exists a point q which is opposite p.

(I3) For a point p and a line L not containing p, we have that
(i) if p is collinear with 2 points of L, then p is collinear with all points of L,
(ii) if p has distance 2 to 2 points of L, then p has distance at most 2 to all points of L,
(iii) if p has distance 2 to all points of L, then there exists a unique point q such that $p \perp q \perp L$,
(iv) if p is opposite some point of L, then there exists exactly one point on L at distance 2 from p.

Our next goal is to show that the planes of icosahedral geometries in which every line contains at least three points are projective planes. In the following, Δ will always denote an icosahedral geometry. At first, we only assume that every line in Δ contains at least two points.

Lemma 4.10.1.2. *For every line L, there exist two points p and q such that*

$$L = p^\perp \cap q^\perp.$$

PROOF. Let L be a line, let y be some point on L, and let z be a point at distance 3 from y. Then $z^{\perp\perp} \cap y^\perp$ is a generalized pentagon. Since z cannot have distance 3 to all points of L and $L \subset y^\perp$, there exists some point a on L contained in $z^{\perp\perp} \cap y^\perp$. Let M and N be two lines in $z^{\perp\perp} \cap y^\perp$ through a. Then there exist two points m and n on M and N, respectively, which are not collinear. Since m and n are both collinear to y and a, they are collinear to all of L, and thus $L = m^\perp \cap n^\perp$. □

Lemma 4.10.1.3. *For every pair of different points $a, b \in \mathscr{P}$, there exists a point $p \in \mathscr{P}$ that is collinear to a but not to b.*

PROOF. **Case 1:** $\mathsf{dist}(a, b) = 3$. Take any point in $a^\perp \cap b^{\perp\perp}$.

Case 2: $\mathsf{dist}(a, b) = 2$. Let L be the line $a^\perp \cap b^\perp$. Let x be a point at distance 3 from a. Then we can take any point in the generalized pentagon $a^\perp \cap x^{\perp\perp}$, except the points of L, if $L \subset a^\perp \cap x^{\perp\perp}$.

Case 3: $a \perp b$. Let x be a point at distance 3 from a. Then there exists some point c on ab at distance 2 from x. Consider the generalized pentagon $x^{\perp\perp} \cap a^\perp$. If there did not exist a point not collinear to b in $x^{\perp\perp} \cap a^\perp$, then every point in $x^{\perp\perp} \cap a^\perp$ would be collinear to the whole line ac. This contradicts the axioms of a generalized pentagon. □

Lemma 4.10.1.4. *Let p and q be two opposite points. Let a and b be two collinear points in $\Gamma := p^\perp \cap q^{\perp\perp}$. Let x be a point on pa, and let y be a point on pb such that $x \neq y$, and $xy \neq ab$. Then ab and xy intersect in a unique point.*

PROOF. If xy conincides with ap or pb, then $ab \cap xy$ is either the unique point a or the unique point b. In the following we may assume that x is dinstinct from both p and a, and y is distinct from both p and b.

The point q has distance 2 to all points of ab. Thus, there exists a unique point c such that $q \perp c \perp ab$. Suppose the point c had distance 2 to all points of xy. Then there would have to exist a unique point v such that $c \perp v \perp xy$. But every point on ab is collinear to both c and every point of xy. It follows that the assumption was wrong, and c has to be collinear to a point of xy. But since $p^\perp \cap c^\perp$ is exactly ab, this point has to be on ab. Thus, xy and ab intersect in a point, which is unique because of axiom (I1)(i). □

Lemma 4.10.1.5. *Let p and q be two opposite points. Let a and b be two collinear points in $p^\perp \cap q^{\perp\perp}$. Let x be a point on pa distinct from p, and let y be a point on pb distinct from p. Then there exists a point q' opposite p such that q' has distance 2 to every point on xy.*

PROOF. **(i)** If every line in Δ contains only two points, then $x = a$ and $y = b$, and we are done.

(ii) Now suppose every line in Δ has at least two points and $x = a$, but $y \neq b$. Then q has distance 3 to y, and we put

$$\Gamma_{q,y} := q^\perp \cap y^{\perp\perp}, \quad \text{and} \quad \Gamma_{q,p} := q^\perp \cap p^{\perp\perp}.$$

Let c be the unique point collinear to q and ab. Let d be a point on $a^\perp \cap q^\perp$ distinct from c. Then d is contained in $\Gamma_{q,y} \cup \Gamma_{q,p}$. Let K be a line through d in $\Gamma_{q,y}$ not equal to cd, and let q' be a point on K not equal to d. Then q' has distance 2 to a and y. Suppose q' were not opposite p. Then q' would have distance at most two to all points on pb.

a) Suppose q' were collinear to some point q'' on pb. Since $q \perp q'$, we would have that $q'' \in q^{\perp\perp} \cap p^\perp$. But if $q'' \in q^{\perp\perp} \cap p^\perp$ and $q'' \in pb$, then $q'' = b$. It follows that $q' \in b^\perp \cap q^\perp$. That means the line K through d intersects $b^\perp \cap q^\perp$ in q'.

Now either $q' = c$, but then $K = cd$, which contradicts the fact that K was chosen different from cd, or $q' \neq c$. But if $q' \neq c$, then, d is collinear to c and q' and thus to all points of $b^\perp \cap q^\perp$. Since all of the points d, c, and q' are contained in $\Gamma_{q,p}$, this yields a contradiction to the fact that $\Gamma_{q,p}$ is a generalized pentagon. It follows that q' is opposite p.

b) Suppose q' had distance 2 to all points of pb. Then b would have distance 2 to all points of $q'd$, and there would have to exist a unique point e collinear to b and $q'd$. We have that

$$b^\perp \cap d^\perp = ca.$$

Thus, e has to be on ca. If $e \neq c$, then e and q have distance 2 and

$$e^\perp \cap q^\perp = cd$$

would contain q'. But q' can not be on cd since $K \neq cd$. If $e = c$, then q' is collinear to c and d and there would be a triangle in $\Gamma_{q,p}$, contradicting the fact that it is a generalized pentagon.

Thus, q' is a point opposite p and at distance 2 to all points of xy (cf. Fig. 1).

(iii) Now suppose every line in Δ has at least two points and $x \neq a$ and $y \neq b$. According to Lemma 4.10.1.4, ab and xy intersect. Let z be their intersection point. Then we can apply the steps of (ii) again, interchanging a and z. □

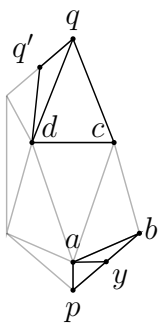


FIG. 1

Lemma 4.10.1.6. *Let L be a line in Δ. Then there exist two opposite points p and b such that $L \subset p^\perp \cap b^{\perp\!\perp}$. In particular, every line L in Δ is contained in a generalized pentagon.*

PROOF. Let L be a line in Δ. According to Lemma 4.10.1.2, there exist two points a and b such that $a^\perp \cap b^\perp = L$. Let p be a point opposite b. Then for every point ℓ_i on L, p has distance 2 to a unique point p_i on $b\ell_i$. If these points p_i are not the points ℓ_i on L, then, according to Lemma 4.10.1.5, we can find a point p' such that $p'^{\perp\!\perp} \cap b^\perp$ contains L. □

Lemma 4.10.1.7. *Let b, c, d, e be four different points that are pairwise collinear, but no three of which are on a common line. If $bc \cap de$ is a unique point a, then $bd \cap ce$ is also a unique point.*

PROOF. Let p be a point opposite a. Then p has distance 2 to a unique point p_1 on bc and to a unique point p_2 on de. Since b is collinear to a and e, b is collinear to p_2. Then p_2 is collinear to b and a and thus to p_1. With that, p has distance 2 to all points of the line p_1p_2. That means there exists a unique point q such that $p \perp q \perp p_1p_2$. Since p and a have distance 3, q and a have distance 2, and $q^\perp \cap a^\perp$ is exactly p_1p_2.

Suppose $p_1 = b$ and $p_2 = d$, and suppose $bd \cap ce = \varnothing$. Then q had distance 2 to all points of ce, and there has to exist a unique point y such that $q \perp y \perp ce$. But that contradicts the fact that any point on bd is collinear to q and any point of ce. Thus, q has to be collinear to a point of ce. But since $q^\perp \cap a^\perp$ is exactly the line bd, this point has to be on ce, and it follows that bd and ce intersect in a unique point.

The same argument goes through if $p_1 = c$ and $p_2 = e$.

Now suppose $p_1 \neq b, c$ and $p_2 \neq d, e$.

According to Lemma 4.10.1.5, we can find a point p' opposite a such that p' has distance 2 to all points of bd. That means bd is a line of the generalized pentagon $a^\perp \cap p'^{\perp\!\perp}$. Since c is a point of ba, e is a point of da, and $c \perp e$, the line ce has to intersect bd in a unique point according to Lemma 4.10.1.4. □

Corollary 4.10.1.8. *If every line in Δ contains at least three points, then the maximal singular subspaces of Δ are projective planes.*

Observation 4.10.1.9. Let Δ be an icosahedral geometry in which every line contains at least three points. Let p be a point, let q be a point opposite p, and let π be a projective plane through q. Since p has distance 2 to at least one point on every line through q, $p^{\perp\perp} \cap \pi$ is a line.

Definition 4.10.1.10. Let Δ be an icosahedral geometry in which every line contains at least three points. Let p be a point in Δ. Then we can define a new point-line geometry $\mathsf{Res}_\Delta(p)$ as follows.

(i) The set of points of $\mathsf{Res}_\Delta(p)$ is given by all lines through p.

(ii) The set of lines of $\mathsf{Res}_\Delta(p)$ is given by all projective planes in Δ.

(iii) Two points of $\mathsf{Res}_\Delta(p)$ lie on a common line if the corresponding lines in Δ lie in a common projective plane.

We call $\mathsf{Res}_\Delta(p)$ the *point residue of p in Δ*.

Observation 4.10.1.11. Let Δ be an icosahedral geometry in which every line contains at least three points. For a point p and a point q opposite p in Δ, we have

$$\mathsf{Res}(p) \cong p^\perp \cap q^{\perp\perp}.$$

This is the case since q has distance 2 to a unique point of every line through p. In particular, this means that all point residues are generalized pentagons.

Definition 4.10.1.12. Let Δ be an icosahedral geometry. Then we call Δ

(i) *thick* if every line is contained in at least 3 planes.

(ii) *thin* if every line is contained in exactly 2 planes.

Lemma 4.10.1.13. *If Δ is thick, then every line has at least 3 points, and every point residue is a thick generalized pentagon.*

PROOF. Suppose Δ is thick. Let L be a line in Δ, and let a be a point on L. Let p be a point opposite a and at distance 2 to some point b on L.

Since Δ is thick, there exist at least three (non-projective) planes through L. Let K_1, K_2, and K_3 be three lines through a not equal to L. The point p must have distance 2 to b and to a point c_i on each of the lines K_i, for $i \in \{1, 2, 3\}$.

Furthermore, $a^\perp \cap p^{\perp\perp}$ is a generalized pentagon according to (I1)(iii).

In $a^\perp \cap p^{\perp\perp}$, b is contained in the 3 lines bc_1, bc_2, and bc_3. Now let x be any other point in $a^\perp \cap p^{\perp\perp}$. Since Δ is thick, the line ax is contained in three planes, and again, p has to have distance 2 to a unique point of any of the lines through a in these planes. It follows that x lies on three edges in $a^\perp \cap p^{\perp\perp}$, and, since x was arbitrary, $a^\perp \cap p^{\perp\perp}$ is a thick generalized pentagon.

Hence, every line in $a^\perp \cap p^{\perp\perp}$ contains at least 3 points. □

4.10.2. Apartments

Let Δ be a thick icosahedral geometry.

Definition 4.10.2.14. Let α be a projective plane in Δ. Let P be a set of three points of α not contained in a common line. Let

$$L := \{(p_i, p_j) \mid p_i, p_j \in P,\ i \neq j\}.$$

We define a *triangle of* α as the point-line geometry given by (P, L). Note that any such set P generates α in Δ.

Definition 4.10.2.15 (**Apartments**)**.** Let Δ be an icosahedral geometry, and let p and q be two opposite points. Let Γ_q be an apartment of $q^\perp \cap p^{\perp\perp}$. Then, by projecting every point and every line of Γ_q to the point p, we obtain an apartment Γ_p of $p^\perp \cap q^\perp$. We call the subgeometry consisting of

(i) the points and lines of Γ_p and Γ_q,
(ii) the points p and q,
(iii) the lines between p and points of Γ_p,
(iv) the lines between q and points of Γ_q,
(v) the lines between points of Γ_p and Γ_q,
(vi) and the triangles appearing naturally,

an *apartment of* Δ.

Observation 4.10.2.16. The apartments of an icosahedral geometry are icosahedra. Every apartment is a thin icosahedral geometry with exactly two points on each line.

Observation 4.10.2.17. Let α be a plane in Δ. Then there exist two opposite points p and b such that α contains p and intersects $p^\perp \cap b^{\perp\perp}$ in a line. With Lemma 4.10.1.5, it follows that any triangle of any plane in Δ is contained in an apartment.

Notation 4.10.2.18. By abuse of language, we will also say that *the plane α is contained in an apartment* Σ if some triangle of α is contained in Σ. Similarly, we will say that a line is contained in Σ if two distinct points of L are contained in Σ.

Lemma 4.10.2.19. *Given a point p and a line L contained in $p^\perp$, we can always find an apartment containing p and L.*

Proof. If L does not contain p, we can use Lemma 4.10.1.6 to find a point q opposite p such that $L \subset p^\perp \cap q^{\perp\perp}$. Then we choose an apartment Γ_p of $p^\perp \cap q^{\perp\perp}$ containing L and proceed as in Definition 4.10.2.15.

If L contains p, then let q be any point opposite p. Let a be the unique point of L at distance 2 to q. Then we choose an apartment Γ_p of $p^\perp \cap q^{\perp\perp}$ containing a and proceed as in Definition 4.10.2.15. □

Definition 4.10.2.20. We say two planes α and β in Δ with $\alpha \cap \beta = \varnothing$ are

(i) *opposite* if every point of α has distance 2 to a unique line of β, and distance 3 to all other points of β,

(ii) *far* if there exists a unique point a of α collinear with a unique point b of β, all other points of α have distance 2 to a unique line of β through b, and are opposite all other points of β,

(iii) *close* if there exists a unique point a in α collinear with a unique line L of β, and there exists a unique line K of α through a collinear with a unique point b of L in β. In that case, the points in α that are not on K have distance 2 to the points of L and distance 3 to the points in $\beta \setminus L$.

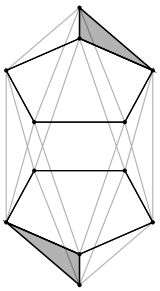


FIG. 2. Opposite planes

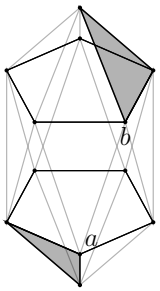


FIG. 3. Far planes

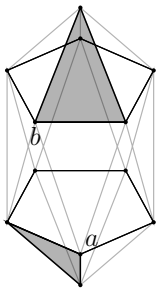


FIG. 4. Close planes

Lemma 4.10.2.21. *If two planes α and β in Δ do not intersect, then they are either opposite, far, or close.*

PROOF. Since planes are maximal singular subspaces in Δ, no point of $\alpha \cup \beta$ is collinear to all of $\alpha \cup \beta$. Thus, if a is a point of α, then $a^\perp \cap \beta$ can at most be a line.

Close planes: Suppose there exists a point $a \in \alpha$ such that $a^\perp \cap \beta$ is a line L. Then every point of $\alpha \setminus \{a\}$ has distance at most 2 to all points of L. Let c be a point in $\beta \setminus L$. Then a and c have distance 2 and $a^\perp \cap c^\perp = L$. The point c cannot be collinear to any point x of α, since otherwise x would have to be contained in L, forcing α and β to intersect. The point c can also not be opposite to all points of $\alpha \setminus \{a\}$, since otherwise there would exist a line $J \subset \alpha$ not containing a such that all points of J have distance 3 to c. Thus, c has distance 2 to at least a line K of α through a.

Suppose c had distance 2 to all points of $\alpha \setminus \{a\}$.

Let $x \in \alpha$ be a point distinct from a, and let ℓ be a point on L. Then $x^\perp \cap c^\perp$ and $x^\perp \cap \ell^\perp$ are lines, intersecting in a unique point z such that $x \perp z \perp c\ell$. Since $x \perp a \perp \ell$, the line $x^\perp \cap \ell^\perp$ contains a and z, and thus $a \perp z$. Then $z \in a^\perp \cap c^\perp$, and thus $z \in L$.

Now let y be a point in α not on the line ax. Then we can use the same arguments as before to see that y must be collinear to a point z' of L. In $\mathsf{Res}_\Delta(a)$ we can see the lines yx, xz, yz', and zz', and they would form a quadrangle, unless $x \perp z'$ or $y \perp z$ or $z = z'$ – but in these cases $\mathsf{Res}_\Delta(a)$ would contain triangles, which is even worse.

It follows that c has distance 2 to all points of a unique line K of α through a, and distance 3 to all other points of α. Let b be the unique point such that $c \perp b \perp K$. Since $a^\perp \cap c^\perp = L$, b has to be contained in L.

Now we have $d \perp b \perp K$, and it follows that $d^{\perp\!\perp} \cap \alpha = L$, for all $d \in \beta \setminus L$. One shows analogously that every point e of $\alpha \setminus K$ has distance 2 to K, and distance 3 to all other points of β.

Suppose another point $e \in K$ were collinear to a point l of L. Then e, a, and l would form a triangle in $\mathsf{Res}_\Delta(b)$, contradicting the fact that all point residues are generalized pentagons. Analogously, no point of K except b is collinear to a point of L. Thus α and β are close.

Far planes: Suppose no point of α is collinear to a line of β, but there exists a point $a \in \alpha$ such that $a^\perp \cap \beta$ is a point b.

Suppose there existed another point a' collinear to a point b' of β. If $b = b'$, then $b \perp aa'$ and we are in the case of close planes with the roles of α and β switched. So suppose $b \neq b'$. Then $b' \in a^{\perp\!\perp} \cap \beta$, and $b'^\perp \cap a^\perp$ contains a' and b, forcing a' and b to be collinear. But then $a' \perp bb'$, contradicting the fact that no point of α is collinear to a line of β. It follows that all points of $\alpha \setminus \{a\}$ have distance at least 2 to all points of β.

Let $x \in \alpha$ be a point distinct from a and suppose x had distance 2 to all points of β. Let K be a line in β not containing b. Then there exists a unique point z such that $b \perp z \perp K$. Let c and d be two distinct points on K. Then there also has to exist a unique point z' collinear to x and bc, and a unique point z'' collinear to x and bd. Then

$$x^\perp \cap c^\perp = zz', \quad x^\perp \cap b^\perp = z'z'', \quad x^\perp \cap d^\perp = zz''.$$

If z, z', and z'' were all distinct, they would form a triangle in $\mathsf{Res}_\Delta(x)$. If at least two of them coincided, say $z = z'$, then z would be collinear to the entire plane β, contradicting the fact that planes are maximal singular subspaces.

Opposite planes: Suppose no point of α is collinear to any point of β. No point a of $\alpha \cup \beta$ can be opposite all points of either α or β, or have distance 2 to only a unique point of either α or β, since otherwise there would exist a line all of whose points at distance 3 to a. □

Corollary 4.10.2.22. *Let α be a plane, and p a point in Δ. Then p cannot have distance 2 to all points of α.*

Definition 4.10.2.23. A *chamber* in Δ is a triple (a, L, α), consisting of a point a, a line L, and a plane α such that $a \in L \subseteq \alpha$.

Proposition 4.10.2.24. *Any two chambers of Δ are contained in a common apartment.*

Proof. Let α and β be two planes. Let L be a line in α, and a a point on L. Let K be a line in β, and b a point on K.

Planes intersecting in a line: Suppose α and β intersect in a line M. If $K = L = M$, we can find a point p opposite a and an apartment Σ through a and p containing $K = L$, α, and β (according to Lemma 4.10.2.19). If b is not contained in Σ, we can apply Lemma 4.10.1.5 to find another point p' opposite a and at distance 2 to b and an apartment Σ' through a and p' containing b, $L = K$, α and β (again, according to Lemma 4.10.2.19).

In the following we use Lemma 4.10.2.19 without mention.

Suppose $L \neq M$. Let m be a point on M and L. Then there exists a point n opposite m having distance 2 to all points of a unique line of α, and to all points of a unique line of β. Now we can find an apartment Σ_0 containing m, n, $n^{\perp\perp} \cap \alpha$, and $n^{\perp\perp} \cap \beta$, and thus containing α and β as well.

Suppose L and K are not contained in Σ_0.

For convenience, we now label all points of Σ_0. Let p_0, p_1, p_2, p_3, and p_4 be the points of $m^\perp \cap n^{\perp\perp}$ such that p_4 and p_0 are in α, p_0 and p_1 are in β, and $p_4 \perp p_3 \perp p_2 \perp p_1$. Let b_0, b_1, b_2, b_3, and b_4 be the points of $n^\perp \cap m^{\perp\perp}$ such that $p_i \perp b_{i-1} b_i$ for $i \in \mathbb{Z}/5\mathbb{Z}$ (cf. Fig. 5).

Let p_4' be the intersection point of L and $p_0 p_4$. Then p_4' is opposite b_1, and we have $p_0, m \in p_4'^{\perp} \cap b_1^{\perp\perp}$, $p_1 \in p_4'^{\perp\perp} \cap b_1^{\perp}$. Thus, we can find an apartment Σ through p_4' and b_1 containing L, α, and β.

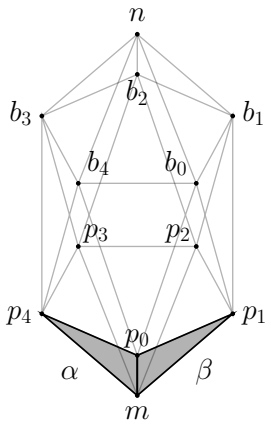


FIG. 5

Now suppose K is not contained in Σ. By abuse of notation, we label all the points of Σ as we did before for Σ_0 (cf. Fig. 5). Then L is the line mp_4.

Put $p_1' := K \cap p_0 p_1$. Then p_1' and b_3 are opposite, and we can find an apartment Σ' through p_1' and b_3 containing α, β, L, and K.

Again, by abuse of notation, we label all the points of Σ' as we did before for Σ_0 (cf. Fig. 5). Then L is either the line $p_0 p_4$ or the line mp_4, and K is either the line mp_1 or $p_0 p_1$.

We still have to show that we can find apartments containing L, K, α, and β that also contain the points a and b.

If $K = p_0 p_1$ and $b \neq p_0, p_1$, then we can do exactly the same as in (ii') and find an apartment that contains b. If $L = p_0 p_4$ and $a \neq p_0, p_4$, then — because of symmetry — we can also do exactly the same as in (ii') with roles switched.

Suppose $K = mp_1$ and $b \neq m, p_1$. Then b is opposite b_3, $b^\perp \cap b_3^{\perp\perp}$ contains p_4, and $b_3^\perp \cap b^{\perp\perp}$ contains p_0 and m, and thus we can find an apartment through b and b_3 containing b, L, K, α, and β. If $L = mp_4$ and $a \neq m, p_4$, we do exactly the same.

Planes intersecting in a point: Suppose α and β intersect in a point i. Let j be a point opposite i. Then j has distance 2 to a unique line of α and a unique line of β. As before, we can find an apartment Σ containing i, j, $j^{\perp\!\!\perp} \cap \alpha$, $j^{\perp\!\!\perp} \cap \beta$, and thus α and β.

Suppose L and K are not contained in Σ.

(i) *Suppose L is disjoint from i.* Then we can use Lemma 4.10.1.5 to change j to a point j' opposite i such that $j'^{\perp\!\!\perp} \cap \alpha = L$. Then we can find an apartment Σ' containing i, j', L, $j'^{\perp\!\!\perp} \cap \beta$, α, and β.

(ii) *Suppose L is a line through i.* Then we can find an apartment Σ' through i and j containing L, α and β.

So in both cases (i) and (ii) we found an apartment Σ' containing α, β, and L.

For convenience, we now label all points of Σ'. Let b_0, b_1, b_2, b_3, and b_4 be the points of $j'^{\perp\!\!\perp} \cap i^{\perp}$ such that b_0 and b_1 are in β, b_3 and b_4 are in α, $b_0 \perp b_4$, and $b_1 \perp b_2 \perp b_3$. Let p_0, p_1, p_2, p_3, and p_4 be the points of $j'^{\perp} \cap i^{\perp\!\!\perp}$ such that $p_i \perp b_{i-1}b_i$ for $i \in \mathbb{Z}/5\mathbb{Z}$ (cf. Fig. 6). Now we know that the line L is either equal to b_3b_4 or to ib_3, according to (i) and (ii).

Suppose Σ' does not contain K. Then we have to consider two cases again.

(i') *Suppose K is disjoint from i.* Let b'_0 be the intersection point of K with ib_0, and let b'_1 be the intersection point of K with ib_1 (cf. Fig. 7).

Note that b'_1 and p_4 are opposite. Hence, $b'^{\perp}_1 \cap p_4^{\perp\!\!\perp}$ is a generalized pentagon, and $b'^{\perp\!\!\perp}_1 \cap p_4^{\perp}$ is a generalized pentagon. The line b_3b_4 is contained in $b'^{\perp\!\!\perp}_1 \cap p_4^{\perp}$. The line ib'_0 is contained in $b'^{\perp}_1 \cap p_4^{\perp\!\!\perp}$. Hence, we can find an apartment Σ'' through p_4 and b'_1 containing i, b'_0, b_3, b_4, α, and β (cf. Fig. 8). In particular, since b'_0 and b'_1 are contained in Σ'', K is contained in Σ'', and since ib_3 and b_3b_4 are contained in Σ'', L is contained in Σ''.

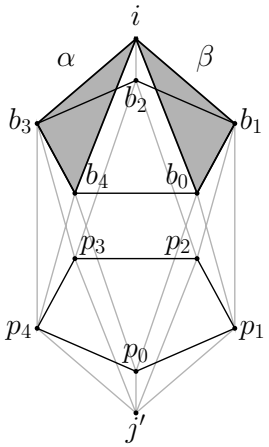


FIG. 6.
Σ' with all points labeled

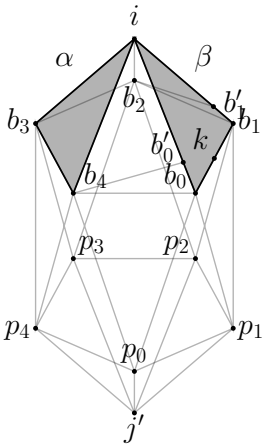


FIG. 7.
Σ' in Δ in case (i')

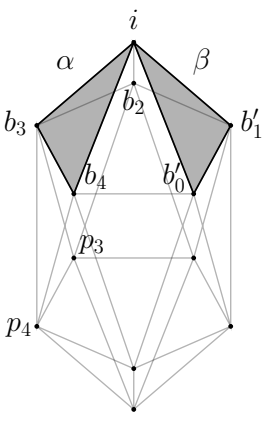


FIG. 8.
Apartment Σ'' of (i')

(ii') *Suppose K is a line through i.* Let b'_1 be the intersection point $K \cap b_0b_1$ (cf. Fig. 9). Then b'_1 is opposite both p_3 and p_4. Both $b'^{\perp}_1 \cap p_4^{\perp\!\!\perp}$ and $b'^{\perp\!\!\perp}_1 \cap p_4^{\perp}$ are generalized quadrangles. The points b_3 and b_4 are contained in $b'^{\perp\!\!\perp}_1 \cap p_4^{\perp}$, and b_0 and

i are contained in $b_1'^{\perp} \cap p_4^{\perp\!\!\perp}$. Hence, we can find an apartment Σ'' through p_4 and b_1' containing b_3, b_4, i, and b_0. Then Σ'' contains L, K, α, and β.

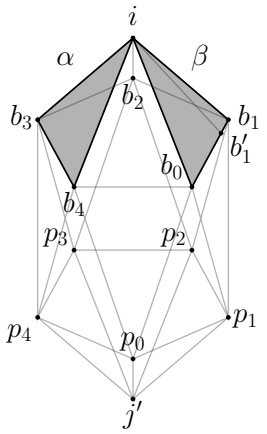


FIG. 9.
Σ' in Δ in case (ii')

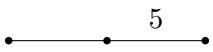
Now that we have found an apartment again containing L, K, α, and β, it follows from similar arguments as before that we can also include a and b.

Non-intersecting planes: Suppose α and β are either close, far, or opposite. Let $a \in \alpha$ and $b \in \beta$ be two opposite points. Then there exists an apartment through a and b containing α and β.

With similar arguments as above, one can adjust the apartment such that it contains L, K, a, and b. □

4.10.3. Lie Incidence Geometries of Type H_3 Are Icosahedral

With our observations about icosahedral geometries, it is now easy to see why Lie incidence geometries of type H_3 are icosahedral. Recall that the Coxeter diagram for H_3 is the following:



Since the proper subdiagrams of this diagram are of types A_2 and $\mathsf{I}_2(5)$, we deduce that a Lie incidence geometry Δ of type H_3 has projective planes and generalized pentagons as rank-2 residues and icosahedra as apartments (for details, see [**BC13**, §2.1, § 2.4]). Points can coincide or be at distances 1, 2, or 3. Consequently, one can infer from the diagram that Lie incidence geometries of type H_3 satisfy the axioms of icosahedral geometries (cf. Definition 4.10.1.1), and moreover that thick buildings of type H_3 correspond to thick icosahedral geometries.

5. Deeper Point-Line Affairs

After having deduced more specific properties from the diagrams of the Lie incidence geometries most relevant to us, we now move beyond consulting the diagrams and explore some deeper concepts defined for Lie incidence geometries.

First, we discuss the classification of polar spaces and deduce further properties for them, which we will mainly use in Sections 9.4, 10.7, and 10.8. After that, we introduce the notions of *hyperbolic lines*, *ideal subquadrangles*, *reguli*, *ovoids*, *spreads*, *tangents*, *secants*, *conics*, and *nuclei*, which we will need in Chapter 9 and Section 10.4.4.

We then turn to *long root* and *hexagonic geometries*. These are certain classes of Lie incidence geometries with shared properties. Introducing them is convenient for us, as we can use these notions to prove results for entire classes of geometries instead of only for one specific type. We will already establish some of their shared properties here and then mainly use them in Chapter 8, Sections 9.7 and 9.8.

Summarizing certain geometries into classes can be practical, but investigating certain subgeometries can also make proofs much easier, since some properties extend to ambient spaces, while in other cases proving properties of an ambient space requires properties of its residues. We will discuss geometries of type $\mathsf{E}_{6,1}(\mathbb{K})$ and $\mathsf{E}_{7,7}(\mathbb{K})$ in more detail, as they appear as subgeometries (e.g., point residues) in higher-rank exceptional geometries. We will use the observations made here in Chapter 8. Moreover, we will discuss *homologies* and *similitudes* of certain Lie incidence geometries and introduce some groups, including $\overline{\mathsf{PGO}}_{2r}(\mathbb{K})$, $\overline{\mathsf{P\Omega}}_{2r}(\mathbb{K})$, $\mathsf{PGE}_6(\mathbb{K})$, and $\mathsf{PGE}_7(\mathbb{K})$. We will need these in Section 10.6.

We then move on to *equator geometries*, another useful concept that allows us to see certain subgeometries in higher-rank geometries, and we will use them in Chapter 8, Sections 10.6.3 and 10.9, and also here when we subsequently prove some properties of Lie incidence geometries of type $\mathsf{E}_{8,8}(\mathbb{K})$, which we will need in Section 10.6.3.

Finally, we present models of apartments of some exceptional Lie incidence geometries — as promised in Section 3.7 — since they make it much easier to visualize the structures of the corresponding geometries. In particular, Section 10.9 is far easier to understand by considering an apartment.

Before proceeding, we record a few general facts that help to clarify the common background underlying some of the main topics of this chapter, and recall some basic definitions.

Fact 5.0.0.1. All projective spaces, polar spaces and strong parapolar spaces can be described as null sets of some form on some vector space.

Definition 5.0.0.2 (**Split building**)**.** A building is *split* if it arises from a linear algebraic group over a field, and if it has the same rank as the linear algebraic group

over the algebraic closure of that field. In that case, the linear algebraic groups are Chevalley groups (which are non-twisted).

Fact 5.0.0.3. Due to [**Bri85**] and [**Ram87**], the Grassmannians of the split spherical buildings can be described as the intersection of null sets of different quadratic forms on the same vector space.[1] In this case, none of the quadratic forms change over any extension of the base field. One can show that this implies that all of the quadratic forms are either hyperbolic or parabolic.

Example 5.0.0.4 (**Quadrics**)**.** Let V be a vector space of dimension $n+1 \geq 2$ over a field $\mathbb{K}$, and let $\mathsf{PG}(V)$ be the projective space on V.

Let κ be a non-trivial quadratic form on V, that is, a non-trivial map $\kappa\colon V \to \mathbb{K}$ satisfying

$$\kappa(v\lambda) = \lambda^2\kappa(v), \quad \text{for all } v \in V \text{ and all } \lambda \in \mathbb{K},$$

and such that the associated form

$$V \times V \to \mathbb{K}\colon (v, w) \mapsto \kappa(v+w) - \kappa(v) - \kappa(w)$$

is bilinear (cf. [**DTV25**, Definition 5.1.27]).

We define a geometry Γ, that we will call the *quadric* corresponding to κ, as follows. The point set of Γ will be given by

$$\mathscr{P} := \{\, \langle x\rangle \in \mathsf{PG}(V) \mid \kappa(x) = 0 \,\},$$

and the set of lines $\mathscr{L}$ of Γ will be the set of lines of $\mathsf{PG}(V)$, whose points are all contained in $\mathscr{P}$. Observe that $\mathscr{P}$ can be empty, for instance, if we take $\mathbb{K} = \mathbb{R}$ and κ positive definite. Suppose that $\mathscr{P}$ and $\mathscr{L}$ are not empty. Then one can verify that Γ is a polar space embedded in $\mathsf{PG}(V)$ (cf. [**BC13**, Example 7.8.1]).

We will demonstrate that, if Γ contains a point p and a line L, then there exists a point $q \in L$ such that p is collinear to q. For that, let f be the symmetric bilinear form associated with κ. If $p = \langle x\rangle \in \mathscr{P}$, then

$$f(x,x) \;=\; \kappa(2x) - 2\kappa(x) \;=\; 2\kappa(x) \;=\; 0.$$

If L is a line of $\mathscr{L}$, then there exists a point $q \in L$ of the form $q = \langle y\rangle$ with $f(x,y) = 0$. For an arbitrary point $\langle \alpha x + \beta y\rangle$ on the line $pq = x\mathbb{K} + y\mathbb{K}$ of $\mathsf{PG}(V)$, we have

$$\kappa(\alpha x + \beta y) = f(\alpha x, \beta y) + \kappa(\alpha x) + \kappa(\beta y) = \alpha f(x,y)\beta + \alpha^2\kappa(x) + \beta^2\kappa(y) = 0,$$

and thus, $pq \in \mathscr{L}$.

Note 5.0.0.5. Note that, if the characteristic of $\mathbb{K}$ is distinct from 2, then

$$\kappa(x) = \tfrac{1}{2} f(x,x),$$

for $x \in V$ (cf. [**DTV25**, Definition 5.1.27]).

[1] The references only consider the cases of split spherical buildings related to semi-simple algebraic groups over algebraically closed fields. However, the result also holds for non-algebraically closed fields.

5.1. The Classification of Polar Spaces

Polar spaces of rank at least 3 have been classified and we will give an overview in the following. Most of the polar spaces we will review are embeddable into some projective space; in fact, every polar space of rank at least 4 is embeddable. When Jacques Tits classified all thick irreducible spherical buildings of rank at least 3 (cf. [**Tit74**]), he also obtained a classification of all polar spaces, which had previously been introduced and studied by Veldkamp (cf. [**Vel60**]). Veldkamp had already classified the polar spaces of rank at least 3 that contained only Desarguesian planes, constructed over fields of characteristic distinct from 2. Tits developed a method to treat the remaining cases, introducing pseudo-quadratic forms. These cases included polar spaces with non-Desarguesian Moufang planes, which are not embeddable into any projective space.

In 1987, Ronan and Tits described a construction of these polar spaces, along with all other spherical buildings without a subdiagram of type H_3 (cf. [**RT87**]). Three years later, Bernhard Mühlherr showed that the thick non-embeddable polar spaces arise as fixed point sets of involutions in buildings of type E_7 (cf. [**Müh90**]). Afterwards, Bart De Bruyn and Hendrik Van Maldeghem provided a coordinate construction of the thick non-embeddable polar spaces, in the spirit of the coordinatization of the non-Desarguesian Moufang planes (cf. [**DV14**]).

We will now recall a few definitions before turning to the classification results. Our main references are [**Tit74**, Section 8.3] and [**Van24**, Section 3.2].

Definition 5.1.0.1 (**Witt index**)**.** The *Witt index* of a quadratic form κ on a vector space V is the maximum dimension of a subspace of V on which κ vanishes. Such a form κ is called *anisotropic* if its Witt index is zero. For polar spaces, the Witt index of the form coincides with its rank.

Definitions 5.1.0.2 (**Polarities**, **absolute subspaces**, **radicals**, ...)**.**

- A *polarity* of a projective space is a symmetric relation $\square$ between the points such that $p^\square := \{q \mid q \square p\}$ is a hyperplane for each point p.
- A polarity is *non-degenerate* if $p^\square$ is always a proper hyperplane.
- For a subspace S, we define $S^\square$ to be the intersection of all $x^\square$ for $x \in S$.
- An *absolute subspace* for $\square$ is a subspace U with the property that $U \subseteq U^\square$.
- The *radical* of a polarity $\square$ is the image of the full point set under $\square$. The radical is the intersection of all $x^\square$ for x varying over all points.
- A polarity is called a *null-polarity* or *symplectic polarity* if each point is absolute.

Definitions 5.1.0.3 (**Anisotropic**, **associated polarities**)**.**

- If a subspace U is disjoint from that quadric, then the quadratic form is said to be *anisotropic over* U.

- For a $\mathbb{K}$-vector space and bilinear form $b\colon V \times V \to \mathbb{K}$, the *associated polarity* $\square$ is defined as $p \square q$ if $b(v, w) = 0$, where p and q are the 1-spaces of V generated by v and w, respectively.

We start by describing the different embeddable polar spaces.

5.1.1. Symplectic Polar Spaces

Symplectic polar spaces arise from non-degenerate null-polarities, that is, non-degenerate polarities in finite-dimensional projective spaces such that all points of the projective space are absolute points. The dimension of the projective space is always odd, say $2n-1$, and the rank of the polar space is n. The coordinatizing field is always commutative. These polar spaces are completely and uniquely determined by giving the collinearity relation between arbitrary points of the projective space. In standard form, this is given by the alternating bilinear form

$$\begin{aligned} f((x_{-n}, \ldots, x_{-1}, x_1, \ldots, x_n), (y_{-n}, \ldots, y_{-1}, y_1, \ldots, y_n)) \\ = x_{-n}y_n - y_{-n}x_n + \cdots + x_{-1}y_1 - y_{-1}x_1. \end{aligned}$$

5.1.2. Orthogonal Polar Spaces

Orthogonal polar spaces arise from non-degenerate quadratic forms of Witt index at least 1. Recall that a quadratic form is non-degenerate if it is anisotropic over the radical of the associated bilinear form. In characteristic different from 2, this just means that the associated bilinear form is non-degenerate (has trivial radical). The polarity associated to the bilinear form is also non-degenerate. In characteristic 2, however, we distinguish between two different kinds of quadratic forms:

Definitions 5.1.2.4 (**Separable & inseparable quadratic forms**)**.** In characteristic 2, we call the non-degenerate quadratic forms with degenerate associated bilinear form *inseparable*, and those with non-degenerate associated bilinear form *separable*. We extend this terminology to the other characteristics by calling every non-degenerate quadratic form in characteristic not 2 *separable*. We further extend this terminology to the orthogonal polar spaces themselves in the obvious way.

In characteristic 2, the associated polarity of a separable orthogonal polar space is a null-polarity. The one of an inseparable orthogonal polar space is a degenerate null-polarity. Separable orthogonal polar spaces admit, up to isomorphism, a unique embedding in some projective space.

Definition 5.1.2.5 (**Anisotropic corank**)**.** The difference of the dimension of the ambient projective space with twice the rank, plus one, will be called the *anisotropic corank* of the polar space. It is the (maximal) vector dimension of an anisotropic form that is needed to describe the corresponding quadric.

Definition 5.1.2.6 (**Hyperbolic quadrics**, **parabolic polar spaces**)**.** Quadrics with anisotropic corank equal to 0 are *hyperbolic polar spaces* and correspond to buildings of type D_n. We also sometimes call them *hyperbolic quadrics* if we want to emphasize that we view them as a subspace of a projective space. If the anisotropic corank is 1, we speak about *parabolic quadrics* and *parabolic polar spaces*.

According to Theorems 3.4.3 and 4.4.4 of [**Van24**], the standard equation of a quadric in $\mathsf{PG}(V)$, for V a $\mathbb{K}$-vector space that corresponds to a polar space of rank r, is given as follows.

Let $V = V' \oplus V_0$, $\dim V' = 2r$, and $\{e_{-r}, \dots, e_{-1}, e_1, e_2, \dots, e_r\}$ a basis of V'. Let $v_0 \in V_0$, and f_0 an anisotropic quadratic form in V_0. Then the standard equation is

$$X_{-r}X_r + \cdots + X_{-2}X_2 + X_{-1}X_1 = f_0(v_0, v_0),$$

and a vector $v \in V$ is written as

$$v = x_{-r}e_{-r} + \cdots + x_{-1}e_{-1} + x_1e_1 + x_2e_2 + \cdots + x_re_r + v_0.$$

In the finite-dimensional case, if $\dim V_0$ is even, the above equation becomes the equation of a hyperbolic quadric over a *splitting field*, that is, a field extension of $\mathbb{K}$ over which f_0 becomes completely reducible (one can take the algebraic or quadratic closure of $\mathbb{K}$). We will use this and note that hyperbolic quadrics have two natural systems of maximal singular subspaces. Two maximal singular subspaces belong to the same system if and only if the intersection has even codimension in each of them (the codimension of $U \cap V$ in U is $\dim U - \dim(U \cap V)$).

Notation 5.1.2.7. For a hyperbolic quadric Q of Witt index r, associated to the quadratic form $g\colon V \to \mathbb{K}$ with associated bilinear form $f\colon V \times V \to \mathbb{K}$, we denote by $\mathsf{PGO}_{2r}(\mathbb{K})$ the group of all elements of $\mathsf{PGL}_{2r}(V)$ that preserve f and g. The unique subgroup of index 2 preserving each class of maximal singular subspaces will be denoted by $\mathsf{PGO}^\circ_{2r}(\mathbb{K})$. Note that $\mathsf{PGO}^\circ_6(\mathbb{K})$ is isomorphic to $\mathsf{PSL}_4(\mathbb{K}, 2)$.

Definition 5.1.2.8 (**Parabolic polarities**)**.** A *parabolic polarity* of Q is the involution fixing a given parabolic subquadric P of Witt index $r-1$ and interchanging each two maximal singular subspaces of Q containing a common maximal singular subspace of P. Each parabolic polarity belongs to $\mathsf{PGO}_{2r}(\mathbb{K})$, as in V, it is given by $V \to V\colon v \mapsto v - \frac{f(v,w)}{g(w)}w$, for some $w \in V$ with $g(w) \neq 0$.

The following lemma is a reformulation of Theorem 1.5.1 of [**Che54**].

Lemma 5.1.2.9. *The group* $\mathsf{PGO}_{2r}(\mathbb{K})$ *is generated by the parabolic polarities.*

5.1.3. Pseudo-quadratic Polar Spaces

Pseudo-quadratic polar spaces arise from σ-quadratic forms, where σ is an involution of a skew field $\mathbb{L}$ (and $\sigma \neq \mathsf{id}$). We briefly describe these. Let V be a right vector space over $\mathbb{L}$. Let $g\colon V \times V \to \mathbb{L}$ be a (σ, id)-linear form (meaning g is additive in both variables and $g(vk, w\ell) = k^\sigma g(v, w)\ell$, for all $k, \ell \in \mathbb{L}$ and $v, w \in V$).

We define the following:

$$\mathbb{L}_\sigma := \{t - t^\sigma \mid t \in \mathbb{L}\},$$
$$f\colon V \times V \to \mathbb{L},\ (v,w) \mapsto g(v,w) + g(w,v)^\sigma,$$
$$\mathfrak{q}\colon V \to \mathbb{L}/\mathbb{L}_\sigma,\ v \mapsto g(v,v) + \mathbb{L}_\sigma.$$

Then f is (σ, id)-linear and $f(v,w)^\sigma = f(w,v)$, and we call f a *Hermitian sesquilinear form*.

Definitions 5.1.3.10.

- With the above notation, the *radical of the Hermitian sesquilinear form* f is given by
$$R_f := \{v \in V \mid f(v,w) = 0, \forall w \in V\}.$$
- We say that f *is associated to* $\mathfrak{q}$.
- $\mathfrak{q}$ is said to be *anisotropic over* R_f if
$$\mathfrak{q}(v) = 0 \text{ if and only if } v = 0, \text{ for all } v \in R_f.$$

Suppose $\mathfrak{q}$ is anisotropic over R_f. Let X be the set of vectors $v \in V$ for which $\mathfrak{q}(v) = 0$. Suppose the subspaces of V of maximal dimension that are contained in X have finite dimension r. Then the point set $\mathscr{P} = \{\langle v\rangle \mid v \in X\}$ of $\mathsf{PG}(V)$, together with the lines induced from $\mathsf{PG}(V)$, is a polar space of rank r, called a *pseudo-quadratic polar space*. This description also makes sense for $\sigma = 1$ (and all embeddable polar spaces, except for the symplectic ones in characteristic distinct from 2). We call it the *pseudo-quadratic description*.

Definition 5.1.3.11 (**Hermitian polar spaces**)**.** In the commutative case, the Hermitian sesquilinear form f determines the polar space (without using the associated pseudo-quadratic form), and we call the polar spaces *Hermitian*. Sometimes the form $V \to V$ with $v \mapsto f(v,v)$ is called a *Hermitian form*.

5.1.4. Non-embeddable Polar Spaces

There are two different kinds of non-embeddable polar spaces. Non-embeddable polar spaces of one kind are top-thin and arise as the line-Grassmannian of a 3-dimensional projective space over a non-commutative skew field. The other non-embeddable polar spaces are thick, and each of them arises as the fixed point structure of an involution in a building of type E_7. In fact, for each non-Desarguesian Moufang plane π, there exists a unique polar space of rank 3 whose planes are isomorphic to π. We will not need an explicit construction of those polar spaces and refer to [**DV14**] for an elementary one. We will use some properties of these polar spaces derived in [**PV24**], where thick non-embeddable polar spaces are called *Freudenthal-Tits polar spaces*. However, we will need the standard description of the Moufang plane associated to a Cayley division algebra $\mathbb{O}$ over a field $\mathbb{K}$. We denote that plane as $\mathsf{PG}(2,\mathbb{O})$. Recall that this is an 8-dimensional, alternative, quadratic, non-associative composition algebra with a standard involution $x \mapsto \overline{x}$, such that, for all $x \in \mathbb{O}$, we have $x\overline{x} \in \mathbb{O}$ and $x + \overline{x} \in \mathbb{O}$. An affine plane can be described as

the set of ordered pairs $(x, y) \in \mathbb{O} \times \mathbb{O}$, with lines of two types: (1) For $m, k \in \mathbb{O}$, the line $[m, k]$ contains the points $(x, mx + k)$, for all $x \in \mathbb{O}$; (2) for $x \in \mathbb{O}$, the line $[x]$ contains the points (x, y), for all $y \in \mathbb{O}$.

5.1.5. Orders

We conclude our brief excursion into different polar spaces with a useful fact about finite polar spaces of higher rank.

Fact 5.1.5.12. Let Δ be a finite polar space of rank at least 3 and let q be some prime power.

- If Δ is symplectic, then it has order (q, q).
- If Δ is orthogonal, then it has order $(q, 1)$, (q, q) or (q, q^2).
- If Δ is Hermitian, then it has order (q^2, q) or (q^2, q^3).

This is [**Van24**, Exercise 5.3]. Furthermore,

- If Δ has order (q, q), then it is symplectic or parabolic.
- If Δ has order $(q, 1)$, then it is hyperbolic.
- If Δ has order (q, q^2), then it is elliptic.
- If Δ has order (q^2, q), then it is Hermitian in odd dimension.
- If Δ has order (q^2, q^3), then it is Hermitian in even dimension.

5.2. More Properties of Polar Spaces

We prove various results, which we will use in the proof of Lemma 10.4.2.5, but which are also interesting in their own right. Note that, to avoid technical issues, we do not always assume the most general conditions, but content ourselves with situations that are applicable to our case. The content of this section is based on [**BSV25**].

Lemma 5.2.0.1. *Let Γ be a polar space of rank at least 3 such that each line contains at least 4 points. Then Γ is not the union of three (proper) geometric hyperplanes H_1, H_2, H_3.*

Proof. Let p_1 be a point in $H_1 \setminus H_2$ such that $H_1 \neq p_1^\perp$. Then H_1 induces a proper geometric hyperplane in $\mathsf{Res}_\Gamma(p_1)$, and H_3 has to contain the complement, which is ridiculous. □

Lemma 5.2.0.2. *Let Γ be a polar space of rank at least 4, such that each line contains at least 4 points. If Γ is the union of 4 proper geometric hyperplanes H_1, H_2, H_3, H_4, then $H_i \cap H_j = H_k \cap H_\ell$, for $\{i, j, k, \ell\} \subseteq \{1, 2, 3, 4\}$, $i \neq j$ and $k \neq \ell$.*

PROOF. As before, we can find a point $p_1 \in H_1 \setminus H_2$, such that $H_1 \neq p_1^\perp$. We may also assume $H_3 \neq p_1^\perp \neq H_4$, as this only excludes at most two more points. If $p_1 \in H_3$, then in order to cover $p_1^\perp$, we need to cover $\mathsf{Res}_\Gamma(p_1)$ with the three proper hyperplanes H_1, H_2, H_3, contradicting Lemma 5.2.0.1. Hence, $p_1 \notin H_3 \cup H_4$. If $H_3 = p_3^\perp$, with $p_3 \in H_1 \setminus H_2$, then an arbitrary line L in H_1 through p_3 contains a point p of $H_3 \cap H_1$ that does not belong to H_2, and such that $p^\perp \neq H_4$, as L contains at least 4 points. This contradicts what we just argued. Likewise, there is no point $p_4 \in H_1 \setminus H_2$, such that $p_4^\perp = H_3$. We conclude that no point of $H_1 \setminus H_2$ belongs to $H_3 \cup H_4$. But since $H_1 \cap H_2$ is a geometric hyperplane of H_1, this implies that $H_1 \cap H_2 \subseteq H_3 \cap H_4$ and $H_1 \cap H_2 = H_1 \cap H_3 = H_1 \cap H_4$. The assertion now follows. □

Lemma 5.2.0.3. *Let Γ be a polar space of rank at least 4, such that each line contains at least 4 points. Then the complement C of the union of two geometric hyperplanes H_1, H_2 of Γ is a connected geometry.*

PROOF. Indeed, let x and y be two non-collinear points of C. If $x^\perp \cap y^\perp$ is neither contained in H_1 nor in H_2, then by Lemma 5.2.0.1 it is not contained in $H_1 \cup H_2$, and so there exists a point in $x^\perp \cap y^\perp \cap C$. So, we may assume that $x^\perp \cap y^\perp \subseteq H_1$. Since H_1 contains lines, it contains at least one point $p_2 \in H_2$. Let L be a line through y opposite the line xp_2. On L we can find a point z distinct from y, and not belonging to $H_1 \cup H_2$ (since L contains at least 4 points). On the line xp_2 only the point p_2 belongs to $H_1 \cup H_2$, and p_2 is not collinear to z. Hence, the unique point z' on xp_2 collinear to z belongs to C. We have $x \perp z' \perp z \perp y$ all in C, and the assertion is proved. □

Lemma 5.2.0.4. *Let Γ be a polar space of rank at least 4. Then given four maximal singular subspaces M_1, M_2, M_3, and M_4, there exists a line disjoint from*

$$M_1 \cup M_2 \cup M_3 \cup M_4.$$

PROOF. Indeed, it is easy to see that Γ is not the union of M_1 up to M_4 (this is obvious for infinite polar spaces; for finite ones the ovoid number is never less than 5, where the ovoid number indicates the number of maximal singular subspaces needed to partition the point set of Γ). Let p be a point outside $M_1 \cup M_2 \cup M_3 \cup M_4$. Let M_i' be the maximal singular subspace containing p and intersecting M_i in a hyperplane, $i = 1, 2, 3, 4$. The same argument (which would lead to technicalities if Γ had rank 3) as above applied in the residue $\mathsf{Res}_\Gamma(p)$ yields the desired line L through p. □

Lemma 5.2.0.5. *Let Γ be a polar space of rank at least 4. Then given two maximal singular subspaces M_1, M_2 of the same kind, if Γ is hyperbolic, and a line L disjoint from $M_1 \cup M_2$, there exists a maximal singular subspace through L disjoint from $M_1 \cup M_2$.*

PROOF. Looking in the residue $\mathsf{Res}_\Gamma(L)$, it suffices to show that in every polar space of rank at least 2 we can find a maximal singular subspace disjoint from the union of two given maximal singular subspaces of the same kind in the hyperbolic case, which we can call M_1, M_2. By taking subsequently the residue in a point outside these maximal singular subspaces, we may assume that the rank of the residue is equal to 2. Then M_1 and M_2 are lines. Consider a line L intersecting both and distinct from M_i, $i = 1, 2$ (such a line exists since either every point is contained

in at least three lines, or we are dealing with a grid in case M_1 and M_2 are disjoint). Then, since lines have size at least 3, there is a point $x \in L$ not in $M_1 \cup M_2$. Any line through x distinct from L is disjoint from $M_1 \cup M_2$. The claim is proved. □

Notation 5.2.0.6. Let Γ be a hyperbolic polar space of rank d at least 4, and let U and U' be two maximal singular subspaces in Γ. We write $U \sim U'$, if $U \cap U'$ has codimension 2 in both U and U'.

The next lemma follows directly Proposition 2.3.1.13, but we offer an independent and short proof merely in the terminology of polar spaces.

Lemma 5.2.0.7. *Let Γ be a hyperbolic polar space of rank d at least 4, and let M_1, M_2 be two maximal singular subspaces of the same kind. Then, given two maximal singular subspaces W, W' disjoint from both M_1 and M_2, there exists a sequence of maximal singular subspaces*

$$W = W_0 \sim W_1 \sim \cdots \sim W_k = W',$$

for some natural number k, such that all W_i, for $i \in \{0, 1, \ldots, k\}$, are disjoint from both M_1 and M_2.

Proof. Let p be a point of $W \setminus W'$. Let M be the maximal singular subspace through p intersecting W' in a hyperplane. Then $M \cap M_i$ is a point p_i, $i = 1, 2$. It is easy to see that there is a point q in $W \setminus (W' \cup p_1^\perp \cup p_2^\perp)$, since no projective space is the union of three proper subspaces, among which at least one is not a hyperplane. Since q is not collinear to p_i, $i = 1, 2$, the unique maximal singular subspace W'' through pq intersecting W' in a hyperplane is disjoint from $M_1 \cup M_2$ and is closer to W with respect to the relation $\sim$ than W'. An induction argument now concludes the proof of the lemma. □

5.3. More Notable Notions

We will now introduce some more notions that we will need in Chapter 9, Chapter 8 and Section 10.4.4.

5.3.1. Classical Geometries

Definition 5.3.1.1 (**Hyperbolic lines**)**.** A *hyperbolic line* in a polar space Δ is a set of points collinear to every point collinear to two given non-collinear points. In other words, it is a set of points of the form $(\{x, y\}^\perp)^\perp = \{x, y\}^{\perp\perp}$, where x and y are two non-collinear points. It is called *large* if it coincides with $\{u, v\}^\perp$, for u and v any two distinct points of $\{x, y\}^\perp$.

Definition 5.3.1.2 (**Ideal subquadrangle**)**.** A *subquadrangle* Γ' of a generalized quadrangle Γ is a subset X of points which, endowed with the intersections of X with the lines of Γ containing at least two points of X, is a generalized quadrangle. If Γ has order (s, t) and Γ' has order (s', t), then we say that Γ' is an *ideal* subquadrangle.

Definition 5.3.1.3 (**Regulus**)**.** In a polar space associated to a quadric, the set of lines meeting two given opposite lines is called a *regulus*.

Definition 5.3.1.4 (**Ovoid, spread**)**.** An *ovoid* of a polar space is a set of points with the property that every line contains exactly one point of that set. A *spread* is a set of generators that partitions the point set.

With that, a spread in a generalized quadrangle $\Gamma = (\mathscr{P}, \mathscr{L})$ is a partition of $\mathscr{P}$ into members of $\mathscr{L}$. If Γ has order (s, t), then the number of points of an ovoid, and the number of lines of a spread of Γ, is equal to $1 + st$ (see 1.8.1 of [**PT84**]).

Definition 5.3.1.5 (**Tangents & secants in PG(V)**)**.** Suppose X is a set of points of $\mathsf{PG}(V)$. A line L of $\mathsf{PG}(V)$ is called an *exterior line* if $L \cap X = \varnothing$, a *tangent line* (or *tangent*) if $|L \cap X| = 1$ or $L \subseteq X$, and a *secant line* (or *secant*) if $|L \cap X| \geq 2$ and $L \not\subseteq X$.

Definition 5.3.1.6 (**Tangents**, & **tangent hyperplanes to quadrics**, **regular points**, ...)**.** (cf. [**DTV25**, Definition 5.1.27])

- A *tangent line* (or *tangent*), to a quadric is a line either entirely contained in the quadric, or intersecting it in a unique point. This is in conformity with Definition 5.3.1.5.
- Let Q be a quadric and $p \in Q$. Then the *tangent space* at p to Q is the subspace of $\mathsf{PG}(V)$ generated by all tangent lines through p.
- If the tangent space at $p \in Q$ is a hyperplane, we talk about the *tangent hyperplane*, and call p a *regular point*. The set of points at which the tangent space is the whole of $\mathsf{PG}(V)$ is called the *radical* of Q and denoted $\mathsf{Rad}(Q)$. Note that the radical of a quadric does not necessarily coincide with the projective subspace corresponding to the radical of the bilinear form associated to the quadric form defining the quadric.

Definition 5.3.1.7 (**Non-degenerate quadrics**)**.** A quadric is said to be *non-degenerate* if the radical of the associated bilinear form has trivial intersection with the null set of the quadratic form (cf. [**DTV25**, Definition 5.1.27]), or, alternatively, when its radical is trivial.

Definition 5.3.1.8 (**Unitals**, **tangents** & **secants to unitals**)**.** Let V be some 3-dimensional vector space over the finite field $\mathbb{F}_{q^2}$ of order q^2, for some prime power q.[2] Put $\mathsf{PG}(2, q^2) = \mathsf{PG}(V)$. Then $\mathsf{PG}(2, q^2)$ is the finite Desarguesian plane over $\mathbb{F}_{q^2}$. A *unital* is any set of $q^3 + 1$ points in $\mathsf{PG}(2, q^2)$ intersecting each line in either a singleton or a set of $q + 1$ points. Hence, with above terminology, each line is either a tangent or a secant (cf. [**DTV25**, §4.2.5].

Definition 5.3.1.9 (**Conics**, **singular points**)**.** Assume we have chosen homogeneous coordinates in $\mathsf{PG}(2, \mathbb{K})$. Then a set C of points of $\mathsf{PG}(2, \mathbb{K})$ is called a *conic* if it consists of all points whose coordinates satisfy a homogeneous equation of degree 2 over $\mathbb{K}$. In standard notation:

$$ax^2 + a'y^2 + a''z^2 + 2b''xy + 2b'xz + 2byz = 0, \quad \text{if } \operatorname{char} \mathbb{K} \neq 2,$$

and

$$ax^2 + by^2 + cz^2 + fxy + exz + dyz = 0, \quad \text{if } \operatorname{char} \mathbb{K} = 2.$$

[2] In the reference $\mathbb{F}_{q^2}$ is denoted $\mathsf{GF}(q^2)$.

The points whose coordinates satisfy this equation are called *singular points*. The notion of a conic is well-defined, i.e., independent of the chosen coordinate system for the homogeneous coordinates (cf. [**DTV25**, §4.2.5]). Roughly speaking, a conic is just a quadric in a plane.

Definition 5.3.1.10 (**Nuclei**)**.** The *nucleus* of a conic C is the intersection of all tangent lines to C. (cf. [**DTV25**, Definition 4.1.22]).

Note 5.3.1.11. If the field $\mathbb{K}$ has characteristic 2, then the nucleus of a conic is non-empty.

5.3.2. Generalized Hexagons

First, we want to introduce two important classes of thick generalized hexagons.

(**Split Cayley hexagons**, **twisted triality hexagons**)**.** In [**Tit59**], Tits defines two classes of thick generalized hexagons. An arbitrary member Γ of the first class, which exists over an arbitrary field $\mathbb{K}$, is nowadays called a *split Cayley hexagon* and allows an embedding in the parabolic quadric Q of $\mathsf{PG}(6, \mathbb{K})$. In fact, all points of Q are points of Γ, see [**Tit59**, §8.1]. It is denoted as $\mathsf{G}_{2,2}(\mathbb{K}, \mathbb{K})$. The second class consists of the so-called *twisted triality hexagons*, or briefly, *triality hexagons*, and these are the generalized hexagons described in [**Tit59**, §6], see also [**Van98**, §3.5.8], which admit an embedding in a hyperbolic quadric of $\mathsf{PG}(7, \mathbb{K})$, but are not contained in a hyperplane. They exist over any field $\mathbb{K}$ admitting a cubic Galois extension $\mathbb{L}$, and then they are denoted as $\mathsf{G}_{2,2}(\mathbb{K}, \mathbb{L})$. We will say more about split Cayley hexagons and twisted triality hexagons in Section 6.1.5.

Now generally, in generalized hexagons, opposite points are points at distance 3 in the point graph. Opposite lines are lines such that each point of either is opposite at least one point of the other. We use the notation and terminology of parapolar spaces to express the mutual position of two points, with the obvious meaning. In particular, points are special if they are not collinear but are collinear to a unique common point. Let $\Gamma = (X, \mathscr{L})$ be a generalized hexagon. If we call a point of Γ *close to* a line if it is collinear to a unique point of that line.

Definition 5.3.2.12 (**Hyperbolic lines in generalized hexagons**)**.**
Let $x \perp a \perp x'$ be three points of Γ with x and x' not collinear. Let b be any point opposite a and not opposite both x and x'. If the set h of points not opposite b and collinear to a is independent of the choice of b, then we call h a *hyperbolic line* of Γ. (This is called an *ideal line* by Ronan in [**Ron80**].)

We will also need the following notion in generalized hexagons.

Definition 5.3.2.13 (**Distance-3 traces in generalized hexagons**)**.**
A *distance*-3 *trace* in Γ is the set of points close to two given opposite lines L and M. We denote it by $[L, M]_3$. It is called *regular* if $[N, M]_3 = [L, M]_3$ whenever N is opposite M and $|[N, M]_3 \cap [L, M]_3| \geq 2$.

In other words: A distance-3 trace is the set of points collinear to at least one respective point of both L and M. It is regular if it is determined by any pair of its members.

A distance-3 trace in a split Cayley hexagon over a perfect field in characteristic 2 is also called an *imaginary line*. We will explain the reasons for that in the proof of Proposition 8.4.0.5.

5.4. Long Root & Hexagonic Geometries

We now introduce *long root* and *hexagonic geometries*, which we regard as umbrella concepts to summarize certain geometries with shared properties. In general, not all long root geometries satisfy the axioms of proper parapolar spaces, but all hexagonic geometries do.

Long root geometries have been studied since the seventies by — among others — Aschbacher, Cooperstein, and Timmesfeld (e.g., in [**AS76**]). They are examples of certain *root filtration spaces*. However, we will neither need to understand nor define root filtration spaces in this thesis. We refer the interested reader to [**BC13**] and [**Shu10**, §17.1.1].

For context, we will mention that long root geometries are geometries for *long root groups*, which are a subject in group theory. Roughly speaking, these are groups G generated by a conjugacy class A^G consisting of elementary abelian subgroups such that any two distinct subgroups in the class generate one of four possible subgroups (cf. [**Tim01**]). These groups possess BN-pairs. If one takes the conjugates of A in G as points, and defines lines by one of the four classes of the generated subgroups, one obtains a *long root geometry*. The conjugates of A are long root subgroups of the corresponding building. A similar thing can be done for the short roots, and this leads, together with the long roots, to the notion of a *root group geometry* for a spherical building.

The root group geometries are the following.

- (P) Polar spaces — $\mathsf{B}_{n,1}$, $\mathsf{C}_{n,1}$, and $\mathsf{D}_{n,1}$,
- (G) Polar Grassmannians of lines — $\mathsf{B}_{n,2}$, $\mathsf{C}_{n,2}$, $n \geq 4$, and $\mathsf{D}_{n,2}$, $n \geq 5$,
- (M) Metasymplectic spaces — $\mathsf{F}_{4,1}$, $\mathsf{F}_{4,4}$, $\mathsf{B}_{4,2}$, $\mathsf{C}_{4,2}$, and $\mathsf{D}_{4,2}$,
- (E) Exceptional Lie incidence geometries of types $\mathsf{E}_{6,2}$, $\mathsf{E}_{7,1}$, and $\mathsf{E}_{8,8}$,
- (O) Geometries which are not parapolar spaces of symplectic rank at least three — $\mathsf{A}_{n,\{1,n\}}$, $n \geq 2$, where $\{1, n\}$ is the type of a pair consisting of a point and a hyperplane, and $\mathsf{C}_{3,2}$, where the points are the lines of the polar space $\mathsf{C}_{3,1}$ and the lines are the point-plane flags of this polar space. Some authors include the generalized hexagons, which have type $\mathsf{G}_{2,2}$, or $\mathsf{G}_{2,1}$.

The root group geometries of the classes (G), (M) and (E) were characterized by Anna Kasikova and Ernest Shult in [**KS02**] (with special thanks to Antonio Pasini) as a class of parapolar spaces of symplectic rank at least three satisfying the following four axioms.

- (H1) Given a point x not incident with a symp ξ, the space $x^\perp \cap \xi$ is never just a point.
- (H2) Given a projective plane π and a line L meeting π at a point p, either

(i) every line of π on p lies in a common symp with L, or
(ii) exactly one such line incident with (p, π) has this property.

(H3) Given a point-line flag (p, L), there exists a second line N such that

$$L \cap N = \{p\},$$

and no symp contains $L \cup N$ — i.e., (x, y) is a special pair for each

$$(x, y) \in (L - \{p\}) \times (N - \{p\}).$$

(F) If all symplecta have rank at least four, assume every maximal singular subspace has finite projective rank.

Ernest Shult later showed in [**Shu08**] that if one just assumes axioms (H1) and (F) above, then this "only" adds strong parapolar spaces of point-diameter 2 for which $x^\perp \cap \xi$ is never a point for any point–symp pair (x, ξ). These include, for example, line Grassmannians of some (possibly infinite-dimensional) projective space, the half-spin geometry $\mathsf{D}_{5,5}$, and the exceptional Lie incidence geometry $\mathsf{E}_{6,1}$.

The results stated above lead to the definition of *hexagonic geometries*, which were named after a scheme devised by Bernhard Mühlherr.

Definition 5.4.0.1 (**Hexagonic geometries**)**.** Any parapolar space which satisfies (H1), (H2), and (H3) is called a *hexagonic geometry* (cf. [**Shu10**, p. 480]).

Another interesting result characterizes the point-residuals of hexagonic geometries.

Theorem 5.4.0.2 (Theorem 2 of [**KS02**])**.** *Let Γ be a strong parapolar space satisfying the following conditions:*

(i) For every point–symp pair (x, ξ), $x^\perp \cap \xi \neq \varnothing$.
(ii) The set of points at distance at most two from a point forms a geometric hyperplane.
(iii) If there is no symp of rank two, assume every maximal singular subspace has finite projective rank.

Then Γ is one of the following:

(i) $\mathsf{D}_{6,6}, \mathsf{A}_{5,3}$, or $\mathsf{E}_{7,1}$,
(ii) a dual polar space of rank three,
(iii) a product geometry $L \times P$, where L is a line and P is a non-degenerate polar space of rank at least two (possibly of infinite polar rank).

Thus, hexagonic geometries are proper parapolar spaces with certain structural properties, originally motivated by long root geometries. These include Lie incidence geometries of type

$$\mathsf{A}_{n,\{1,n\}}\ (n \geq 3),\ \mathsf{B}_{n,2}\ (n \geq 3),\ \mathsf{C}_{n,2}\ (n \geq 3),\ \mathsf{D}_{n,2}\ (n \geq 4),\ \mathsf{E}_{6,2},\ \mathsf{E}_{7,1},\ \mathsf{E}_{8,8},\ \mathsf{F}_{4,1},\ \text{and}\ \mathsf{F}_{4,4},$$

together with some infinite-dimensional analogues of Lie incidence geometries of type $\mathsf{A}_{n,\{1,n\}}$, and line Grassmannians of infinite-rank polar spaces. To avoid hexagons not being hexagonic, some authors sometimes add all generalized hexagons. In that case, Lie incidence geometries of type

$$\mathsf{A}_{2,\{1,2\}},\ \mathsf{G}_{2,1},\text{ and }\mathsf{G}_{2,2}$$

are included as well.

In this thesis we will mostly be concerned with exceptional hexagonic geometries that do contain symps of rank at least 3. Thus, when we talk about them, we mean Lie incidence geometries of type

$$\mathsf{E}_{6,2},\ \mathsf{E}_{7,1},\ \mathsf{E}_{8,8},\ \mathsf{F}_{4,1},\text{ or }\mathsf{F}_{4,4}.$$

We will now clarify which long root geometries will be relevant to us in this thesis and include a proper way of defining them.

Definition 5.4.0.3 (**Long root geometry**)**.** If J corresponds to the set of fundamental roots not perpendicular to the highest root of the root system corresponding to the Dynkin diagram, then $\mathsf{X}_{n,J}(\mathbb{K})$ is a *long root (Lie incidence) geometry* ([**Tim01**, Ch. III, §9]).

We will only be concerned with the *split case* if the Dynkin diagram contains a double bond; that is:

(1) for type B_n, the building associated with a parabolic quadric (viewed as a polar space) in $\mathsf{PG}(2n,\mathbb{K})$ with standard equation
$$X_0^2 = X_{-1}X_1 + X_{-2}X_2 + \cdots + X_{-n}X_n,$$
(2) for type C_n, the building associated with a non-degenerate alternating form in $\mathsf{PG}(2n-1,\mathbb{K})$,
(3) for type F_4, the building whose residue of type B_2 is precisely the case $n=2$ in (1) above.

We give a list of the long root Lie incidence geometries of split types in Table 1. For more details, we also refer to [**DSV24**] and [**Jan19**, §7.2].

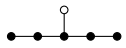
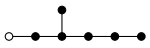
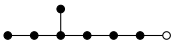
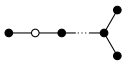
Type	Diagram	Type	Diagram
$\mathsf{A}_{n,\{1,n\}}$, for $n \geq 3$		$\mathsf{E}_{6,2}$	
$\mathsf{B}_{n,2}$, for $n \geq 3$		$\mathsf{E}_{7,1}$	
$\mathsf{C}_{n,2}$, for $n \geq 3$		$\mathsf{E}_{8,8}$	
$\mathsf{D}_{n,2}$, for $n \geq 4$		$\mathsf{F}_{4,1}$	
		$\mathsf{G}_{2,2}$	

TABLE 1. The long root geometries we will consider in this thesis.

Long root geometries and hexagonic geometries are two distinct classes of geometries. We can say that any long root geometry which is a parapolar space of symplectic rank at least 3 is a hexagonic geometry (cf. [**Shu10**, p. 606]). However, there are clearly hexagonic geometries which are not long root geometries, and vice versa.

We hope to make their distinction and relation clearer by presenting a result of Arjeh Cohen and Gábor Ivanyos, which captures long root geometries. In [**CI07**], they showed that every *non-degenerate root filtration space* of finite *singular rank* is isomorphic to a *shadow space* of type

$\mathsf{A}_{n,\{1,n\}}$ $(n \geq 2)$, $\mathsf{B}_{n,2}$ $(n \geq 3)$, $\mathsf{C}_{n,2}$ $(n \geq 3)$, $\mathsf{D}_{n,2}$ $(n \geq 4)$, $\mathsf{E}_{6,2}$, $\mathsf{E}_{7,1}$, $\mathsf{E}_{8,8}$, $\mathsf{F}_{4,1}$, or $\mathsf{G}_{2,2}$.

The only root shadow spaces missing from the above list are polar spaces, that is, the shadow spaces of type $\mathsf{B}_{n,1}$ and $\mathsf{C}_{n,1}$ (cf. [**CI07**]). A characterization of polar spaces in terms of degenerate root filtration spaces is given by Cuypers in [**Cuy06**], who extended earlier results of Timmesfeld [**Tim01**].

Thus, this result encompasses many of the same geometries as the result by Kasikova and Shult, but uses an entirely different paradigm. To quote from [**Shu08**]:

"*It is a truly amazing result for two reasons: (1) its axioms are so apparently outside the realm of parapolar spaces, that it is a surprise that these spaces surface at all, and (2) their approach plumbs the depths of that murky world of low symplectic rank — even its subrealm where no symplecta exist at all.*"

In recent and unpublished work ([**MV26**]), Bernhard Mühlherr and Hendrik Van Maldeghem approach the hexagonic geometries via an axiom very much in the point-line spirit of the one-or-all axiom for polar spaces [**BS74**], completely avoiding the theory of parapolar spaces. The resulting geometries, called *hexagonic spaces*, are precisely the root filtration spaces but without the condition of finite singular rank. Thanks to the main result of [**JV23**], there are no additional examples.

5.4.1. Some Properties of Hexagonic Geometries, Parapolar Spaces of Type $\mathsf{E}_{6,1}(\mathbb{K})$ & $\mathsf{E}_{7,7}(\mathbb{K})$

In this subsection, we will prove some results, which we will later use in Chapter 8 and Chapter 9.

The content of this section is based on [**BV25b**].

Lemma 5.4.1.4. *Let x be a point and ξ a symp of $\mathsf{E}_{6,1}(\mathbb{K})$. Then x is opposite ξ if and only if, for some point $y \in \xi$, the symp $\xi(x,y)$ intersects ξ only in y; if and only if, for all points $y \in \xi$, the symp $\xi(x,y)$ intersects ξ only in y.*

Proof. If x is opposite ξ, then $\xi \cap \xi(x,y)$ cannot be more than just the point y, for every $y \in \xi$, because otherwise $\xi \cap \xi(x,y)$ would have to be a 4-space and x would have to be collinear to a 3-space of that 4-space, contradicting the fact that x is opposite ξ. Now suppose that $\xi \cap \xi(x,y) = \{y\}$ for some fixed $y \in \xi$. We want to see that x has to be opposite ξ. Suppose there exists some $z \in \xi$ such that $\xi \cap \xi(x,z)$ is a 4-space. Then y and x have to be collinear to 3-spaces U_y and U_x, respectively, of that 4-space, which will necessarily intersect in at least a plane π. But then π will also be contained in $\xi(x,y)$ and hence $\xi \cap \xi(x,y)$ will be more than just the point y, which is a contradiction. So for every point $z \in \xi$, $\xi(x,z) = \{z\}$. That means that x is not collinear to any point in ξ and thus x is opposite ξ. □

One direction of the following assertion is stated as Fact 4.6.2.13(ii). Here, we provide a full proof using more basic facts. .

Lemma 5.4.1.5. *Let x, y be two points of $\mathsf{E}_{7,7}(\mathbb{K})$. Then x and y are opposite if and only if they are contained in respective symps intersecting in a line L such that x and y are collinear to unique respective distinct points of L.*

PROOF. First suppose that x and y are opposite. Let ξ_x be an arbitrary symp containing x. Fact 4.6.2.12(ii) implies that y is collinear to a unique point $z \in \xi_x$. Let ξ_y be an arbitrary symp through y and z. If ξ_x and ξ_y intersected in more than a line, they would intersect in a 5-space (cf. Fact 4.6.2.13), and both x and y would have to be collinear to 4-spaces of that 5-space. These would necessarily intersect, meaning that there would exist points collinear to both x and y, contradicting the fact that x and y are opposite. Hence $\xi_x \cap \xi_y$ is a line L. Now, clearly x and y are not collinear to a common point on L, and the assertion follows.

Next suppose, conversely, that x and y are contained in respective symps ξ_x and ξ_y intersecting in a line L such that x and y are collinear to unique respective distinct points x' and y' of L. Suppose, for a contradiction, that x is collinear to a $5'$-space U of ξ_y. Then $y'^{\perp} \cap x^{\perp}$ contains points of U that do not belong to ξ_x, a contradiction. Hence x is far from ξ_y, and the assertion follows from Fact 4.6.2.12(ii). □

Lemma 5.4.1.6. *Each point of a singular subspace U of a Lie incidence geometry with a trivial opposition map is opposite some point of a given singular subspace U' opposite U.*

PROOF. This follows from the definition of opposite singular subspaces. □

Lemma 5.4.1.7. *Let x and u be two points of a hexagonic Lie incidence geometry. Let $x \perp y \perp z \perp u$. Then x and u are opposite if and only if both $\{x, z\}$ and $\{y, u\}$ are special pairs.*

PROOF. This is [**CI07**, Lemma 2(v)]. □

Lemma 5.4.1.8. *Let Δ be a hexagonic Lie incidence geometry. Let x_1 and x_2 be two points in Δ and let ξ be a symp containing x_1. If x_2 is line-collinear to ξ and x_1 is collinear to a unique point of $x_2^{\perp} \cap \xi =: L$, then x_1 and x_2 are special.*

PROOF. Suppose x_1 and x_2 were symplectic. The symps ξ and $\xi(x_1, x_2)$ have a line in common, hence they share a plane α, which has to contain L. Since also $x_1 \in \alpha$, x_1 is collinear to all points of L, which contradicts the assumptions. □

Lemma 5.4.1.9. *Let x be a point of a hexagonic Lie incidence geometry, and suppose x is special to y_1 and y_2, with $y_1 \perp y_2$. Then also the unique points z_1 and z_2 with $x \perp z_i \perp y_i$, $i = 1, 2$, are collinear.*

PROOF. By Lemma 5.4.1.7, z_1 and y_2 are symplectic. The point x is collinear to z_1, and hence to a line L of $\xi(z_1, y_2)$. Then y_2 has to be collinear to a point of L, but this point can only be z_2, since x and y_2 are special. Note that it is possible that $z_1 = z_2$. □

Lemma 5.4.1.10. *Let x and L be a point and line, respectively, of a hexagonic Lie incidence geometry, and suppose that x is special to each point of L. Then there exists a line M consisting of the points collinear to x and some point of L.*

PROOF. Let y_1 and y_2 be two points on L and z_1 and z_2 the unique points with $x \perp z_i \perp y_i$, $i = 1, 2$. Again, by Lemma 5.4.1.7, z_1 and y_2 are symplectic and the symp $\xi(z_1, y_2) =: \xi$ contains y_1. Since x has to be collinear to a line M of ξ, z_2 is contained in M, and hence in ξ as well. Suppose $z_1 = z_2$. Let p be a point on M

distinct from z_1 and let q be the projection of p onto L. Since z_1 is collinear to y_1 and y_2, z_1 is collinear to every point on L, including q. Thus, q is collinear to every point of M and it follows that $x^\perp \cap q^\perp \supseteq M$, and thus, x and q are symplectic, which contradicts the assumptions. Therefore, z_1 and z_2 are distinct. By Lemma 5.4.1.9 it follows that z_1 and z_2 are collinear and $z_1 z_2 = M$. Now, for every point b on L, the unique point a, for which $x \perp a \perp b$, has to be $\mathsf{proj}_M(b)$. □

Similarly, we can say something about a point being symplectic to all points of a line.

Lemma 5.4.1.11. *Let x and L be a point and line, respectively, of a hexagonic Lie incidence geometry, and suppose that x is symplectic to each point of L. Then there exists a maximal singular subspace W not contained in a symp, containing L, such that $x^\perp \cap W$ is complementary to L in W and each symp containing x and a point of L contains $x^\perp \cap W$.*

Proof. Pick $x_1, x_2 \in L$ and set $\xi_1 := \xi(x, x_1)$. Then $x_2^\perp \cap \xi_1$ is either a line or a maximal singular subspace. If it were a line M, then x would be special to x_2, as $x_1 \in M$ and x_2 is not collinear to x. Hence $x_2^\perp \cap \xi_1$ is a maximal singular subspace. Defining $W = \langle x_2, x_2^\perp \cap \xi_1 \rangle$, and noting that x_1 was essentially arbitrary, the assertions follow. □

Lemma 5.4.1.12. *Let ξ_1 and ξ_2 be two non-disjoint symps and $x_i \in \xi_i$, $i = 1, 2$, two points of a hexagonic Lie incidence geometry. Then $x_1 \equiv x_2$ if and only if $\xi_1 \cap \xi_2$ is a point z, the symps ξ_1 and ξ_2 are locally opposite at z, and $\{x_i, z\}$ is a symplectic pair, $i = 1, 2$.*

Proof. This follows from Lemma 5.4.1.7 and Lemma 5.4.1.8, taking into account that ξ_1 and ξ_2 are locally opposite at an intersection point z if and only if each point $z_i \in \xi_i \setminus \{z\}$ is line-collinear to ξ_j, $\{i, j\} = \{1, 2\}$. □

Lemma 5.4.1.13. *Let ξ_1 and ξ_2 be two opposite symps and $L_1 \subseteq \xi_1$ a line. Then the set of points of ξ_2 symplectic to some point of L_1 is a line L_2 of ξ_2. All symps having a point on L_1 and a point on L_2 share a unique common point x, which is collinear to both L_1 and L_2.*

Proof. By Lemma 5.4.1.12, being symplectic is an isomorphism between ξ_1 and ξ_2. Let $x_1 \in L_1$ be symplectic to $x_2 \in L_2$. Let $x_1' \in L_1 \setminus \{x_1\}$. Then there is a unique point $x \in \xi(x_1, x_2)$ collinear to x_2 and x_1'. Clearly, $x \perp L_2$. Standard arguments imply that x is independent of x_1 and x_2, and so x is collinear to each point of $L_1 \cup L_2$. It is now easy to see that x is contained in each symp containing a point of L_1 and a point of L_2. Uniqueness of x follows from the fact that L_1 contains points that are special to points of L_2. □

Lemma 5.4.1.14. *Let Δ be a hexagonic Lie incidence geometry and let x_1 and x_2 be two symplectic points in Δ. Let x_2' be another point in Δ symplectic to x_1 and collinear to x_2. Let x_1' be a point such that $x_1 \perp x_1' \perp x_2'$. Then x_1' and x_2 cannot be special.*

PROOF. Consider the symp $\xi(x_1, x_2')$. Then $x_1' \in \xi(x_1, x_2')$. We may assume $x_2 \notin \xi(x_1, x_2')$, as otherwise the assertion is trivial. The point x_1' is collinear to (at least) a line L of $\xi(x_1, x_2)$, which contains a point y collinear to x_2. If x_1' were special to x_2, then y would be equal to x_2', contradicting $\xi(x_1, x_2) \neq \xi(x_1, x_2')$. □

5.5. Equator Geometries

This subsection is largely based on [**VV19**].

It is often the case that the residue of a point in a Lie incidence geometry Δ is not merely a quotient geometry, but can be realized naturally as a subgeometry of Δ itself.

Example 5.5.0.1. Let p be a point of a Lie incidence geometry Δ of rank at least 3 containing planes, and let X be an object opposite to p. For each line L incident with p, there exists a unique point $p_L \in L$ that is not opposite to X. The set of all such points p_L, endowed with the lines induced by the planes through p, forms a Lie incidence geometry isomorphic to the residue of p in the building associated with Δ.

Note 5.5.0.2. There exist Lie incidence geometries of rank at least 3 that do not contain planes; these correspond to dual polar spaces. In this situation, the residue can again be recovered as a subgeometry by taking the same set of points as above and defining the lines to be the hyperbolic lines contained in it.

We now generalize this construction.

By Lemma 5.4.1.8, if two points x and y of an exceptional hexagonic Lie incidence geometry Δ are opposite, then every symp through x contains a unique point symplectic to y. Consequently, the resulting point set is precisely

$$x^{\perp\!\!\!\perp} \cap y^{\perp\!\!\!\perp}.$$

This set forms a subspace of Δ, which we call the *equator geometry with poles x and y*, and denote it by $E(x, y)$. Whenever this subspace contains lines — which is the case for all simply laced types A_n, D_n, E_6, E_7, and E_8 — we regard $E(x, y)$ as a point-line geometry.

In these cases, equator geometries can be defined uniformly for the corresponding hexagonic geometries, namely for Lie incidence geometries of types

$$\mathsf{A}_{n,\{1,n\}}, \quad \mathsf{D}_{n,2}, \quad \text{and the exceptional types, excluding } \mathsf{F}_4.$$

Note 5.5.0.3. We have the following sequences of equator geometries (where $\Delta \to \Delta'$ indicates that Δ' is an equator geometry of Δ) can be deduced from [**DV23**]:

$$\begin{cases} \mathsf{E}_{8,8}(\mathbb{K}) \to \mathsf{E}_{7,1}(\mathbb{K}) \to \mathsf{D}_{6,2}(\mathbb{K}), \\ \mathsf{E}_{6,2}(\mathbb{K}) \to \mathsf{A}_{5,\{1,5\}}(\mathbb{K}) \to \mathsf{A}_{3,\{1,3\}}(\mathbb{K}). \end{cases}$$

To define equator geometries in full generality, we must proceed somewhat more abstractly.

Definition 5.5.0.4 (**Traces**, **trace geometries**)**.** Let Ω be a building of rank n with type set I. Let Δ be the K-Grassmannian of Ω, with $K \subseteq I$. Let F_1 and F_2 be two opposite flags of Ω, say of common type J. Let F be a flag of type J', incident with F_1, and suppose that the projection of F_2 onto F contains a point p_F. Then the set

$$T(F_1, F_2; J') := \{p_F \mid F \text{ is a flag of type } J' \text{ incident with } F_1\}$$

is called a (J, J')-*trace in* Δ.

Let J^* be the union of J with the type sets of the connected components of $I \setminus J$ disjoint from J'. Suppose $|J^*| < n - 1$. Then endowing $T(F_1, F_2; J')$ with the lines inherited from Δ' turns $T(F_1, F_2; J')$ into a Lie incidence geometry, called the $(J, J')_K$-*trace geometry* (and $|J^*| < n - 1$ ensures a non-trivial set of lines).

We have that $T(F_1, F_2; J')$ is naturally in bijective correspondence with the point set of the Lie incidence geometry Δ' of the J^*-residue Ω' of Ω, corresponding to the J'-Grassmannian.

Note 5.5.0.5. In many cases, but not always, the lines inherited from Δ' are just the lines of Δ contained (as point sets) in $T(F_1, F_2; J')$. Thus, trace geometries in Lie incidence geometries are in fact either subgeometries, or point sets endowed with a set of abstract lines turning them into Lie incidence geometries.

Definition 5.5.0.6 (**Equator geometries**)**.**
A trace geometry with point set $T(F_1, F_2; J')$ is called an *equator geometry*, if

$$T(F_1, F_2; J') = T(F_2, F_1; J').$$

We refer to this as the $(J, J')_K$-*equator geometry*. If $J = K$, then we omit the subscript K and write (J, J')-*equator geometry*.

Example 5.5.0.7. Let Δ be a polar space of rank n, corresponding to a building over the type set $I = \{1, 2, \ldots, n\}$. Let $J = \{1\}$ and $J' = \{2\}$. Then $J^* = J$. Let p_1 and p_2 be two opposite points. Then

$$T(p_1, p_2; J') = p_1^\perp \cap p_2^\perp = T(p_2, p_1; J').$$

The corresponding $(1, 2)$-equator geometry is a polar space of rank $n - 1$, and its lines are precisely the lines of Δ, that are contained in $p_1^\perp \cap p_2^\perp$.

Example 5.5.0.8. Again, let Δ be a polar space of rank n, corresponding to a building over the type set $I = \{1, 2, \ldots, n\}$. Let $j \in I$ with $j \leq n - 2$. Put $J = \{i\}$ and $J' = \{i + 1\}$. Then $J^* = \{1, 2, \ldots, i\}$. The corresponding trace geometry is again an equator geometry, coinciding with the subspace with point set $U_1^\perp \cap U_2^\perp$, for singular subspaces U_1 and U_2 of projective dimension $i - 1$.

Example 5.5.0.9. Let Δ be a Lie incidence geometry of type $\mathsf{B}_{n,i}(\mathbb{K})$, with $i \geq 1$, $n \geq 2i$, or any Lie incidence geometry obtained from the i-Grassmannian of a polar space of rank $n \geq 2i$. Set $J = \{i\}$ and $J' = \{2i\}$. Then the (J, J')-trace geometry is an equator geometry and isomorphic to $\mathsf{B}_{n-i,i}(\mathbb{K})$, or the Lie incidence geometry obtained from the i-Grassmannian of a polar space of rank $n - i$, respectively.

The case $i = 2$ in the previous example is a long root geometry. More generally, we have the following proposition.

Proposition 5.5.0.10. *Let Δ be a long root geometry $X_{n,J}(\mathbb{K})$, with $n \geq 4$. Let Y_m be the connected Dynkin diagram of longest length m, obtained by removing the vertices of types J from the Dynkin diagram X_n. Let J' be, such that $Y_{m,J'}(\mathbb{K})$ is a long root geometry. Then there exists a (J, J')-equator geometry, and it is isomorphic to $Y_{m,J'}(\mathbb{K})$.*

REFERENCE. This is [**VV19**, Proposition 4.7]. □

Corollary 5.5.0.11. *It follows that*

- *the $(8,1)$-equator geometry of $\mathsf{E}_{8,8}(\mathbb{K})$ is $\mathsf{E}_{7,1}(\mathbb{K})$,*
- *the $(1,6)$-equator geometry of $\mathsf{E}_{7,1}(\mathbb{K})$ is $\mathsf{D}_{6,2}(\mathbb{K})$,*
- *the $(2,\{1,6\})$-equator geometry of $\mathsf{E}_{6,2}(\mathbb{K})$ is $\mathsf{A}_{5,\{1,5\}}(\mathbb{K})$,*
- *and the $(1,4)$-equator geometry of $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{K})$ is $\mathsf{C}_{3,1}(\mathbb{K})$.*

Facts 5.5.0.12.

- The $(4,1)$-equator geometry of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$ is $\mathsf{B}_{3,1}(\mathbb{K})$ (cf. [**DSV18**, §5]).
- The $(1,3)_4$-equator geometry of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$ is $\mathsf{C}_{3,2}(\mathbb{K})$ (cf. [**VV19**, §4.2]).
- The $(2,5)_1$-equator geometry of $\mathsf{E}_{6,1}(\mathbb{K})$ is $\mathsf{A}_{5,2}(\mathbb{K})$ (cf. [**VV19**, §4.3]).
- The $(3,\{1,4\})_2$-equator geometry of $\mathsf{A}_{5,2}(\mathbb{K})$ is $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$ (cf. [**VV19**, §4.3]).
- The $(1,2)_7$-equator geometry of $\mathsf{E}_{7,7}(\mathbb{K})$ is $\mathsf{D}_{6,6}(\mathbb{K})$ (cf. [**VV19**, §4.4]).
- The $(5,3)_6$-equator geometry of $\mathsf{D}_{6,6}(\mathbb{K})$ is $\mathsf{A}_{5,3}(\mathbb{K})$ (cf. [**VV19**, §4.4]).
- The $(7,2)_1$-equator geometry of $\mathsf{E}_{7,1}(\mathbb{K})$ is $\mathsf{E}_{6,2}(\mathbb{K})$ (cf. [**VV19**, §4.4]).
- The $(2,\{1,8\})_8$-equator geometry of $\mathsf{E}_{8,8}(\mathbb{K})$ is $\mathsf{A}_{7,\{1,7\}}(\mathbb{K})$ (cf. [**VV19**, §4.5]).
- The $(1,3)_8$-equator geometry of $\mathsf{E}_{8,8}(\mathbb{K})$ is $\mathsf{D}_{7,2}(\mathbb{K})$ (cf. [**VV19**, §4.5]).

We will now define *extended equator geometries*, which we will mention again in the proofs of Proposition 7.5.0.1, Theorem 8.3.0.3, Lemma 10.9.0.5, and Lemma 10.9.0.7. They are only defined for metasymplectic spaces $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$. For details we refer to [**LV25**, §2.6].

Definition 5.5.0.13 (**Extended equator geometry**)**.** Let p, q be two opposite points of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$. Then, denoting $\widehat{E}(p,q)$ as the point-line geometry with point set

$$\bigcup \{E(x,y) \mid x,y \in E(p,q),\ x \text{ opposite } y\},$$

and line set consisting of all hyperbolic lines (cf. 5.3.1.1) contained in this point set, we say that $\widehat{E}(p,q)$ is the *extended equator geometry* of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ (cf. [**LV25**, Definition 2.6.12]).

Note 5.5.0.14. Alternatively, for $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$, we can define extended equator geometries as follows. Let E be a $(4,1)$-equator geometry of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$. Then the union of the sets $T(p,q;\{1\})$, for opposite p and q ranging over E, is an extended equator geometry $\widehat{E}$, if we endow $\widehat{E}$ with all the lines contained in some $(4,1)$-equator geometry contained in it.

Fact 5.5.0.15. An extended equator geometry of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$ is isomorphic to $\mathsf{B}_{4,1}(\mathbb{K})$.

REFERENCE. This is [**DSV18**, Proposition 5.18]. □

5.5.1. More Properties of $\mathsf{E}_{8,8}$

Now we prove three more results about Lie incidence geometries of type $\mathsf{E}_{8,8}$ that we will need later, and make use of equator geometries in the proof of Lemma 5.5.1.17. The first lemma should be compared with Fact 4.6.2.19 to notice the similarity with the $\mathsf{E}_{7,7}$ case.

The content of this section is based on [**BSV25**].

Lemma 5.5.1.16. *Let Δ be a parapolar space of type $\mathsf{E}_{8,8}$. Let ξ and ξ' be two symps of Δ intersecting in a 6-space U. If L is a line intersecting ξ in a point x and ξ' in a point x', with $x \neq x'$, then the map $\zeta \mapsto \zeta \cap L$ defines a bijection between the set of symps containing U and the set of points on L.*

Proof. If there were a point u of U collinear to x but not to x', then the symp ξ' through x' and u would contain x, a contradiction. Hence there is a 7-space W containing L and a hyperplane H of U. A symp ζ containing U shares the 5-space H with W. A symp sharing a 5-space with a 7-space intersects that 7-space in a $6'$-space (cf. Section 4.7). Hence, ζ intersects W in a $6'$-space U_ζ. The latter has a unique point in common with L.

Conversely, let q be any point on L. Then the $6'$-space $\langle q, H\rangle$ is contained in a unique symp ζ, which clearly intersects L in q. □

Lemma 5.5.1.17. *Let Δ be a parapolar space of type $\mathsf{E}_{8,8}$. Let U and U' be two opposite 6-spaces of Δ. Let $\xi \supseteq U$ and $\xi' \supseteq U'$ be two symps that are not opposite. Then there exist $6'$-spaces $V \subseteq \xi$ and $V' \subseteq \xi'$ such that no point of $V \cup V'$ is opposite any point of $\xi \cup \xi'$. The set of points of ξ' symplectic to a given point of V forms a 6-space intersecting V' in a 5-space. Likewise, the set of points of ξ symplectic to a given point of V' forms a 6-space intersecting V in a 5-space. Also, every point x of $\xi \setminus V$ is far from ξ', and the unique point of ξ' symplectic to x is contained in U'. Likewise, every point x' of $\xi' \setminus V'$ is far from ξ, and the unique point of ξ symplectic to x' is contained in U.*

Proof. Since U is opposite U', the symp ξ is opposite some symp $\xi^* \supseteq U'$, and hence every point of U' is symplectic to a unique point of ξ. The set of points of ξ thus obtained is a $6'$-space V, since being symplectic defines an isomorphism between ξ and ξ^* by Lemma 4.7.0.10. Clearly, V is independent of ξ^*, and consequently it is the unique set of points of ξ symplectic to some point of U'. Likewise, let V' be the set of points of ξ' symplectic to some point of U. Pick a point $v \in V$. Let u' be the unique point of U' symplectic to v. Let u be an arbitrary point of U not collinear to v. Then u and u' are opposite. Let $E(u, u')$ be the equator geometry. Then $E(u, u')$ consists of all points symplectic to both u and u', and with the induced line set, it is a Lie incidence geometry of type $\mathsf{E}_{7,1}$. Note $v \in E(u, u')$. Let v' be the unique point of V' symplectic to u, and note $v' \in E(u, u')$. The set of symps through U corresponds to a line $L \ni v$ in $E(u, u')$; likewise, the set of symps through U' corresponds to a line $L' \ni v'$ in $E(u, u')$. Since U and U' are opposite, the lines L and L' are opposite. It follows that the points v and v' are special. Since u' is not collinear to v', Lemma 4.7.0.10(*iii*) implies that the point v is not opposite any point of ξ'. So, we have shown that no point of V is opposite any point of ξ'. Suppose

now some point $x \in \xi \setminus V$ is not opposite every point of ξ'. Since ξ is hyperbolic, x is contained in a line K which intersects both U and V non-trivially. It then follows that also $K \cap U$ is not opposite every point of ξ', a contradiction. Hence V is precisely the set of points of ξ not opposite any point of ξ'. Likewise, V' is exactly the set of points of ξ' not opposite any point of ξ. It follows from Fact 4.7.0.3 that each point of V is symplectic to each point of a 6-space U'_v of ξ'. We now show that $U'_v \cap V'$ is a 5-space.

To that aim, let $v \perp z \perp v'$, and note $z \in E(u, u')$. Let ζ be the symp through u and z, and let ζ' be the symp through u' and z. Let Z be the intersection of ξ and ζ, and let Z' be the intersection of ξ' and ζ'. Set $Z_0 = z^\perp \cap Z$ and $Z'_0 = z^\perp \cap Z'$. By Fact 4.7.0.9, each point of Z_0 is symplectic to all points of at least a 4-space of Z'_0, and hence $Z_0 \subseteq V$. Likewise, $Z'_0 \subseteq V'$.

We now also deduce that each point of V (for which we can take v again without loss of generality) is symplectic to all points of a 5-space W' of ξ', where $u' \in W'$ and $\dim(W' \cap V') = 4$. Let U'_v be the unique 6-space containing W', then it follows from Fact 4.7.0.3(iv) that v is symplectic to all points of U'_v, and moreover, by the properties of hyperbolic polar spaces, $V' \cap U'_v$ is 5-dimensional. Likewise, v' is symplectic to each point of a 6-space $U_{v'}$ of ξ which intersects V in a 5-space.

We have already noted that every point x of $\xi \setminus V$ is far from ξ'. By the properties of hyperbolic polar spaces, x is contained in a 6-space intersecting U in a point (for which we can take u without loss of generality) and V in a 5-space. Hence $x \in U_{v'}$, and the last assertions follow. □

Lemma 5.5.1.18. *Let Δ be a parapolar space of type $\mathsf{E}_{8,8}$. Let U and U' be two opposite singular 4-spaces of Δ, and let W and W' be two singular 7-spaces of Δ containing U and U', respectively, which are not mutually opposite. Then there exist unique planes $\alpha \subseteq W$ and $\alpha' \subseteq W'$ such that no point of α is opposite any point of W', and no point of α' is opposite any point of W.*

Proof. Pick $x \in U$ and $x' \in U'$ opposite. Set $U^* = \mathsf{proj}_x^{x'}(U)$ and $W^* = \mathsf{proj}_x^{x'}(W)$ (cf. Fact 2.5.0.2). We interpret U, W, U^*, and W^* as 3- and 6-spaces in the residue at x, which is a geometry of type $\mathsf{E}_{7,7}$. Then, using Proposition 2.5.0.4, the assertion is equivalent to showing that there exists a plane in W no point of which is opposite any point of W^*. Choosing opposite points in $U \cup U^*$, the same argument reduces the assertion to showing in the geometry of type $\mathsf{E}_{6,6}$ over the field $\mathbb{K}$, for given disjoint but not opposite 5-spaces W and W^{**}, there exist planes $\alpha \subseteq W$ and $\alpha^{**} \subseteq W^{**}$ with the property that each point of W is collinear to any point of α^{**}, and each point of W^{**} is collinear to some point of α. But this follows from Proposition 4.5.0.12(2'). □

5.6. Models for Apartements

The content of this section is largely based on the article *Some Combinatorial and Geometric Constructions of Spherical Buildings* by Hendrik Van Maldeghem and

Magali Victoor (cf. [**VV19**]). In this section, we will use the same notation as in [**VV19**], and refer to the Coxeter complex of type X_n as $\mathsf{X}_n(1)$.

5.6.1. Apartments of type E_6

To clarify the structure of an E_6-apartment, different models will be described.

We begin by describing a construction of the thin Lie incidence geometry $\mathsf{E}_{6,1}(1)$ considered as a graph. For that we first consider the unique generalized quadrangle of order $(2,4)$, denoted $\mathsf{GQ}(2,4)$.

An explicit construction of $\mathsf{GQ}(2,4)$ runs as follows (cf. [**PT84**, §6.1]).

Construction 5.6.1.1 ($\mathsf{GQ}(2,4)$). Let $\mathscr{P}'$ be the set of all 2-subsets of the 6-set $\{1,2,3,4,5,6\}$, and define

$$\mathscr{P} = \mathscr{P}' \cup \{1,2,3,4,5,6\} \cup \{1',2',3',4',5',6'\}.$$

For the 2-subset $\{i,j\}$ we will write ij for short, for all appropriate i,j. Let $\mathscr{L}'$ be the set of partitions of $\{1,2,3,4,5,6\}$ into 2-subsets and define

$$\mathscr{L} = \mathscr{L}' \cup \{\{i,j',ij\} \mid i,j \in \{1,2,3,4,5,6\}, i \neq j\}.$$

Then $\Gamma = (\mathscr{P}, \mathscr{L})$ is a model of $\mathsf{GQ}(2,4)$.

The sets $\{1,2,3,4,5,6\}$ and $\{1',2',3',4',5',6'\}$ have the property that they both do not contain any pair of collinear points, and that non-collinearity is a *pairing*[3] between the two sets. Such a pair of 6-sets is usually called a *double six*.

Define the following set $\mathscr{S}$ of lines of $\mathsf{GQ}(2,4)$.

$$\begin{aligned}\mathscr{S} \;=\; & \{\{14,25,36\},\{15,26,34\},\{16,24,35\},\\ & \{12,2,1'\},\{23,3,2'\},\{13,1,3'\},\\ & \{45,4,5'\},\{56,5,6'\},\{46,6,4'\}\}.\end{aligned}$$

Then $\mathscr{S}$ is a spread.

The complement of the collinearity graph of $\mathsf{GQ}(2,4)$ (cf. Fig. 1[4]) is exactly the thin Lie incidence geometry $\mathsf{E}_{6,1}(1)$ considered as a graph. This graph is called the *Schläfli graph* (cf. [**DTV25**, §9.1.2]), and we provide a picture in Fig. 2[5], and an explicit construction in 5.6.1.2.

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[3]That just means that non-collinearity defines a natural bijection between the two sets: for every vertex v of the one set, there is a unique vertex of the other set not collinear to v. Every vertex is *paired* with a unique vertex of the other set.

[4]Claudio Rocchini created an SVG file for the collinearity graph of the $\mathsf{GQ}(2,4)$ generalized quadrangle, CC-BY 4.0, `https://creativecommons.org/licenses/by/4.0/`, which can be found on his homepage [**Roc10**], and which we used and slightly adjusted for Fig. 1 by coloring all vertices black.

[5]Claudio Rocchini created a drawing of the Schläfli graph in 2010, which can be found on his homepage (cf. [**Roc10**]), and which we aimed to roughly recreate here using TikZ. For that, it was helpful to consider Figure 1 of [**APS12**].

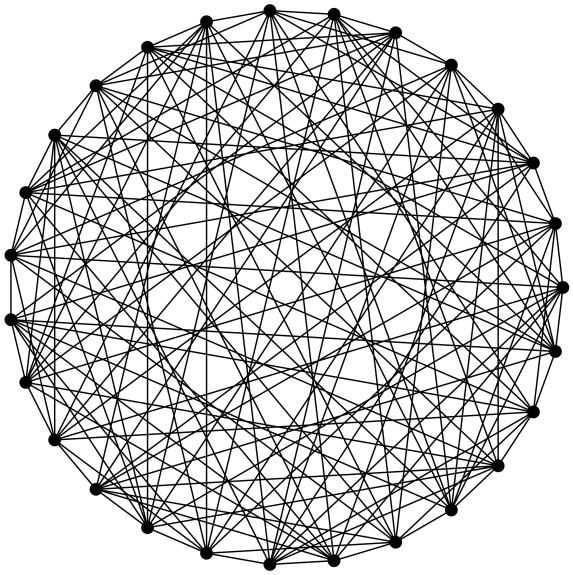


FIG. 1. The collinearity graph of $\mathsf{GQ}(2,4)$.

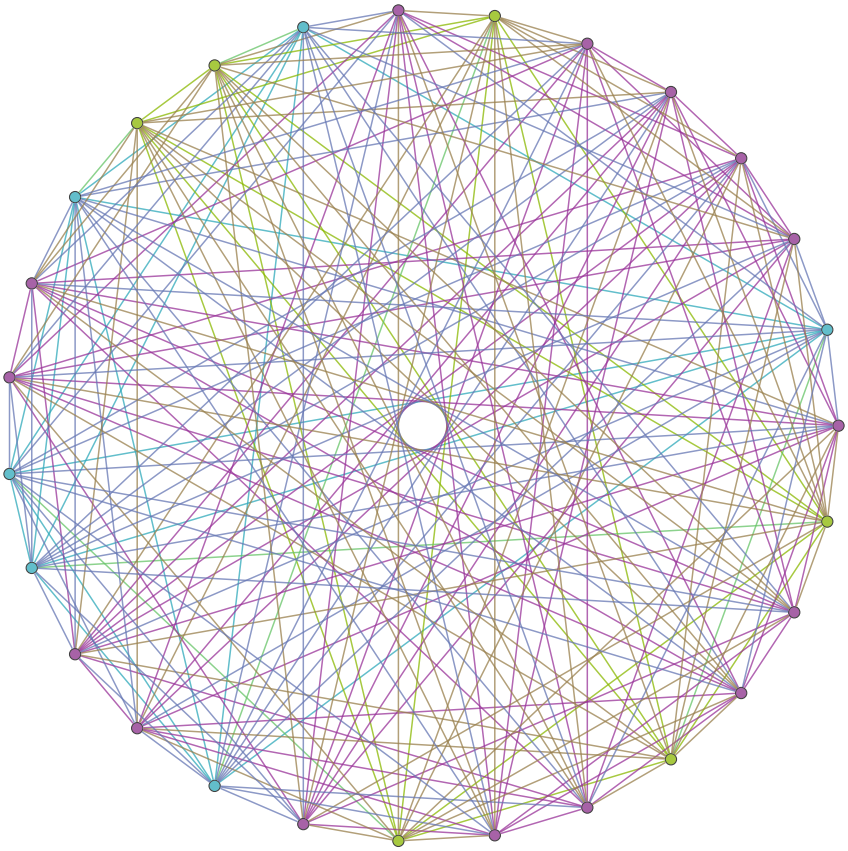


FIG. 2. **The Schläfli graph.** Using the notation of Construction 5.6.1.2, we draw blue vertices for every element of N, green vertices for every element of N', and purple vertices for every element of $\binom{N}{2}$. Edges are drawn according to the relation $\sim$ of Construction 5.6.1.2.

Construction 5.6.1.2 (**Schläfli graph**).
Let $N = \{1,2,3,4,5,6\}$, $N' = \{1',2',3',4',5',6'\}$ and $V = N \cup N' \cup \binom{N}{2}$, where $\binom{N}{2}$

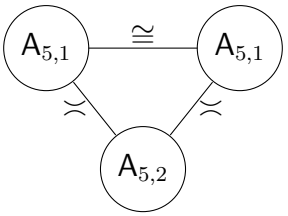
is the set of al 2-subsets of N. Let adjacency be given by $a \sim b$, $a' \sim b'$, $a \sim a'$ and $a \sim \{b, c\} \sim a'$ and $\{a, b\} \sim \{a, c\}$, for all distinct $a, b, c \in N$.

As described in [**VV19**, §7.1], the previous example for a model of an E_6-apartment can be expressed in a more systematic and general way as follows.

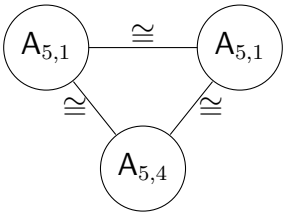
First, note that the graphs induced on N and N' are complete graphs; more precisely, they are isomorphic to $\mathsf{A}_{5,1}(1)$. The graph induced on $\binom{N}{2}$ is isomorphic to $\mathsf{A}_{5,2}(1)$. The labeling using $1, 2, \ldots$ and $1', 2', \ldots$ corresponds to the choice of an isomorphism between the various underlying $\mathsf{A}_5(1)$. To determine adjacency between vertices of N and N', we use the chosen isomorphism and connect corresponding vertices. We say that *adjacency is induced by natural isomorphism* (denoted by the symbol $\cong$).

A vertex a of N is adjacent to a vertex $\{b, c\}$ of $\binom{N}{2}$ if $\{b, c\}$, regarded as a line of $\mathsf{A}_5(1)$, is incident with the hyperplane of $\mathsf{A}_5(1)$ obtained from a by applying the natural duality $x \leftrightarrow N \setminus \{x\}$. We say that *adjacency is induced by natural duality* (denoted by the symbol $\asymp$).

We can picture the construction as follows.



This way of writing the construction is not unique and we will depict another option in the following.



This diagram represents three disjoint graphs, namely two copies of $\mathsf{A}_{5,1}(1)$ and one copy of $\mathsf{A}_{5,4}(1)$. A standard thin building of type A_5 is fixed by defining the set of elements of type 1 as $\{a, b, c, d, e, f\}$. The vertices of each copy of $\mathsf{A}_{5,1}(1)$ are precisely a, b, c, d, e, f, while the vertices of the graph $\mathsf{A}_{5,4}(1)$ are the 4-subsets of $\{a, b, c, d, e, f\}$. Adjacency between an element $x \in \{a, b, c, d, e, f\}$ of a copy of $\mathsf{A}_{5,1}(1)$ and an element $X \subseteq \{a, b, c, d, e, f\}$ with $|X| = 4$ of $\mathsf{A}_{5,4}(1)$ is defined by $x \in X$. Adjacency between an element $x \in \{a, b, c, d, e, f\}$ of one copy of $\mathsf{A}_{5,1}(1)$ and an element $y \in \{a, b, c, d, e, f\}$ of the other copy of $\mathsf{A}_{5,1}(1)$ is defined by $x = y$.

In this situation the "$\cong$" symbols can be omitted, and this convention will be adopted. In this thin geometry it is straightforward to observe that $\mathsf{A}_{5,1}(1)$ is the $(2, 1)_1$-trace geometry of $\mathsf{A}_{5,2}(1)$, and that $\mathsf{A}_{5,2}(1)$ is the $(2, 5)_1$-equator geometry of $\mathsf{E}_{6,1}(1)$.

A common feature of the constructions of Coxeter complexes in this section is that the various parts appear as trace and equator geometries. Exceptions occur, however, at least if no generalization of the notion of trace geometry with respect to opposite flags of different types is considered. An example is given below.

Example 5.6.1.3. Let $\mathsf{D}_{5,1}(1)$ be the thin polar space of rank 5, i.e., the complete graph on the vertex set

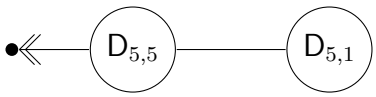
$$\{1, 2, 3, 4, 5, 1', 2', 3', 4', 5'\}$$

with the matching

$$\{(1, 1'), (2, 2'), (3, 3'), (4, 4'), (5, 5')\}$$

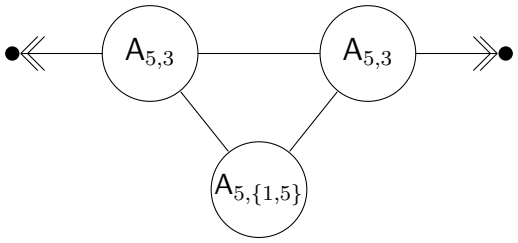
removed. Let $\mathsf{D}_{5,5}(1)$ be the thin half-spin geometry with vertex set consisting of the 5-cliques of $\mathsf{D}_{5,1}(1)$ that intersect $\{1, 2, 3, 4, 5\}$ in an odd number of vertices, with adjacency defined by intersection in three vertices. Define Γ to be the graph with vertex set consisting of a symbol ∞, the vertices of $\mathsf{D}_{5,1}(1)$, and the vertices of $\mathsf{D}_{5,5}(1)$, with adjacency as follows. The vertex ∞ is adjacent to every vertex of $\mathsf{D}_{5,5}(1)$; adjacency within the vertex sets of $\mathsf{D}_{5,1}(1)$ and $\mathsf{D}_{5,5}(1)$ is the natural one; and a vertex x of $\mathsf{D}_{5,1}(1)$ is adjacent to a vertex $\{a, b, c, d, e\}$ of $\mathsf{D}_{5,5}(1)$ if $x \in \{a, b, c, d, e\}$. Then Γ is isomorphic to $\mathsf{E}_{6,1}(1)$.

Again, the adjacency between $\mathsf{D}_{5,1}(1)$ and $\mathsf{D}_{5,5}(1)$ can be regarded as incidence defined by the natural isomorphism arising from the labeling of the points of $\mathsf{D}_{5,1}(1)$. This construction is illustrated as follows.

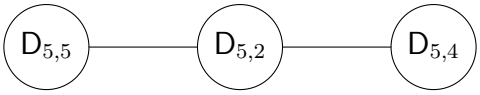


For $\mathsf{E}_{6,2}(1)$, the underlying structure is a thin long root geometry. All thin long root geometries display similar behavior. Consequently, the same pattern described here also arises in $\mathsf{E}_{7,1}(1)$, $\mathsf{E}_{8,8}(1)$, and $\mathsf{F}_{4,1}(1)$. A characteristic property of long root geometries is the existence of equator geometries, and this feature is reflected in the construction of the corresponding apartments.

The first construction is obtained by using trace and equator geometries with respect to two opposite vertices.



The second construction is obtained by using trace and equator geometries with respect to an opposite pair of half-spin geometries of rank 4, i.e., opposite elements of type 1 and 6.



5.6.2. Apartments of type E_7

The thin Lie incidence geometry $\mathsf{E}_{7,7}(1)$, when viewed as a graph, is isomorphic to the *Gosset graph*, which we show in Fig. 3. Several constructions of this graph exist. We begin with one similar to Construction 5.6.1.2.

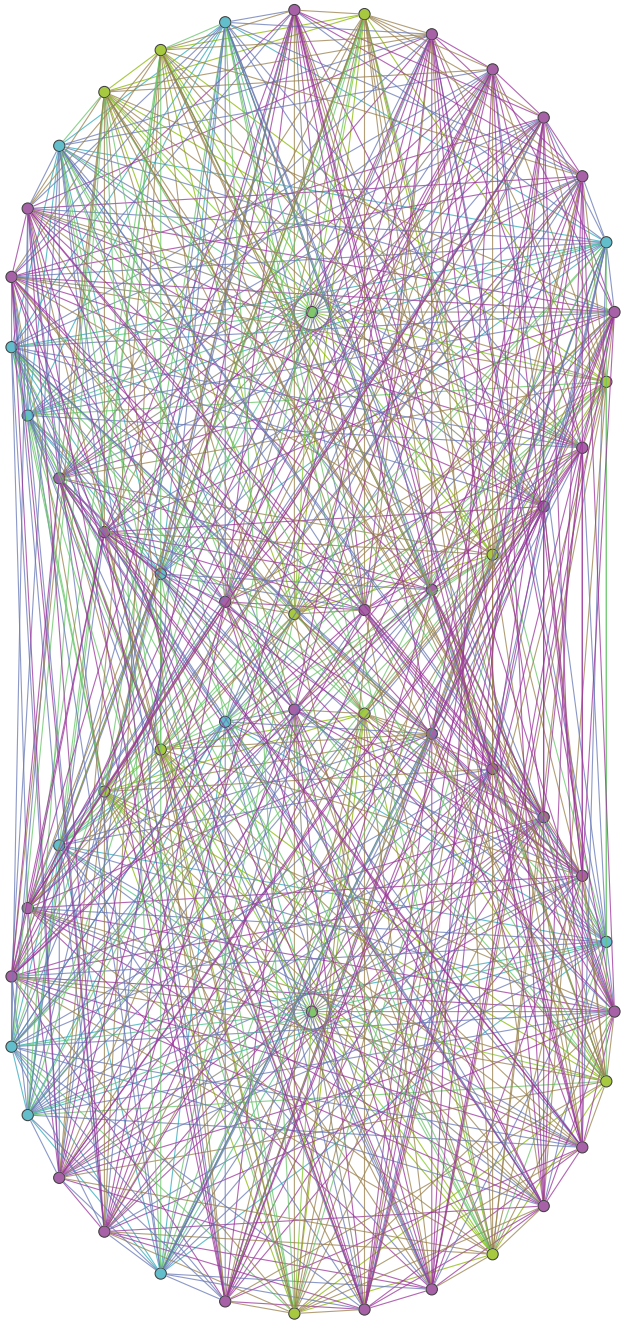


Fig. 3. **The Gosset graph.**

Construction 5.6.2.4. Let $N = \{n_1, n_2, \ldots, n_6\}$, $N' = \{n'_1, n'_2, \ldots, n'_6\}$, and

$$\binom{N}{2} = \{s_{ij} := \{n_i, n_j\} \mid n_i, n_j \in N,\ i \neq j\}.$$

Similarly, let $M = \{m_1, m_2, \ldots, m_6\}$, $M' = \{m'_1, m'_2, \ldots, m'_6\}$, and

$$\tbinom{M}{2} = \{t_{ij} := \{m_i, m_j\} \mid m_i, m_j \in M,\ i \neq j\}.$$

Put $V = N \cup N' \cup \binom{N}{2} \cup M \cup M' \cup \binom{M}{2} \cup \{x\} \cup \{y\}$, where x and y are just supposed to be two other vertices not contained in any of the aforementioned sets.

We will now describe which vertices are adjacent.

(1) For $i, j, k \in \{1, 2, \ldots, 6\}$ with $i \neq j$, $i \neq k$, $j \neq k$, we have:
- $n_i \sim n_j$, and $m_i \sim m_j$,
- $n'_i \sim n'_j$, and $m'_i \sim m'_j$,
- $n_i \sim n'_i$, and $m_i \sim m'_i$,
- $n_i \sim (\{n_j, n_k\} = s_{jk}) \sim n'_i$, and $m_i \sim (\{m_j, m_k\} = t_{jk}) \sim m'_i$,
- $s_{ij} = \{n_i, n_j\} \sim \{n_i, n_k\} = s_{ik}$, and $t_{ij} = \{m_i, m_j\} \sim \{m_i, m_k\} = t_{ik}$.

Like that we obtain two copies of the Schläfli graph.

(2) The vertex x is adjacent to all vertices $N \cup N' \cup \binom{N}{2}$ (i.e., to all vertices of one Schläfli graph).

(3) The vertex y is adjacent to all vertices $M \cup M' \cup \binom{M}{2}$ (i.e., to all vertices of the other Schläfli graph).

(4) n_i is adjacent to m'_j if and only if n_i and n'_j are not adjacent. Since n_i and n'_j are non-adjacent precisely when $i \neq j$, we have:
$$n_i \sim m'_j \quad \Longleftrightarrow \quad i \neq j.$$

(5) n_i is adjacent to t_{jk} if and only if n_i and s_{jk} are not adjacent. The vertices n_i and s_{jk} are non-adjacent exactly when $i = j$ or $i = k$, hence:
$$n_i \sim t_{jk} \quad \Longleftrightarrow \quad i \in \{j, k\}.$$

(6) n'_i is adjacent to m_j if and only if n'_i and n_j are not adjacent. The vertices n'_i and n_j are non-adjacent precisely when $i \neq j$, so:
$$n'_i \sim m_j \quad \Longleftrightarrow \quad i \neq j.$$

(7) n'_i is adjacent to t_{jk} if and only if n'_i and s_{jk} are not adjacent. The vertices n'_i and s_{jk} are non-adjacent exactly when $i = j$ or $i = k$, thus:
$$n'_i \sim t_{jk} \quad \Longleftrightarrow \quad i \in \{j, k\}.$$

(8) s_{ij} is adjacent to m_k if and only if s_{ij} and n_k are not adjacent. The vertices s_{ij} and n_k are non-adjacent precisely when $k = i$ or $k = j$, giving:
$$s_{ij} \sim m_k \quad \Longleftrightarrow \quad k \in \{i, j\}.$$

(9) s_{ij} is adjacent to m'_k if and only if s_{ij} and n'_k are not adjacent. The vertices s_{ij} and n'_k are non-adjacent exactly when $k = i$ or $k = j$, hence:
$$s_{ij} \sim m'_k \quad \Longleftrightarrow \quad k \in \{i, j\}.$$

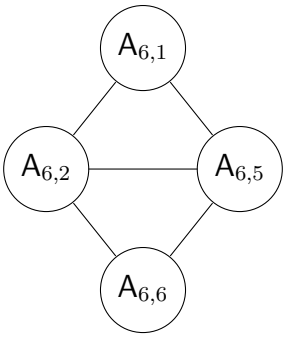
(10) s_{ij} is adjacent to t_{kl} if and only if s_{ij} and s_{kl} are not adjacent. Two s-vertices satisfy s_{ij} non-adjacent to s_{kl} precisely when their index sets are disjoint:
$$s_{ij} \sim t_{kl} \quad \Longleftrightarrow \quad \{i, j\} \cap \{k, l\} = \varnothing.$$

This conculdes the first construction.

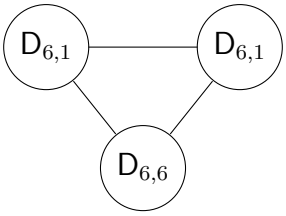
The following more systematic depictions are again based on [**VV19**], and the next one is obtained with respect to two opposite vertices. In this case, two trace geometries appear, neither of which is an equator geometry.

A similar construction exists for the thin Lie incidence geometry $\mathsf{D}_{6,6}(1)$ as follows.

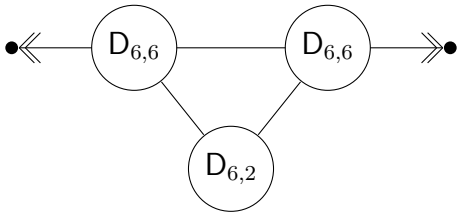
A third construction of $\mathsf{E}_{7,7}(1)$ is obtained with respect to two opposite elements of type 2, yielding two subgeometries isomorphic to $\mathsf{A}_{6,1}(1)$. This leads to the following diagram.



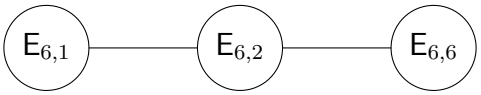
The next construction displays the $(1,2)_7$-equator geometry and is based on two opposite subgeometries isomorphic to $\mathsf{D}_{6,1}(1)$.



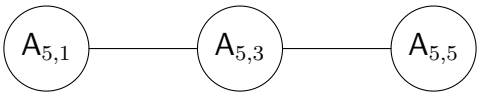
The geometry $\mathsf{E}_{7,1}(1)$, being a long root geometry, admits a canonical construction with respect to two opposite vertices. From this, both the residue (as a trace geometry) and the equator geometry can be identified.



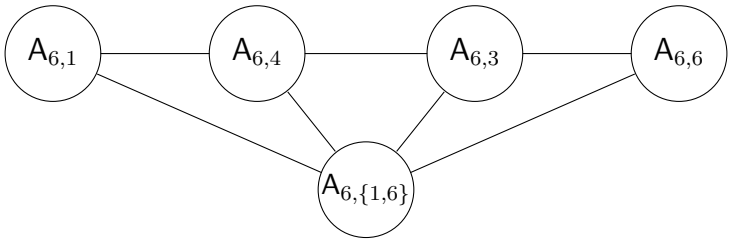
A second construction is obtained with respect to two subgeometries isomorphic to $\mathsf{E}_{6,1}(1)$. This reveals the $(7,2)_1$-equator geometry.



This construction closely resembles the construction of $\mathsf{E}_{6,2}(1)$ using $\mathsf{D}_{5,5}(1)$ and $\mathsf{D}_{5,2}(1)$ described earlier. Another case of the same shape will be encountered below in the discussion of apartments of type F_4. A further example is provided by a construction of $\mathsf{D}_{6,6}(1)$, obtained from two opposite thin projective 5-spaces, and depicted as follows.



From this construction, a third construction of the thin long root geometry $\mathsf{E}_{7,1}(1)$ can be derived.



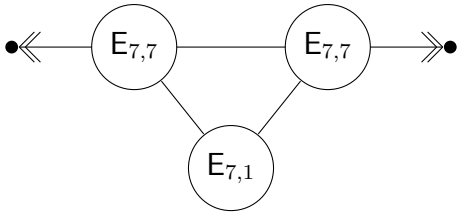
This suggests that $\mathsf{A}_{6,\{1,6\}}(\mathbb{K})$ should be regarded as the $(2,\{1,7\})_7$-equator geometry of $\mathsf{E}_7(\mathbb{K})$ for any field $\mathbb{K}$.

Finally, we mention a construction of $\mathsf{E}_{7,2}$ with 576 vertices. The simplest construction is based on two opposite elements of type 7, that is, two subgeometries isomorphic to a thin long root geometry of type E_6.

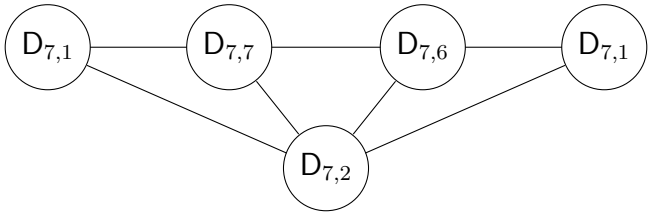
5.6.3. Apartments of type E_8

The focus here is on the thin Lie incidence geometry $\mathsf{E}_{8,8}(1)$. Three constructions are presented, one corresponding to each end node of the diagram.

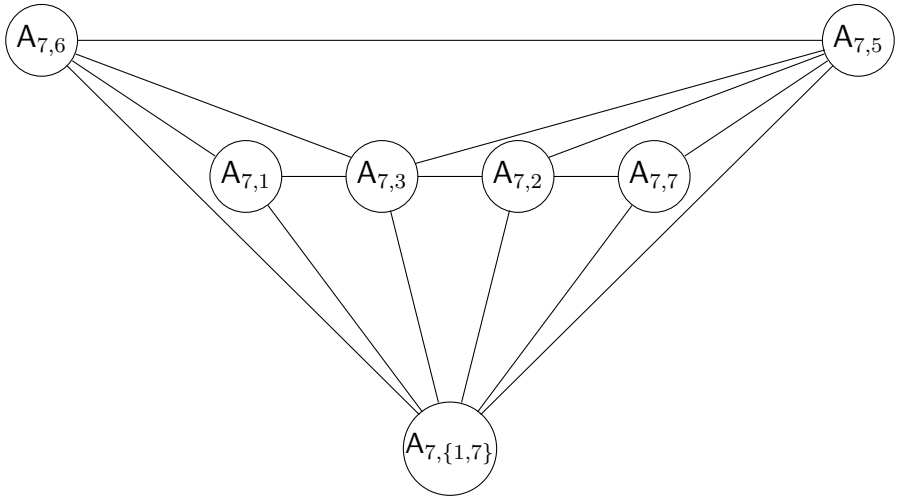
The first construction is obtained with respect to two opposite vertices (elements of type 8). This yields the standard long root configuration.



The second construction is obtained with respect to two opposite elements of type 1, that is, two opposite thin subgeometries isomorphic to $\mathsf{D}_{7,1}(1)$. This construction is analogous to the third construction of $\mathsf{E}_{7,1}(1)$ and is represented as follows.

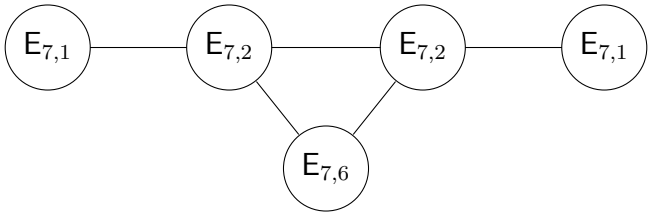


The third construction involves two opposite elements of type 2. This produces a configuration in which all geometries $\mathsf{A}_{7,i}(1)$ with $1 \leq i \leq 7$ appear, except that $\mathsf{A}_{7,4}(1)$ is replaced by the thin long root geometry $\mathsf{A}_{7,\{1,7\}}(1)$.



The resulting diagram is not entirely symmetric. There exists an isomorphism between $\mathsf{A}_{7,3}(1)$ and $\mathsf{A}_{7,5}(1)$, but no edges occur between $\mathsf{A}_{7,2}(1)$ and $\mathsf{A}_{7,6}(1)$.

For an arbitrary field $\mathbb{K}$, the Lie incidence geometry $\mathsf{E}_{8,1}(\mathbb{K})$ appears in several characterization theorems of parapolar spaces. The construction of $\mathsf{E}_{8,1}(1)$ given below provides insight into its structure (see [**VV19**] for further details). It is obtained with respect to two opposite elements of type 8, namely two opposite subgeometries isomorphic to $\mathsf{E}_{7,1}(1)$. This geometry contains 2160 vertices and has degree 64.



This suggests that $\mathsf{E}_{7,6}(\mathbb{K})$ is a $(1,6)_8$-equator geometry of $\mathsf{E}_8(\mathbb{K})$.

5.6.4. Apartments of type F_4

Turning to F_4, we note that the two end nodes of the F_4-diagram play analogous roles, but the arrow on the double bond breaks the symmetry. Nevertheless, the graphs $\mathsf{F}_{4,1}(1)$ and $\mathsf{F}_{4,4}(1)$ are isomorphic. For convenience, the focus is placed on $\mathsf{F}_{4,1}(1)$, since it corresponds to long root geometries.

We begin by showing an F_4-apartment viewed as a graph (Fig. 4).

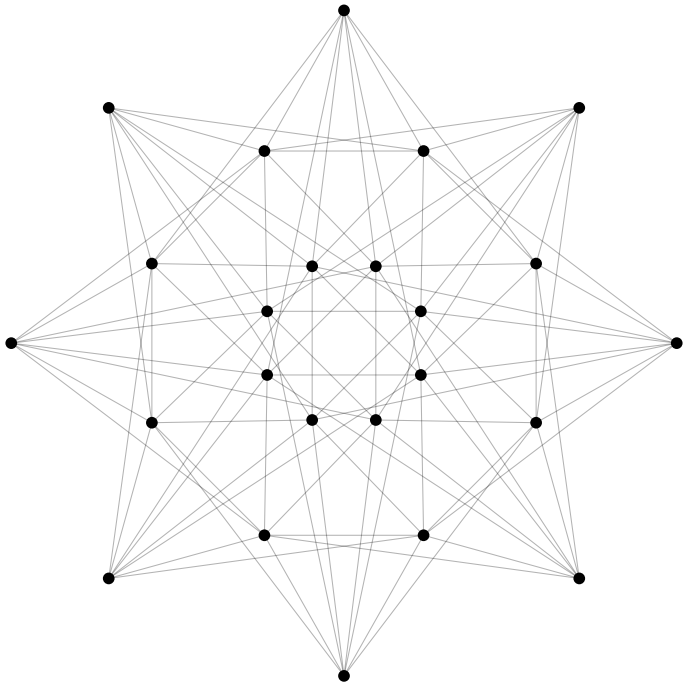


Fig. 4. An F_4-apartment viewed as a graph.

Before proceeding, we discuss a special feature of dealing with non-simply laced Dynkin diagrams, concerning the long root geometries of buildings of type B_n ($n \geq 2$) and C_n ($n \geq 3$). In general, a polar space is regarded as a Lie incidence geometry of type $\mathsf{B}_{n,1}$, and the corresponding long root geometry has type $\mathsf{B}_{n,2}$. The latter is a non-strong parapolar space of diameter 3. Strictly speaking, however, long root geometries are only well-defined for split buildings. In the case of a symplectic polar space — corresponding to type C_n — the long root geometry coincides with the polar space itself. All other long root geometries have diameter 3, except this one.

To resolve this, the long root geometry of a symplectic polar space is instead viewed as a geometry without lines, where points are at distance 2 or 3: collinear points in the polar space are at distance 2, and all others at distance 3. The lines of the polar space are then interpreted as the symplecta of the long root geometry. In this setting, the thin long root geometry $\mathsf{C}_{3,1}(1)$ is represented by an edgeless graph with six vertices, equipped with an opposition relation forming a matching.

This opposition relation allows well-defined isomorphisms between such structures and incidence-preserving maps to other Lie incidence geometries of the same main type. For instance, if the vertices of the graph are labeled $1, 2, 3, 1', 2', 3'$ with a

opposite a' for each $a \in \{1,2,3\}$, then the point set of $\mathsf{C}_{3,3}(1)$ is

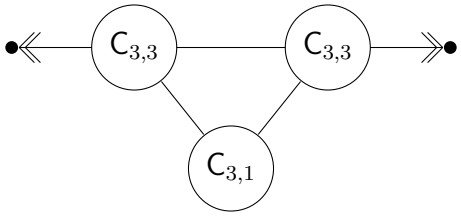
$$\{123, 1'23, 12'3, 123', 12'3', 1'23', 1'2'3, 1'2'3'\}.$$

The natural isomorphism maps

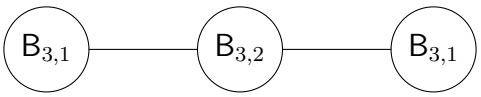
$$1 \text{ to } \{123, 123', 12'3, 12'3'\}, \quad \text{and} \quad 123 \quad \text{to} \quad \{1,2,3\}.$$

It is in this sense that the diagrams below should be interpreted.

With these conventions, the following construction of $\mathsf{F}_{4,1}(1)$ is obtained with respect to two opposite vertices, as a long root geometry.



With respect to two opposite symplecta (elements of type 4), the following construction arises:



Here, $\mathsf{B}_{3,2}(1)$ corresponds to the ordinary thin long root geometry of type B_3.

5.6.5. An Application

As an application of the preceding sections, we now prove the following lemma, which was previously invoked in Proposition 2.10.0.2 without proof.

Lemma 5.6.5.5. *Let α be a root of an irreducible spherical Coxeter complex $\Sigma(W,S)$. Let F be the center of α. Let v be a vertex joinable to a vertex u that is joinable to F. Then v lies in α.*

Proof. For the classical types A_n, B_n, and D_n, this can be verified directly. Indeed, for type A_n, viewing $\Sigma(W,S)$ as the point-line geometry of a projective space with two points per line, F is an incident point–hyperplane pair. Then v is either incident with the point or with the hyperplane, and the result follows. For types B_n and D_n, $\Sigma(W,S)$ can be seen as a polar space with two points per line. Two possibilities arise: if F is a point, then v is a subspace collinear to F, and the assertion follows; if F is a line, then v is either a subspace collinear to F or contains a point of F, and the assertion follows.

Now consider $\Sigma(W,S)$ of exceptional type. The rank 2 cases can be checked directly, so it will be assumed that the type is E_i $(i = 6,7,8)$, F_4, or H_i $(i = 3,4)$. The latter types are not essential here, as they do not correspond to thick buildings, and we leave them to the reader. For the remaining cases, $\Sigma(W,S)$ is represented as a thin long root subgroup geometry Γ, where points correspond to the long roots of the associated root system and lines (or edges, when viewed as a graph) correspond

to pairs of roots forming a 60° angle; see also [**BC13**]. We have depicted such geometries for all exceptional types in the previous subsections, using the results of [**VV19**].

In this representation, F corresponds to a point of Γ (for type F_4, one has to consider the analogous construction with short roots, which yields an isomorphic geometry). Furthermore, α is induced by all points corresponding to roots forming an angle of at most 90° with the root corresponding to F, i.e., by points collinear or symplectic to F.

Each vertex of $\Sigma(W, S)$ corresponds either to a singular subspace of Γ, to a symplecton of Γ, or to a convex subspace isomorphic to a Lie incidence geometry containing no special pairs and having diameter 2. If at least one of u or v is not a convex subspace distinct from a symplecton, then the assertion follows, since v is incident only with points collinear or symplectic to F.

The only case in which both u and v are convex subspaces arises for type E_6, where, up to duality, u is a vertex of type 1 and v of type 6 (the corresponding convex subspaces are geometries of type $\mathsf{D}_{5,5}$). Since u and v are incident, they share a symp, which may be assumed to be opposite F in the convex subspace u. One verifies that v contains four points collinear to F, eight points symplectic to F, and four points special to F. This is a symmetric configuration with respect to F and its opposite point; hence u lies in $\partial\alpha$. □

5.7. The Groups $\overline{\mathsf{PGO}}_{2r}(\mathbb{K})$, $\overline{\mathsf{P\Omega}}_{2r}(\mathbb{K})$, $\mathsf{PGE}_6(\mathbb{K})$ & $\mathsf{PGE}_7(\mathbb{K})$

In this section we introduce some notation and describe several groups that will be be used later in Chapter 10. For the types D_n, E_6, and E_7, we consider suitable vector spaces equipped with forms and define the notion of *similitudes* of these forms. We then introduce groups of similitudes and related groups derived from them, and establish several results concerning their generation.

For thick polar spaces and metasymplectic spaces, we will have to do something similar. However, since in these cases the relevant group-theoretic considerations are closely intertwined with the determination of the corresponding projectivity groups, we defer their discussion to Chapter 10.

The content of this section is based on [**BSV25**].

Definition 5.7.0.1 (**Similitudes**)**.** A *similitude* of a form is a linear transformation preserving the form up to a non-zero constant, and in case of a quadratic form, it is called *direct* if the form is hyperbolic over a splitting field and the similitude preserves each natural system of maximal singular subspaces.

5.7.1. Groups of type D_n

Notation 5.7.1.2. For a hyperbolic quadric Q of Witt index r, associated to the quadratic form $g\colon V \to \mathbb{K}$ with associated bilinear form $f\colon V \times V \to \mathbb{K}$ (cf. Definition A.3.0.4), we denote by $\overline{\mathsf{PGO}}_{2r}(\mathbb{K})$ the group of all elements of $\mathsf{PGL}_{2r}(V)$, preserving f and g up to a scalar multiple. It is the full linear algebraic group of automorphisms of Q, seen as a building of type D_r (cf. Definition 2.8.0.2). The unique subgroup of $\overline{\mathsf{PGO}}_{2r}(\mathbb{K})$ of index 2, preserving each class of maximal singular subspaces, will be denoted by $\overline{\mathsf{PGO}}^{\circ}_{2r}(\mathbb{K})$. It is elementary to see that $\overline{\mathsf{PGO}}_{2r}(\mathbb{K})$ is obtained from $\mathsf{PGO}_{2r}(\mathbb{K})$ by adjoining the appropriate *diagonal automorphisms*, that is, if we assume g in standard form (after introducing coordinates):

$$\begin{aligned} g\colon \mathbb{K}^{2r} \to \mathbb{K}\colon (x_{-r}, x_{-r+1}, \ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots, x_{r-1}, x_r) \\ \mapsto x_{-r}x_r + x_{-r+1}x_{r-1} + \cdots + x_{-2}x_2 + x_{-1}x_1, \end{aligned}$$

then we adjoin the linear automorphisms of Q induced by

$$\begin{aligned} \varphi_k\colon \mathbb{K}^{2r} \to \mathbb{K}^{2r}\colon (x_{-r}, x_{-r+1}, \ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots, x_{r-1}, x_r) \\ \mapsto (x_{-r}, x_{-r+1}, \ldots, x_{-2}, x_{-1}, kx_1, kx_2, \ldots, kx_{r-1}, kx_r), \end{aligned}$$

for all $k \in \mathbb{K}^\times$ (and we may assume k is not a square, as otherwise the given automorphism is already in $\mathsf{PGO}_{2r}(\mathbb{K})$). We denote the commutator subgroup of $\mathsf{PGO}^{\circ}_{2r}(\mathbb{K})$ by $\mathsf{P\Omega}_{2r}(\mathbb{K})$. The latter is the simple group $\mathsf{D}_r(\mathbb{K})$ of type D_r over the field $\mathbb{K}$ (cf. [**Die63**]). The group obtained from $\mathsf{P\Omega}_{2r}(\mathbb{K})$ by adjoining the diagonal automorphisms as above, is denoted by $\overline{\mathsf{P\Omega}}_{2r}(\mathbb{K})$.

If r is even and $\mathbb{K}$ is not quadratically closed, then $\overline{\mathsf{P\Omega}}_{2r}(\mathbb{K})$ does not coincide with $\overline{\mathsf{PGO}}^{\circ}_{2r}(\mathbb{K})$, as we will demonstrate later (cf. Remark 10.6.3.5).

We will call any automorphism of a hyperbolic quadric Q, that pointwise fixes two opposite maximal singular subspaces, a *homology of* Q. The automorphisms φ_k, $k \in \mathbb{K}^\times$ (mentioned above) are homologies. If r is even, then there are two types of such, due to the fact that there are two different kinds of maximal singular subspaces. If r is odd, then it is always the case that one maximal singular subspace of each type is fixed pointwise. We now have the following result, which can be proved using standard arguments, similarly to, but simpler than, Lemma 5.1.2.9, Lemma 5.7.2.6 and Lemma 5.7.3.10.

Lemma 5.7.1.3. *Let Q be a (non-degenerate) hyperbolic quadric of Witt index r, corresponding to the building of type D_r over the field $\mathbb{K}$. Then the following hold.*

- (*i*) *The set of all homologies generates $\overline{\mathsf{PGO}}^{\circ}_{2r}(\mathbb{K})$.*
- (*ii*) *If r is even, then the set of homologies, pointwise fixing two opposite maximal singular subspaces of only one given type, generates $\overline{\mathsf{P\Omega}}_{2r}(\mathbb{K})$.*
- (*iii*) *If r is even, then the homologies, pointwise fixing two opposite maximal singular subspaces of only one given type, and the elements of $\mathsf{PGO}^{\circ}_{2r}(\mathbb{K})$ together generate $\overline{\mathsf{PGO}}^{\circ}_{2r}(\mathbb{K})$.*
- (*iv*) *The set of all automorphisms fixing two opposite points p, q, and pointwise fixing $p^\perp \cap q^\perp$, generates $\mathsf{PGO}^{\circ}_{2r}(\mathbb{K})$.*

5.7.2. Groups of type E_6

We now introduce some notation concerning the exceptional groups of types E_6 and E_7. There does not seem to be standard notation (some people use $\tilde{E}$ and $\hat{E}$, others $\mathrm{SE}_6(\mathbb{K})$, for some of the following groups). The following is partly based on [**Ste20**].

Notation 5.7.2.4. Let V be a 27-dimensional vector space over the commutative field $\mathbb{K}$, written as the direct sum of three 1-dimensional subspaces and three 8-dimensional subspaces, each of them identified with a split octonion algebra $\mathbb{O}$ over $\mathbb{K}$. We thus write $V = \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O}$. Let $\mathfrak{C}\colon V \to \mathbb{K}$ be the cubic form defined as

$$\mathfrak{C}(x, y, z; X, Y, Z) = -xyz + xX\overline{X} + yY\overline{Y} + zZ\overline{Z} - (XY)Z - \overline{(XY)Z}.$$

Then we denote by $\mathsf{GE}_6(\mathbb{K})$ the *similitudes* of $\mathfrak{C}$, that is, the subgroup of $\mathsf{GL}(V)$ preserving $\mathfrak{C}$ up to a multiplicative constant. The subgroup of $\mathsf{GE}_6(\mathbb{K})$ preserving $\mathfrak{C}$ is denoted by $\mathsf{SE}_6(\mathbb{K})$ (it is a subgroup of $\mathsf{SL}(V)$), and the quotients with the respective centers (consisting of scalar matrices) are $\mathsf{PGE}_6(\mathbb{K})$ and $\mathsf{PSE}_6(\mathbb{K})$. The latter is also denoted briefly by $\mathsf{E}_6(\mathbb{K})$ and is simple. The group $\mathsf{PGE}_6(\mathbb{K})$ is the full linear group. The group obtained by adjoining a diagram automorphism is denoted by $\mathsf{PGE}_6(\mathbb{K}).2$.

The cubic form $\mathfrak{C}$ above can also be written without the use of octonions, but using the unique generalized quadrangle $\mathsf{GQ}(2, 4)$ of order $(2, 4)$ (cf. Construction 5.6.1.1). The following equivalent description of the cubic form $\mathfrak{C}$ can be found in Section 2 of [**VV22**].

Let V be the vector space of dimension 27 over $\mathbb{K}$, where the standard basis B is labelled, using the elements of $\mathscr{P}$, say $B = \{e_p \mid p \in \mathscr{P}\}$. We denote a generic vector $v \in V$ by $\sum_{p\in\mathscr{P}} x_p e_p$, with $x_p \in \mathbb{K}$. Then

$$\mathfrak{C}(v) = \sum_{\{p,q,r\}\in\mathscr{S}} x_p x_q x_r - \sum_{\{p,q,r\}\in\mathscr{L}\setminus\mathscr{S}} x_p x_q x_r.$$

The projective null set[6] of $\nabla\mathfrak{C}$ is a set of points, denoted $\mathscr{E}_6(\mathbb{K})$, and endowed with the lines contained in it. It is a point-line geometry isomorphic to the Lie incidence geometry of type $\mathsf{E}_{6,1}$ over the field $\mathbb{K}$, and called the *Schläfli variety* (cf. [**DTV25**, §9.1]).

Definition 5.7.2.5 (**Symplectic polarities in buildings of type E_6**)**.** Let ρ be an automorphism of $\mathsf{E}_6(\mathbb{K})$ interchanging the types 1 and 6, and 3 and 5. Then, by [**Van12**], the following conditions are equivalent.

- ρ does not map any vertex of type 2 to an opposite;
- ρ does not map any chamber to an opposite chamber;
- if the image under ρ of any vertex v of type 1 is not an opposite vertex, then it is joinable to v, and at least one vertex of type 1 is mapped onto a joinable one.

[6] projective null set: the points of the projective space such that the corresponding vectors become zero under the form.

Due to the main results of [**Van12**], if one of these equivalent conditions is satisfied, then ρ has order 2. In this case, we call it a *symplectic polarity*. The motivation for this terminology stems from the fact that it shares some properties with ordinary symplectic polarities in projective spaces, and that the polarity induced in the residue of any fixed vertex of type 2 is a symplectic polarity of a 5-dimensional projective space.

With these constructions and notations at hand, we are able to prove the following generation results.

Lemma 5.7.2.6. *The products of an even number of symplectic polarities of the building Δ of type E_6 over the field $\mathbb{K}$ generate the Chevalley group $\mathsf{PGE}_6(\mathbb{K})$. The symplectic polarities themselves generate $\mathsf{PGE}_6(\mathbb{K}).2$.*

Proof. We first claim that diagonal automorphisms φ necessarily have a ninth power as determinant. Indeed, φ acts as

$$\varphi\colon V \to V\colon \sum_{p\in\mathscr{P}} x_p e_p \mapsto \sum_{p\in\mathscr{P}} \lambda_p x_p e_p,$$

with $\lambda_p \in \mathbb{K}$. Then φ is a similitude of $\mathfrak{C}$ only if the product $\lambda_p\lambda_q\lambda_r =: \lambda$ is a constant across all lines $\{p,q,r\} \in \mathscr{L}$. Then the determinant of φ is obtained by multiplying this constant over the spread $\mathscr{S}$ and therefore the determinant equals λ^9. The claim is proved.

Now, a symplectic polarity σ of Δ induces a symplectic polarity in every fixed 5-space of the corresponding Lie incidence geometry Γ of type E_6. Since all symplectic polarities are conjugate, every symplectic polarity of a given 5-space extends to a symplectic polarity of Δ. By the strong transitivity of $\mathsf{Aut}\,\Delta$, we may even assume that two given opposite 5-spaces are fixed (and then, since the fixed building has type F_4 and its polar type corresponds to the fixed 5-spaces (as is apparent from [**DSV18**]), all 5-spaces in the span of the two given ones in $\mathsf{PG}(V)$ are fixed). Now, two opposite 5-spaces W and W' are given by the span of the base points, corresponding to the two respective 6-sets of a double six. One can see that the product of the symplectic polarities, corresponding to the symplectic forms

$$x_{-3}y_3 + x_{-2}y_2 + x_{-1}y_1 - x_1y_{-1} - x_2y_{-2} - x_3y_{-3},$$
$$\text{and } \lambda x_{-3}y_3 + x_{-2}y_2 + x_{-1}y_1 - x_1y_{-1} - x_2y_{-2} - \lambda x_3y_{-3},$$

for $\lambda \in \mathbb{K}$, corresponds to a diagonal collineation of $\mathsf{PG}(5,\mathbb{K})$ with diagonal

$$(\lambda, 1, 1, 1, 1, \lambda).$$

Letting these coordinates correspond naturally to the bases

$$(e_1, \dots, e_6) \text{ and } (e_{1'}, \dots, e_{6'})$$

of the subspaces of V, corresponding to W and W', respectively, we first derive that the product θ of the corresponding symplectic polarities of Δ acts on $\langle W, W'\rangle$ as

$$(x_1, x_2, \dots, x_6, x_{1'}, x_{2'}, \dots, x_{6'}) \mapsto (\lambda x_1, x_2, x_3, x_4, x_5, \lambda x_6, \lambda x_{1'}, x_{2'}, x_{3'}, x_{4'}, x_{5'}, \lambda x_{6'}).$$

Secondly, since each point $\langle e_{ij}\rangle$ is the unique point of Γ collinear to all $\langle e_\ell\rangle$ except for $\langle e_i\rangle$ and $\langle e_j\rangle$, and to all $\langle e_{\ell'}\rangle$ except for $\langle e_{i'}\rangle$ and $\langle e_{j'}\rangle$ (as follows from Lemma 3.5 in [**DV23**]), we see that θ is a diagonal automorphism. Now one calculates that θ is uniquely determined by its restriction to $\langle W, W'\rangle$ and maps e_{ij} to e_{ij}

if $|\{i,j\} \cap \{1,6\}| = 1$, to λe_{ij} if $\{i,j\} \cap \{1,6\} = \varnothing$, and to $\lambda^{-1} e_{ij}$ if $\{i,j\} = \{1,6\}$. Correspondingly, the determinant of θ is λ^9. It follows that the diagonal automorphisms generate non-trivial elements of $\mathsf{PSE}_6(\mathbb{K})$. Since the latter is simple, and since the subgroup of $\mathsf{E}_6(\mathbb{K}).2$, generated by all symplectic polarities of Δ, is normal, the group generated by arbitrary products of an even number of symplectic polarities contains $\mathsf{PSE}_6(\mathbb{K})$. Since it also contains all diagonal automorphisms by the above, the assertions follow. □

Lemma 5.7.2.7. *With the notation of* Section 2.9.1*, we have* $\mathsf{E}_6(\mathbb{K}).T_1 = \mathsf{PGE}_6(\mathbb{K})$.

Proof. A chamber C of a building of type E_6 over $\mathbb{K}$ is given by a 6-tuple of pairwise incident elements of the Lie incidence geometry of type $\mathsf{E}_{6,1}$ over $\mathbb{K}$ described above, consisting of a point, a line, a plane, a 5-space, a 4-space intersecting the 5-space in a 3-space, and a hyperplane of the 5-space. In the above description, we can take, with obvious notation,

$$C = (\langle e_1\rangle, \langle e_1, e_2\rangle, \langle e_1, e_2, e_3\rangle, \langle e_1, e_2, \ldots, e_6\rangle, \langle e_1, \ldots, e_4, e_{\{5,6\}}\rangle, \langle e_1, \ldots, e_5\rangle).$$

Denote $W = \langle e_1, \ldots, e_6, e_{\{5,6\}}\rangle$. An arbitrary diagonal automorphism of W inducing an element of T_1, that is, acting trivially on the rank-1 residues defined by C except for the type 1 rank-1 residue, is given by

$$\begin{cases} e_1 \mapsto \lambda e_1, & \\ e_i \mapsto e_i, & i = 2,3,4,5,6, \\ e_{\{5,6\}} \mapsto e_{\{5,6\}}. & \end{cases}$$

One can check that this can be extended to a unique diagonal automorphism preserving the cubic form $\mathfrak{C}$ by defining

$$\begin{cases} e_{1'} \mapsto e_{1'}, & \\ e_{i'} \mapsto \lambda^{-1} e_{i'}, & i = 2,3,4,5,6, \\ e_{\{i,j\}} \mapsto e_{\{i,j\}}, & 1 \notin \{i,j\} \subseteq \{1,2,\ldots,6\}, i \neq j, \\ e_{\{1,j\}} \mapsto \lambda^{-1} e_{\{1,j\}}, & j = 2,3,4,5,6. \end{cases}$$

Now one calculates that the determinant of the corresponding diagonal matrix is λ^{-9}, and the result follows similarly as in the proof of Lemma 5.7.2.6 above. □

5.7.3. Groups of type E_7

Notation 5.7.3.8. Let V be a 56-dimensional vector space over the commutative field $\mathbb{K}$, written as the direct sum of two 1-dimensional subspaces and two 27-dimensional subspaces. We briefly recall the construction given in Section 10 of [**DSVV24**], and we refer the reader there for details. Let $\Gamma_1 = (V_1, E_1)$ be the Schläfli graph (cf. 5.6.1.2, 2) and let $\Gamma_2 = (V_2, E_2)$ be the Gosset graph (cf. 3). Note that the Schläfli graph is the graph with vertices the points of $\mathsf{GQ}(2,4)$, adjacent if not collinear in $\mathsf{GQ}(2,4)$. One can describe Γ_2 in terms of Γ_1 as follows.

Let $\Gamma_1' = (V_1', E_1')$ and $\Gamma_1'' = (V_1'', E_1'')$ be two disjoint copies of Γ_1 and consider two symbols ∞' and ∞''. Then the vertices of Γ_2 are the vertices of Γ_1' and Γ_1'', together with ∞' and ∞''. The vertex ∞' (resp. ∞'') is adjacent to all vertices of

Γ_1' (resp. Γ_1''). Adjacency inside Γ_1' and Γ_1'' is as in Γ_1, and a vertex of Γ_1' is adjacent to the vertex of Γ_1'' if the corresponding vertices of Γ_1 are at distance 2 from one another. Now, we fix a Hermitian spread $\mathscr{S}$ of Γ_1, and denote by $\mathscr{S}'$ and $\mathscr{S}''$ the copies of $\mathscr{S}$ in Γ_1' and Γ_1'', respectively.

We label the basis vectors of V by the vertices of the Gosset graph Γ_2 and define for each cross-polytope, which we will call a *hexacross* of Γ_2, and for each pair of opposite hexacrosses, a quadratic form, determined up to a non-zero scalar. Later on, we will use precisely these quadratic forms to describe a variety — denoted by $\mathscr{E}_7(\mathbb{K})$, and called the *Gosset variety* (cf. [**DTV25**, §9.2]) — which will define the geometry $\mathsf{E}_{7,7}(\mathbb{K})$ (see Theorem 5.7.3.9 below).

We use coordinates relative to the standard basis of V, denoting the variable related to the basis vector corresponding to the vertex v of Γ_2 by X_v. The set of all quadratic forms will (only) depend on Γ_2, the vertex ∞' of Γ_2, and the spread $\mathscr{S}'$ of V_1'. We will refer to the first two classes of quadratic forms below as the *short quadratic forms belonging to* $(\Gamma_2, \infty', \mathscr{S}')$, and to those of the last two classes as the *long quadratic forms belonging to* $(\Gamma_2, \infty', \mathscr{S}')$. Hence, there are four classes in total.

(i) *Let Q be a hexacross, defined by a vertex $v'' \in \Gamma_1''$, that is,*

$$Q = (\Gamma_2(v'') \cap V_1') \cup \{\infty', v''\}.$$

There are exactly two vertices i, j of $\Gamma_2(v'') \cap V_1'$, belonging to a common member of $\mathscr{S}'$. Let P be the partition of $(\Gamma_2(v'') \cap V_1') \setminus \{i, j\}$ in pairs of non-adjacent vertices. We define the quadratic form

$$\beta_Q \colon V \to \mathbb{K} \colon (X_v)_{v \in V_2} \mapsto -X_i X_j + X_{\infty'} X_{v''} + \sum_{\{k,\ell\} \in P} X_k X_\ell.$$

Similarly, one defines 27 quadratic forms using a hexacross, defined by a vertex of Γ_1' and ∞''.

(ii) *Let Q be a hexacross, consisting of the union of a 6-clique W' of Γ_1' and a 6-clique W'' of Γ_1''.*

There are unique 3-cliques $C_1, C_2 \in \mathscr{C}$ with $C_1 \cup C_2 = W'$. For each $w' \in W'$, let $w'' \in W''$ denote the unique vertex of W'' not adjacent to w'. Then we define the quadratic form

$$\beta_Q \colon V \to \mathbb{K} \colon (X_v)_{v \in V_2} \mapsto \sum_{w' \in C_1} X_{w'} X_{w''} - \sum_{w' \in C_2} X_{w'} X_{w''}.$$

(iii) *Let (Q', Q'') be a pair of opposite hexacrosses with $\infty' \in Q'$ and $\infty'' \in Q''$.*

Then Q' and Q'' have a unique vertex v' and v'' in Γ_1'' and Γ_1', respectively. For each $w' \in Q'$, let $w'' \in Q''$ denote the unique vertex of Γ_2 opposite w'. Then we define the quadratic form

$$\beta_{Q',Q''} \colon V \to \mathbb{K} \colon (X_v)_{v \in V_2} \mapsto -X_{\infty'} X_{\infty''} - X_{v'} X_{v''} + \sum_{w' \in Q' \setminus \{\infty', v'\}} X_{w'} X_{w''}.$$

(iv) *Let (Q', Q'') be a pair of opposite hexacrosses with $\infty' \notin Q'$ and $\infty'' \notin Q''$.*

Set $W' = Q' \cap V_1'$ and $W'' = Q'' \cap V_1''$. For each $w \in W' \cup W''$, let w_* be the vertex of Γ_2 opposite w. Then we define the quadratic form

$$\beta_{Q',Q''} \colon V \to \mathbb{K} \colon (X_v)_{v \in V_2} \mapsto \sum_{w' \in W'} X_{w'} X_{w'_*} - \sum_{w'' \in W''} X_{w''} X_{w''_*}.$$

We now recall Theorem 10.6 of [**DSVV24**].

Theorem 5.7.3.9. *The variety $\mathscr{E}_7(\mathbb{K})$, obtained by intersecting the respective null sets in $\mathbb{P}(V)$ of the bundle $\mathscr{B}$ of* 126 *quadratic forms β_Q, for Q ranging over the set of hexacrosses of Γ_2, and the* 63 *quadratic forms $\beta_{Q',Q''}$, with $\{Q', Q''\}$ ranging over the set of pairs of opposite hexacrosses of Γ_2, naturally defines a point-line geometry isomorphic to the Lie incidence geometry of type $\mathsf{E}_{7,7}$ over the field $\mathbb{K}$.*

We denote by $\mathsf{GE}_7(\mathbb{K})$ the group of linear automorphisms of V that stabilize the bundle $\mathscr{B}$, that is, the subgroup of $\mathsf{GL}(V)$ that maps every quadratic form defined by $\mathscr{B}$ to a linear combination of quadratic forms defined by $\mathscr{B}$. The subgroup $\mathsf{GE}_7(\mathbb{K}) \cap \mathsf{SL}_{56}(V)$ is denoted by $\mathsf{SE}_7(\mathbb{K})$, and the quotients with the respective centers (consisting of scalar matrices) are $\mathsf{PGE}_7(\mathbb{K})$ and $\mathsf{PSE}_7(\mathbb{K})$. The latter is also denoted briefly by $\mathsf{E}_7(\mathbb{K})$ and is simple (as follows with similar methods as in [**Ste20**]; however, we will not need this fact and therefore we do not include a proof).

Lemma 5.7.3.10. *The Chevalley group $\mathsf{PGE}_7(\mathbb{K})$ is generated by the automorphisms of the building Δ of type E_7 over the field $\mathbb{K}$ pointwise fixing two opposite residues of type* 7*.*

Proof. We denote the base vector of V, corresponding to a vertex $v \in V_2$, by e_v, and the corresponding coordinate of a generic vector by x_v. Suppose first that we have some diagonal automorphism φ of Δ, which maps the coordinate x_v to $\lambda_v x_v$, for all $v \in V_2$. We may assume $\lambda_{\infty'} = 1$. The projective points $\langle e_{v'} \rangle$, for $v' \in V_1'$, generate a 26-dimensional space, intersecting $\mathscr{E}_7(\mathbb{K})$ in a set of points isomorphic to $\mathscr{E}_6(\mathbb{K})$. It follows from the proof of Lemma 5.7.2.6 that there is a constant $\lambda \in \mathbb{K}^\times$ such that the product $\lambda_{v'_1} \lambda_{v'_2} \lambda_{v'_3}$ is equal to λ, as soon as v'_1, v'_2, v'_3 correspond to a line of the $\mathsf{GQ}(2,4)$ underlying Γ_1'.

Let $v' \in V_1'$ be arbitrary, and let $v'' \in V_1''$ be the corresponding vertex (these are at distance 3 in the Gosset graph). Then, since φ is a similitude of each short quadratic form, we infer that $\lambda_{v''} \lambda_{v'} = \lambda$ (consider the hexacross defined by ∞' and v''). Hence, with obvious notation, we deduce that $\lambda_{v''_1} \lambda_{v''_2} \lambda_{v''_3} = \lambda^2$. Since φ is a similitude of any long quadratic form "containing" ∞' and ∞'', we deduce $\lambda_{\infty''} = \lambda$. Now the determinant of the matrix of φ is equal to $\lambda^9 \cdot \lambda^{18} \cdot \lambda = \lambda^{28}$. Hence, diagonal automorphisms have determinant a 28th power.

Now, let θ be an automorphism of Δ, pointwise fixing the residues of two opposite points. We may take for the latter $\langle e_{\infty'} \rangle$ and $\langle e_{\infty''} \rangle$. Since θ fixes all lines through these points, and since the points $\langle e_{v'} \rangle$, $v' \in V_1'$, are the unique points on lines through $\langle e_{\infty'} \rangle$ at distance 2 from $\langle e_{\infty''} \rangle$, we see that θ is a diagonal automorphism. We assume again that θ maps the coordinate x_v to $\lambda_v x_v$, for all $v \in V_2$, with $\lambda_{\infty'} = 1$. Since θ fixes the residue of $\langle e_{\infty'} \rangle$ linewise, we see that $\lambda_{v'} = \lambda$ is a constant for all $v' \in V_1'$. As above, it now follows that $\lambda_{v''} = \lambda^2$ for all $v'' \in V_1''$, and $\lambda_{\infty''} = \lambda^3$. Hence, the determinant of the matrix of θ is equal to $\lambda^{27} \cdot \lambda^{54} \cdot \lambda^3 = \lambda^{84}$, which is a 28th power modulo a 56th power. Moreover, one can see that θ induces similitudes on all

short and long quadratic forms, for arbitrary $\lambda \in \mathbb{K}$. Hence, all 28th powers occur. Now the assertion follows similarly to the end of the proof of Lemma 5.7.2.6. □

6. The Moufang Property

As mentioned in the introduction, Jacques Tits noticed in his lecture notes from 1974 that all thick irreducible spherical buildings of rank at least three satisfy the Moufang condition, and that thick buildings of types H_3 and H_4 do not exist. He then provided a proof in [**Tit77**]. Jacques Tits's unified and elegant approach was completely new, and his results laid the foundation for classifying all thick, irreducible spherical buildings of rank at least three — and hence all finite-dimensional projective spaces of dimension at least three, and all polar spaces of finite rank at least three.

In his proof, Tits made use of of his extension theorem ([**Tit74**, Theorem 4.1.2], Theorem 2.3.0.12). Recall from the introduction that the proof of the extension theorem is considered rather technical (cf. [**Tit74**, p. 7]), and that therefore, there have been attempts to find alternative proofs of Tits's results.

A new proof for the non-existence of thick buildings of types H_3 and H_4 that does not invoke the extension theorem was given by Hendrik Van Maldeghem in 1995 (cf. [**Van95**, §5]). Van Maldeghem's argument assumes the existence of thick spherical buildings of type H_3, shows that the rank-2 residues that are generalized pentagons are regular, and that regular generalized pentagons do not exist.

Main Results

In this chapter, we present another alternative proof that is more direct and less intricate, as it does not require the notion of regularity.

Furthermore, we present a new proof of Tits's result for polar spaces: we give direct geometric constructions for root elations of generalized quadrangles embedded in thick polar spaces of higher finite rank. These constructions imply the Moufang property for all such quadrangles. We then show that these elations extend to the ambient polar space, which again implies the Moufang property for the entire space.

We also give an equivalent way to define the Moufang condition for polar spaces without using the notion of apartments.

This enables us to describe what the Moufang property looks like for top-thin polar spaces, and for polar spaces of infinite rank, and we then proceed to prove it for them as well.

A corollary of our constructions for generalized quadrangles embedded in finite higher-rank polar spaces is that every root elation can be expressed as a self-projectivity of length 4. This corollary — together with [**Kna88**, Proposition 2.3] — was the initial motivation for proving the more general Proposition 2.10.0.2 about the connection between the little projective group of a thick irreducible spherical building Δ and the special projectivity group of Δ.

We also discuss the Moufang property for buildings of type A_2 embedded in buildings of types $\mathsf{A}_{n_{\geq 3}}$ and $\mathsf{B}_{n_{\geq 3}} = \mathsf{C}_{n_{\geq 3}}$, and for buildings of type A_n for $n \geq 3$.

Thus, overall, this chapter provides alternative, elementary proofs of Tits's results on thick irreducible spherical buildings of rank 3, without using Tits's extension theorem or Tits's classification; or, with other words, on thick, irreducible spherical buildings of classical types and buildings associated to icosahedral geometries.

The research presented in this chapter is based on [**Bus25a**], and [**Bus25b**].

Note 6.0.0.1. It is worth mentioning that a simplified version of Tits's extension theorem exists for polar spaces (cf. [**BC13**, Theorem 8.5.5]). However, this does not give us a direct construction for root elations, and it assumes that we already know that a certain isomorphism exists (cf. [**BC13**, p. 400, l. 2]).

Note 6.0.0.2. Whenever we talk about rank or dimension n in this chapter, $\infty \neq n$ is a natural number.

6.1. Defining the Moufang Property

6.1.1. Projective Spaces

First, we have to define *half-apartments*, and for that, heuristically, we want to cut apartments into halves. Since one can still visualize what is happening quite well in rank 3 (Fig. 1), we will first consider rank 3 and later rank n.

If a projective space Δ has rank 3, an apartment Γ of Δ is an ordinary tetrahedron. A *half-apartment* α of Γ contains

- a plane π_0 with its three lines L_0, L_1, and L_2, and its three points p_0, p_1, and p_2 (say $L_0 = p_0p_1$, $L_1 = p_1p_2$, $L_2 = p_0p_2$),
- two planes π_1 and π_2, which intersect π_0 in one of these lines L_i ($i \in \{0, 1, 2\}$) (Fig. 2-example: $\pi_1 \cap \pi_0 = p_1p_2$, $\pi_2 \cap \pi_0 = p_0p_2$), and the line in the intersection of π_1 and π_2.

That means $\Gamma \setminus \alpha$ consists of a plane, two lines of that plane, and the intersection point of these lines.

The *interior* α^+ of the half-apartment α consists of π_0, two lines of π_0 (Fig. 4-example: p_1p_2 and p_0p_2), and the point in the intersection of these two lines (Fig. 4-example: p_2).

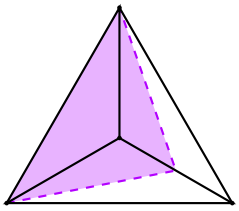


FIG. 1. A visualization of cutting an apartment.

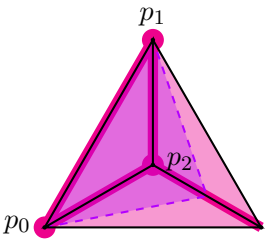


FIG. 2. A visualization of what is contained in a half-apartment.

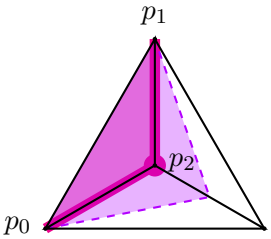


FIG. 3. A visualization of what is contained in the interior of a half-apartment.

Definition 6.1.1.1 (**Half-apartments**, **interiors of half-apartments**)**.** Let Δ be a projective space of dimension n, let $\Gamma^o := \{x_1, x_2, \ldots, x_n, x_{n+1}\}$, be a projective frame of Δ (cf. Definition 3.7.0.2), and let Γ be the apartment spanned by it.

Let c be one of the points of Γ^o, and let C be a hyperplane of Γ through c. Then we obtain a *half-apartment* α *of* Γ if we take Γ and remove from it:

- the hyperplane A of Γ that is disjoint from c,
- all singular subspaces that are contained in A, except those that are contained in $A \cap C$.

We obtain the *interior* α^+ of α if we take α and also remove all singular subspaces that are contained in $A \cap C$.

Now we can define *root elations* and the *Moufang property*.

Definition 6.1.1.2 (**Root elations**)**.** A *root elation of* Δ *with respect to a half-apartment* α (of some apartment Γ of Δ) is a collineation of Δ that fixes each point on each line, and each line through each point of α^+.

Definition 6.1.1.3 (**Moufang property**)**.** If for every half-apartment α (of any apartment Γ of Δ) the group generated by all root elations of Δ with respect to α acts transitively on the set of apartments containing α, then we say that Δ is *Moufang*.

If we want to show that a projective space of dimension 3 is Moufang, this means — let π be a plane in Δ, let c be a point in π, let L be a line through c not contained in π, and let a and a' be two points on L distinct from c — we have to show that we can construct a root elation θ fixing π pointwise, stabilizing all lines through c, and moving a to a' (cf. Fig. 4).

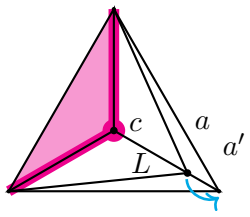


FIG. 4

6.1.2. Polar Spaces

Let Δ be a polar space of rank $n \geq 2$. Given an apartment $\mathscr{A}$ spanned by a polar frame

$$\{p_{-n}, p_{-n+1}, \ldots, p_{-1}, p_1, p_2, \ldots, p_n\},$$

we can obtain a *half-apartment* α in two different ways:

(R1) by removing one point p_i of the polar frame and all singular subspaces of $\mathscr{A}$ that contain p_i.

(R2) by removing two collinear points p_i and p_j of the polar frame, all maximal singular subspaces A of $\mathscr{A}$ that contain p_i and p_j, and all singular subspaces contained in them that either contain p_i or p_j, or

Half-apartments are also called *roots*. We will say that a root is *of the first kind* if we remove one point of the corresponding polar frame, and *of the second kind* if we remove two points. The *interior* α^+ *of a root* α *of the first kind* is obtained when we also remove all submaximal singular subspaces that were contained in one of the removed maximal singular subspaces. The *interior* α^+ *of a root* α *of the second kind* is obtained when we also remove all maximal singular subspaces that contain either p_i or p_j.

Example 6.1.2.4. We consider apartments of a polar space of rank 3 and mark what is contained on the interior of a root in both cases in magenta.

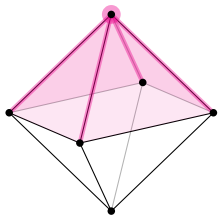


Fig. 5. The interior of a root of the first kind.

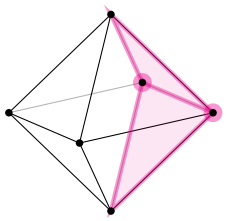


Fig. 6. The interior of a root of the second kind.

Definition 6.1.2.5. Let Δ be a polar space of rank $n \geq 2$. If, for every root α, the group of automorphisms of Δ that fix α^+ pointwise and stabilize all singular subspaces that intersect α^+ non-trivially acts transitively on the set of apartments containing α, then we say that Δ has the *Moufang property*, or that Δ is *Moufang*.

Definition 6.1.2.6 (**Root elations**)**.** The collineations of a polar space that fix the interior of a root pointwise are also called *root elations*.

Thus, the Moufang property ensures the existence of non-trivial root elations. In the following, we want to give descriptions of and names to different automorphisms of polar spaces and express the Moufang property solely through these automorphisms, without needing the notions of apartments and roots.

Definition 6.1.2.7 (**Point elations**)**.** Let Δ be an arbitrary polar space, and let L and L' be two lines through a point c that are not coplanar. A *point elation* (with center (L, c, L')) is a collineation of Δ stabilizing all singular subspaces containing c and fixing L and L' pointwise.

Definition 6.1.2.8 (**Line elations**)**.** Let Δ be an arbitrary polar space, let L be a line, and let p and q be two distinct points on L. A *line elation* (with center (p, L, q)) is a collineation of Δ fixing all points on L and stabilizing all singular subspaces containing p or q (where p and q are two distinct points on L).

Definition 6.1.2.9. Let Δ be an arbitrary polar space. We define the following two properties.

(M1) For any point c and any two noncoplanar lines L and L' through c, the group of point elations with center (L, c, L') acts transitively on the set of points opposite c in $x^\perp \cap x'^\perp$, where $x \in L \setminus \{c\}$ and $x' \in L' \setminus \{c\}$.

(M2) For any line L and any two distinct points p and q on L, the group of line elations with center (p, L, q) acts transitively on the points of the lines through p that are not coplanar with L.

Proposition 6.1.2.10. *Let Δ be a polar space of rank n. Then Δ is Moufang if and only if it satisfies the properties (M1) and (M2).*

PROOF. "$\Rightarrow$":

Let Δ be a polar space of rank n satisfying (M1) and (M2).

(Point elations) Let α_1 be a root of the first kind, let b be a point in α_1^+, let π and π' be two maximal singular subspaces in α_1^+ through b, and let a and a' be two noncollinear points not contained in α_1, both opposite b, and such that $\mathsf{proj}_\pi(a) = \mathsf{proj}_\pi(a')$ and $\mathsf{proj}_{\pi'}(a) = \mathsf{proj}_{\pi'}(a')$. We need to find an elation of Δ pointwise fixing α_1^+, stabilizing all singular subspaces that intersect α_1^+, and mapping a to a'. Let L and L' be two lines in α_1^+ through b in π and π', respectively. According to (M1), we can find a point elation θ_1 with center (L, c, L') mapping a to a'. Using Tits's rigidity theorem (cf. [**Tit74**, Theorem 4.1.1]), we see that θ_1 is the desired root elation.

(Line elations) Let α_2 be a root of the second kind containing two collinear points p and q. Let m and m' be two collinear points both not collinear with q such that mm' contains p. In order to show that Δ is Moufang, we need to find an elation of Δ pointwise fixing α_2^+, stabilizing all singular subspaces that intersect α_2^+, and mapping m to m'. According to (M2), there exists a line elation θ_2 with center (p, pq, q) mapping m to m'. Then θ_2 is the desired root elation.

"$\Leftarrow$":

Let Δ be a polar space of rank n having the Moufang property.

(Line elations) Let L be any line, and let p and q be two distinct points on L. Let m and m' be two collinear points both not collinear with q such that mm' contains p. We need to find a line elation with center (p, L, q) that maps m to m'. Let α_2 be a root of the second kind containing L on its interior. Then there exists a root elation θ_2 fixing α_2^+ pointwise, stabilizing all singular subspaces intersecting α_2^+, and

mapping m to m'. Using Tits's rigidity theorem (cf. [**Tit74**, Theorem 4.1.1]), in the respective residues of p and q, we see that θ_2 is the desired line elation.

(Point elations) Let c be a point, and let M and N be two noncoplanar lines through c. Let a and $\hat{a}$ be two points, both opposite c, and such that

$$\mathsf{proj}_M(a) = \mathsf{proj}_M(\hat{a}) =: m, \quad \mathsf{proj}_N(a) = \mathsf{proj}_N(\hat{a}) =: n.$$

We need to find a point elation with center (M, c, N) that maps a to $\hat{a}$. Note that if the rank of Δ is 2, the Moufang property ensures the existence of a root elation, which is exactly the desired point elation. So suppose Δ has rank $n \geq 3$.

Let U be an arbitrary maximal singular subspace through cm, and let V be a maximal singular subspace through cn locally opposite at c. Put

$$U_a := a^\perp \cap U \quad \text{and} \quad V_a = a^\perp \cap V,$$

and similarly for $\hat{a}$:

$$U_{\hat{a}} := \hat{a}^\perp \cap U \quad \text{and} \quad V_{\hat{a}} = \hat{a}^\perp \cap V.$$

Case 1. $U_a = U_{\hat{a}}$ and $V_a = V_{\hat{a}}$.

Then there exist a root α of the first kind containing c in its interior, and a root elation fixing the interior of α and mapping a to $\hat{a}$.

Case 2. $U_a = U_{\hat{a}}$ and $V_a \neq V_{\hat{a}}$.

We denote $V_a \cap V_{\hat{a}}$ by Y. Then Y is a subspace of dimension $n-2$. Select a line L through c in V not intersecting Y. Let e be the projection of a onto L and $\hat{e}$ the projection of $\hat{a}$ onto L. These are different, as L does not intersect Y. Consider

$$K = U \cap Y^\perp.$$

Then K is a line through c. Consider the line elation with center $(c, K, K \cap a^\perp)$ mapping e to $\hat{e}$. Then a is sent to a point a' that has the same projection onto L as $\hat{a}$. Moreover, Y and U are fixed pointwise. Hence, with self-explanatory notation, we have $U_{\hat{a}} = U_{a'}$, since U_a is fixed pointwise, and $V_{\hat{a}} = V_{a'}$, since $\mathsf{proj}_L(a') = \hat{e}$ and $V_{a'}$ is a hyperplane of V not containing c or e.

Now we can apply Case 1 to a' and $\hat{a}$. The composition of root elations is then the desired point elation.

Case 3. $U_a \neq U_{\hat{a}}$ and $V_a \neq V_{\hat{a}}$.

We denote $U_a \cap U_{\hat{a}}$ by Z, and $V_a \cap V_{\hat{a}}$ by Y.

Let L be the projection of Z onto V, and M the projection of Y onto U.

- If L intersects Y, then we replace U by $\langle L, Z\rangle$ and V by a maximal singular subspace through cn locally opposite $\langle L, Z\rangle$.
- If L does not intersect Y, but M intersects Z, then we replace V by $\langle M, Y\rangle$ and U by a maximal singular subspace through cm locally opposite $\langle M, Y\rangle$.

Then we are back in Case (2).

If L and Y do not intersect, and M and Z do not intersect, L and M are locally opposite at c. Put

$$d := \mathsf{proj}_M(a), \ e := \mathsf{proj}_L(a), \ \hat{e} := \mathsf{proj}_L(\hat{a}).$$

We can consider the line elation with center (c, M, d) mapping e to $\hat{e}$. Then a is mapped onto a point a' such that $\mathsf{proj}_V(\hat{a}) = \mathsf{proj}_V(a')$, and then we continue with Case (2) for the points a' and $\hat{a}$. □

Definition 6.1.2.11 (**Moufang property for polar spaces of infinite rank**)**.** Let Δ be a polar space of infinite rank. Then Δ is *Moufang* if Δ satisfies the conditions (M1) and (M2) of Definition 6.1.2.9.

Observation 6.1.2.12. The line elations of Δ are exactly the root elations of Δ fixing the interior of a root of the second kind pointwise, and we also refer to them as such in the following.

6.1.3. Generalized Polygons

Let $\Gamma = (\mathscr{P}, \mathscr{L})$ be a generalized n-gon for some $n \geq 2$.

Definition 6.1.3.13 (**k-paths**)**.** A *k-path* ($k \geq 0$) is a sequence $(x_0, x_1, \ldots, x_k)$ of points and lines of Γ such that

- if x_i is a point, then x_{i+1} is a line, and if x_i is a line, then x_{i+1} is a point,
- x_{i-1} and x_i are incident,
- $x_{i-1} \neq x_{i+1}$, for $1 \leq i < k$ (cf. [**HV04**, §2]).

Definition 6.1.3.14 (**Apartments**, **half-apartments**, **interiors of roots**)**.** The ordinary n-gons contained in Γ as subgeometries are called the *apartments* of Γ. Thus, according to Definition 3.5.0.2(ii), any two elements $x, y \in \mathscr{P} \cup \mathscr{L}$ are contained in a common apartment of Γ. Every apartment can be viewed as a $2n$-path $(x_0, x_1, \ldots, x_{2n})$ such that x_0 and x_{2n} are incident.

A *root* or *half-apartment* of Γ is an n-path in Γ contained in some apartment

$$\Sigma = (x_0, x_1, \ldots, x_{2n}) \text{ of } \Gamma.$$

For a root $\alpha = (x_0, x_1, \ldots, x_n)$, the *interior α^+ of α* is the path $(x_1, x_2, \ldots, x_{n-1})$.

Note that, if n is even, there are two different kinds of roots: If α is a root in Γ, then α is an n-path $(x_0, x_1, \ldots, x_n)$. Now, x_0 can either be a point or a line. Both options are valid, and this is why we obtain different kinds of roots. These two different kinds represent the fact that there can be long roots and short roots, as mentioned in Definition & Fact 1.5.0.24.

Definition 6.1.3.15 (**Root groups**)**.** For every root α of Γ, we denote the subgroup of $\mathsf{Aut}(\Gamma)$ that fixes every element of Γ incident with some element of α^+ by U_α, and call it the *root group of α*.

Definition 6.1.3.16 (**Moufang polygons**)**.** A *Moufang polygon* (or *Moufang n-gon*) is a thick generalized n-gon Γ for some $n \geq 3$ such that for every root α, the root group U_α acts transitively on the set of apartments of Γ containing α (cf. [**TW02**, Definition 4.2]).

Note 6.1.3.17. In 1976, an article by Jacques Tits was published, showing that Moufang m-gons only exist for $m \in \{2, 3, 4, 6, 8\}$ (cf. [**Tit76**, Théorème 1]).

6.1.4. Generalized Pentagons

We will give an explicit description of what it would means for a generalized pentagon to be Moufang, as this will be used later.

Definition 6.1.4.18. Let Δ be a generalized pentagon, and let Γ be an apartment in Δ.

Let L_0 and L_1 be two intersecting lines of Γ, and let c be their intersection point. Let p be the other point of L_0, and let q be the other point of L_1. Let L_2 be another line of Γ through q. Then we call the set $\{p, L_0, c, L_1, q, L_2\}$ a *half-apartment of* Γ, and denote it by α. We call the set $\alpha^+ := \{L_0, c, L_1, q\}$ the *interior of the half-apartment* α.

Definition 6.1.4.19. A *root elation of* Δ *with respect to a half-apartment* α (of some apartment Γ of Δ) is a collineation of Δ that fixes each point on each line, and each line through each point of α^+.

Definition 6.1.4.20. If for every half-apartment α (of any apartment Γ of Δ) the group generated by all root elations of Δ with respect to α acts transitively on the set of apartments containing α, then we say that Δ is *Moufang*.

6.1.5. A Note on Moufang Hexagons

Generalized hexagons satisfying the *Moufang condition* correspond to thick irreducible spherical buildings of rank at least three and type G_2.

A Moufang hexagon is determined by a field $\mathbb{K}$ and a quadratic Jordan division algebra $\mathbb{J}$ over $\mathbb{K}$ (cf. [**TW02**]). Using the Bourbaki labelling of nodes for Dynkin diagrams, we define $\mathsf{G}_{2,2}(\mathbb{K}, \mathbb{J})$ as the hexagon where the point rows are parametrised by $\mathbb{J}$ and the line pencils by $\mathbb{K}$. There is perhaps ambiguity when $\mathbb{J} = \mathbb{K}$, but we take that ambiguity away by mentioning that in this case, $\mathsf{G}_{2,2}(\mathbb{K}, \mathbb{K})$ is the hexagon arising from a triality of type I_{id}, as defined by Tits [**Tit59**]. Such a hexagon is exactly what we called the split Cayley hexagon in Section 5.3.2 and the corresponding building is referred to as being *split* (cf. [**Van98**]). The triality hexagon is $\mathsf{G}_{2,2}(\mathbb{K}, \mathbb{J})$, where $\mathbb{J}$ is a cubic Galois field extension of $\mathbb{K}$ (cf. Section 5.3.2). We will only need this hexagon in the finite case. The finite triality hexagon has order (q^3, q), for some prime power q. In the finite case, we replace the field and the Jordan algebra with their sizes in the notation.

6.2. Proving The Moufang Property for Projective Spaces

6.2.1. For Projective Planes in Projective Spaces

The following proposition and its proof are well-known and we only provide a proof here for completeness.

Proposition 6.2.1.1. *Let Δ be a projective space of dimension at least 3. Then the planes of Δ are Moufang.*

Proof. Let π_0 be a plane in Δ. We want to show that π_0 is Moufang. Let M be an arbitrary line in π_0 and let c be an arbitrary point on M. Let L be an arbitrary line in π_0 through c not equal to M. We have to construct a collineation of π_0, with axis M and center c, such that we can map an arbitrary point p on L, distinct from c, to any other point p' on L, distinct from c.

Since every line of Δ is contained in at least 3 planes, we can find a plane π_1, that intersects π_0 exactly in M. Let q be some point in π_1 not on M. Since p, p' and c lie on the common line L, the lines cq, pq and $p'q$ all lie in the common plane $\langle L, q\rangle$. Let a be some point on pq distinct from p and q. Let a' be the intersection point of the lines ca and $p'q$.

Now, let

- θ_1 be the map from the point set of π_0 to the line pencil through a that maps every point x of π_0 to the line xa;
- θ_2 be the map from the line pencil through a to the point set of π_1 that maps every line xa to $xa \cap \pi_1$;
- θ_3 be the map from the point set of π_1 to the line pencil through a' that maps every point y of π_1 to the line ya';
- θ_4 be the map from the line pencil through a' to the point set of π_0 that maps every line ya' to $ya' \cap \pi_0$;
- $\theta := \theta_4 \circ \theta_3 \circ \theta_2 \circ \theta_1$.

Then, for a point m on M, we have that $m \mapsto ma$ under θ_1, $ma \mapsto m$ under θ_2, $m \mapsto ma'$ under θ_3 and $ma' \mapsto m$ under θ_4 and thus, θ fixes all points on M.

The point p maps to pa under θ_1, $pa \mapsto q$ under θ_2 (since a is on pq), $q \mapsto qa'$ under θ_3 and $qa' \mapsto p'$ under θ_4 (since a' is on $p'q$), and it follows that θ maps p to p'.

Let L' be an aribtrary line through c in π_0, not equal to M or L. Every point x of L' maps to a line xa through a under θ_1 and all the lines xa form a pencil in the common plane $\langle L', a\rangle$. The planes π_0 and π_1 span a 3-space in Δ and since a and L' are contained in that 3-space, the whole plane $\langle L', a\rangle$ is contained in it. Two planes inside a 3-spaces intersect in a line. Therefore, $\langle L', a\rangle$ and π_1 intersect in some line K, and every line xa maps to a point y on K under θ_2. Note that the point c will be on K. Now, because the lines ca and ca' coincide, the planes $\langle K, a\rangle$ and $\langle K, a'\rangle$

coincide. Under θ_3, every point y on K maps to a line ya' through a' inside the plane $\langle K, a'\rangle$. Since $\langle K, a\rangle = \langle K, a'\rangle$ and $\langle K, a\rangle \cap \pi_0 = L'$, every line ya' maps back to a point on L'.

This shows that θ is the desired elation. □

6.2.2. For Projective Spaces

The following proposition is well known as well. The author expects that this way of proving the Moufang property for higher-dimensional projective spaces was known, but could not find any reference. The usual way to prove it is via Desargues' theorem (cf. [**Bue95**, Ch. 2]). We want to prove it by extending elations to have a unified theme throughout this chapter.

Proposition 6.2.2.2. *Let Δ be a projective space of dimension at least 3. Then Δ is Moufang.*

Proof. Let Δ be a projective space of dimension 3. Let π be a plane in Δ, let c be a point in π, let M be a line through c not contained in π, and let a and a' be two points on M distinct from c. We have to show that we can construct an elation θ fixing π pointwise, stabilizing all lines through c, and moving a to a'.

We first define a map $\theta_{a,a'}$ on an arbitrary point $x \notin M$. If $x \in \pi$, then $\theta_{a,a'}(x) = x$. If $x \notin \pi$, then there is a unique plane α containing M and x, and it intersects π in a line L. Then, since planes are Moufang, there is a unique elation θ_α in α with axis L and center c and mapping a to a'. We define $\theta_{a,a'}(x) = \theta_\alpha(x)$.

Now let $b \notin M \cup \pi$ be an arbitrary point and set $b' := \theta_{a,a'}(b)$. Then we can define $\theta_{b,b'}$ on the points of bb' similarly as we defined $\theta_{a,a'}$ on points of M. We claim that $\theta_{a,a'}$ coincides with $\theta_{b,b'}$ over the set of points of $M \cap bb'$. Indeed, let e be a point not on M or bb'. If $e \in \pi$, then the claim is trivial. Suppose now $e \notin \pi$. If $e \in \langle M, b\rangle$, then this follows from the fact that $\theta_{a,a'}$ coincides with $\theta_{b,b'}$ over $\langle M, b\rangle$ since the latter is Moufang. Assume now $e \notin \langle M, b\rangle$ and set $e' = \theta_{b,b'}(e)$. Note that, by definition of $b' = \theta_{a,a'}(b)$, the line ab intersects the line $a'b'$ in a point $y \in \pi$. Also, by definition of $\theta_{b,b'}(e)$, the line be intersects the line $b'e'$ in a point $z \in \pi$. Since $e \notin \langle M, b\rangle$, the points y and z are distinct, and hence the plane $\langle a, b, e\rangle$ intersects the plane $\langle a', b', e'\rangle$ in the line yz. Now, the lines ae and $a'e'$ are contained in the plane $\langle M, e\rangle$ and hence they intersect in some point x. The point x lies in the intersection of the planes $\langle a, b, e\rangle$ and $\langle a', b', e'\rangle$ and hence on the line yz. Hence, in the plane $\beta := \langle M, e\rangle$, the elation θ_β maps e to e'. The claim is proved.

The previous claim now immediately implies that, for each line K not intersecting M, $\theta_{a,a'}(K)$ is again a line. Interchanging the roles of a and b, we can now define $\theta_{a,a'}$ on M as $\theta_{b,b'}$. It follows that the image under $\theta_{b,b'}$, and hence also under $\theta_{a,a'}$, of each line intersecting M is again a line.

This shows that $\theta_{a,a'}$ is a collineation. Since it fixes π pointwise and c linewise, it is the wanted elation θ.

Now let Δ be a projective space of arbitrary dimension. Let H be a hyperplane in Δ, and let a and a' be two points in Δ not contained in H. We have to find an

automorphism θ of Δ fixing H pointwise and mapping a to a'. For that, let x be an arbitrary point in Δ not in H and not on aa'. Consider the plane $\pi_x := \langle a, a', x\rangle$. Let A_x be an arbitrary 3-space containing π_x. Then there exists a unique elation θ_{A_x} of A_x fixing $A_x \cap H$ pointwise and mapping a to a'. Now let B_x be a different 3-space containing π_x, and let θ_{B_x} be the unique elation of B_x fixing $B_x \cap H$ pointwise and mapping a to a'. Then, since there is a unique elation in π pointwise fixing $\pi \cap H$ and mapping a to a', we conclude $\theta_{A_x}|_{\pi_x} = \theta_{B_x}|_{\pi_x}$. This implies that $\theta_{A_x}(x) = \theta_{B,x}(x)$ and so if we define $\theta(x) = \theta_{A,x}(x)$, then $\theta(x)$ is well-defined and does not depend on A. It is now straightforward to check that θ is also (well-)defined on the line aa' and that θ is a collineation. □

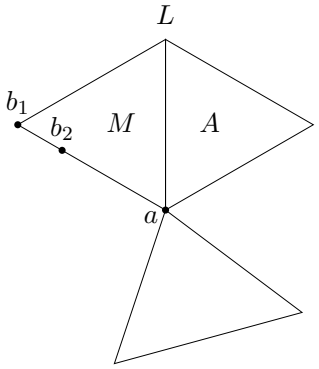
6.3. Proving The Moufang Property for Rank-2 Residues of Polar Spaces

6.3.1. For Projective Spaces in Polar Spaces

The following is Proposition 7.11 of [**Tit74**] and has been proved by Jacques Tits. We will present the same proof here in a bit more detail. We use the exact same notation as in [**Tit74**] for easy comparability. One can also find another version of this proof written down in [**BC13**, Theorem 7.7.10].

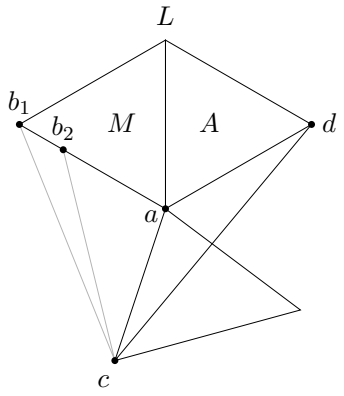
Proposition 6.3.1.1. *The planes of a thick polar space of rank* 3 *are Moufang.*

Proof. Let S be a thick polar space of rank 3. Let M be a plane in S and L a line in M. Let b_1 and b_2 be two arbitrary but distinct points of M not on L. Let a be the intersection point of the line b_1b_2 and L. Let A be a plane of S that intersects M in L. In $\mathsf{Res}(a)$, we can see lines corresponding to the planes A and M and – since S is thick (and therefore $\mathsf{Res}(a)$ as well) – we can find a line in $\mathsf{Res}(a)$ which is opposite both of the lines corresponding to A and M and which corresponds to a plane B in S that intersects both M and A only in a.

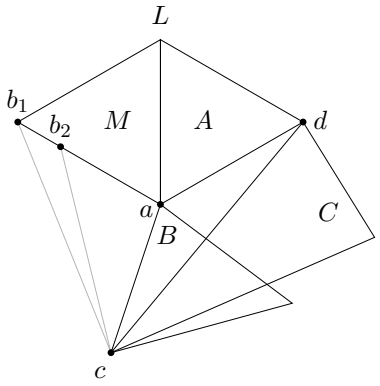


The unique plane containing b_1 and intersecting B is $\langle b_1, \mathsf{proj}_B(b_1)\rangle$. This plane contains a since b_1 is collinear to a and a is a point of B. Because $\langle b_1, \mathsf{proj}_B(b_1)\rangle$ contains b_1 and a, it also contains b_2. Similarly the plane $\langle b_2, \mathsf{proj}_B(b_2)\rangle$ contains b_1. Suppose $\mathsf{proj}_B(b_1)$ and $\mathsf{proj}_B(b_2)$ were two distinct lines through a. Then every point of $\mathsf{proj}_B(b_1)$ would be collinear to b_1 and a. That means every point of $\mathsf{proj}_B(b_1)$ would already have to be collinear to every point of the line b_1a, including b_2. But then $\mathsf{proj}_B(b_1) \subseteq \mathsf{proj}_B(b_2)$. In the same way we get $\mathsf{proj}_B(b_2) \subseteq \mathsf{proj}_B(b_1)$. That means the lines $\mathsf{proj}_B(b_1)$ and $\mathsf{proj}_B(b_2)$ have to be equal.

Let c be a point of the line $\mathsf{proj}_B(b_1) = \mathsf{proj}_B(b_2)$ that is not a. Then $\mathsf{proj}_A(c)$ contains a, since c is collinear to a and a is a point of A. The line $\mathsf{proj}_A(c)$ is not the line L, because: $\langle c, \mathsf{proj}_A(c)\rangle$ is the unique plane containing c intersecting A in a line. Suppose that line were L. Then $\mathsf{proj}_M(c)$ would also have to be L. But that contradicts the fact that c is collinear to b_1 and b_2. Now, let d be another point on the line $\mathsf{proj}_A(c)$ distinct from a. Then the point d does not lie on L because $\mathsf{proj}_A(c) \neq L$. Since the point c is collinear to every point of $\mathsf{proj}_A(c)$ per definition, there is a line between c and d.



The unique plane containing cd and intersecting M is $\langle cd, \mathsf{proj}_M(cd)\rangle = \langle cd, a\rangle$. Since S is a thick polar space, the line cd lies in at least three planes. Let C be a plane containing cd different from $\langle cd, a\rangle$.



The points b_i and d are not collinear, because: The line L lies in $b_i^\perp \cap d^\perp$. If b_i and d were collinear, the unique plane containing b_i and $\mathsf{proj}_A(b_i)$, i.e., $\langle b_i, \mathsf{proj}_A(b_i)\rangle$,

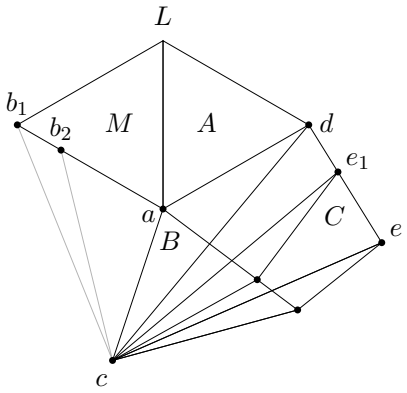
would contain b_i, L, and d. That would contradict the fact that $M \cap A = L$ and b_i and d not on L.

Since b_i and d are not collinear, b_i is not contained in C. The line $\mathsf{proj}_C(b_i)$ contains c but is otherwise not contained in B, because: If $\mathsf{proj}_C(b_i)$ were a line in B, then $\mathsf{proj}_C(b_i)$ and $\mathsf{proj}_B(b_i)$ would already have to be the same line. The line $\mathsf{proj}_B(b_i)$ contains a. It would contradict the fact that C is not $\langle a, cd\rangle$ and does not contain a. Furthermore

$$\mathsf{proj}_C(b_1) \neq \mathsf{proj}_C(b_2),$$

because otherwise, every point of $\mathsf{proj}_C(b_1) = \mathsf{proj}_C(b_2)$ would be collinear to a, which contradicts the fact that $\mathsf{proj}_C(a) = cd$.

Now let e_i be an arbitrary point on $\mathsf{proj}_C(b_i)$ that is not c, and therefore not contained in B. The line $\mathsf{proj}_B(e_i)$ contains c, since e_i and c are both on the line $\mathsf{proj}_C(b_i)$ and c is a point in B.



We will denote the line $\mathsf{proj}_B(e_i)$ by H_i. We define the plane D_i as $\langle e_i, \mathsf{proj}_B(e_i)\rangle$. The plane $\langle e_i, \mathsf{proj}_B(e_i)\rangle$ contains $\mathsf{proj}_C(b_i)$, since it contains c and e_i.

Suppose there were a point p in $M \cup A$ contained in D_i. Then the point p would be collinear to a, and every point of the line H_i would be collinear to a. That means the line H_i and the points p and a would lie in a common plane together. Since p and H_i are contained in D_i, the point a would also have to be in D_i. We know that $\mathsf{proj}_C(b_i) = D_i \cap C$. If a were in D_i, then $\mathsf{proj}_C(a)$ would also equal $D_i \cap C$. But $\mathsf{proj}_C(a) = cd$ and b_i and d are not collinear. It follows that the plane D_i is disjoint from M and A. We set

$$\alpha_i := \mathsf{proj}_A^{D_i} \circ \mathsf{proj}_{D_i}^{M}, \quad \text{for } i = 1, 2,$$
$$\alpha := \alpha_2^{-1} \circ \alpha_1.$$

That means, with α, we basically project from M to D_1 to A to D_2 and back to M. We get that $\alpha_i(x) = x$ for all $x \in L$, since $L = M \cap A$. Furthermore,

$$\alpha_i(b_i) = \mathsf{proj}_A(\mathsf{proj}_{D_i}(b_i)),$$

according to the definition of α_i, and

$$\mathsf{proj}_A(\mathsf{proj}_{D_i}(b_i)) = \mathsf{proj}_A(\mathsf{proj}_C(b_i)),$$

because of the way we defined D_i. Since $\mathsf{proj}_C(b_i)$ is in C, every point of that line is collinear to the point d, which is in C and A. All in all, we get $\alpha_i(b_i) = d$. The

point a maps to the line H_i under proj_{D_i}, since H_i and a are both in the common plane B.

Now let X be an arbitrary line in M distinct from L through a. In the residue of a we see X as a point on the line corresponding to M. Since the lines corresponding to M and B are opposite, X is collinear to a unique point X' on the lines corresponding to B and X' is collinear to a unique point X'' on the line corresponding to A. In S we see X' as a line through a in B and X'' as a line through a in A. The projection of X onto D_i is $X' \cap H_i$ and the projection of $X' \cap H_i$ onto A is X''. That means we get the same line in A, no matter if we map X with α_1 or α_2. It follows that

$$\alpha(X) = \alpha_2^{-1} \circ \alpha_1(X) = \alpha_1^{-1} \circ \alpha_2(X) = \alpha^{-1}(X) = X.$$

The map α is an automorphism $M \to M$ that fixes the line L pointwise, maps b_1 to b_2 and fixes every line through $a = b_1 b_2 \cap L$ linewise. □

The following corollary is known as a direct consequence of Proposition 6.3.1.1.

Corollary 6.3.1.2. *Let Δ be a thick polar space of rank ≥ 3. Then all non-trivial projective spaces (cf. Note 3.2.0.4) in Δ are Moufang.*

Proof. The only projective spaces, that could be non-Moufang, are the projective planes. Projective planes in rank 3 polar spaces are Moufang, as we have just seen. If the rank of Δ is greater than 3, then every projective plane is contained in a higher-dimensional projective space, and projective planes in higher-dimensional projective spaces are Moufang. □

6.3.2. For Generalized Quadrangles in Polar Spaces

Let $\tilde{\Delta}$ be a polar space of rank $n \geq 3$. Let Γ be an arbitrary generalized quadrangle in $\tilde{\Delta}$ arising as the intersection of $U^\perp$ with $V^\perp$, for two opposite singular subspaces of dimension $n-3$. Then Γ is contained in some polar space Δ of rank 3 inside $\tilde{\Delta}$, and there exist two opposite points p and b in Δ such that $\Gamma = p^\perp \cap b^\perp$. Furthermore, Γ is isomorphic to $\mathsf{Res}_\Delta(p)$ and $\mathsf{Res}_\Delta(b)$. In the following, we prove the Moufang condition for Γ. For that, we have to show that the group of automorphisms of Γ that pointwise fix the interior of an arbitrary half-apartment α^+ of an arbitrary apartment α of Γ and stabilize all elements incident with some element on the interior of the half-apartment α^+ acts transitively on the set of apartments containing α^+. We will give direct constructions for the desired automorphisms for both kinds of roots.

6.3.2.1. Second kind of root

Lemma 6.3.2.3. *Let Δ be a polar space of rank 3, and let Γ be the generalized quadrangle obtained from two opposite points p, b of Δ by considering $p^\perp \cap b^\perp$. Then all roots of the second kind of Γ are Moufang. Furthermore, every line elation of Γ can be written as a self-projectivity of length 4.*

PROOF. Let α be an arbitrary apartment of Γ. Let q and d be two collinear points in α. Let u be the point in α collinear to q but not to d, and let n be the point in α collinear to d but not to q. Our goal is to construct an automorphism that fixes qd pointwise, stabilizes the lines that intersect q or d, and moves the line nu to a line $n'u'$, such that $n' \in nd \setminus \{n\}$ and $u' \in uq \setminus \{u\}$. Let j be a point on pq not equal to p or q. The lines ju and pu' intersect in a point that we will denote by i. Set $\ell := \mathsf{proj}_{bd}(i)$. For a visualization, see Figure 7.

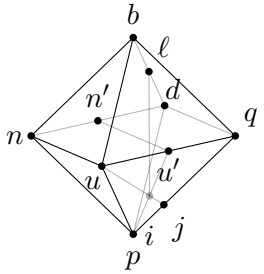


FIG. 7

The points p and b, b and j, j and ℓ, and ℓ and p are opposite. We define

$$\theta \colon \mathsf{Res}(p) \to \mathsf{Res}(p)$$

as follows:

$$\theta := \mathsf{proj}_p^{\ell} \circ \mathsf{proj}_{\ell}^{j} \circ \mathsf{proj}_j^{b} \circ \mathsf{proj}_b^{p}.$$

The map in Γ defined by $x \mapsto \theta(px) \cap b^{\perp}$ is a collineation of Γ that we will also denote by θ (and there is no danger of confusion).

Since qd is in $p^{\perp} \cap b^{\perp} \cap j^{\perp} \cap \ell^{\perp}$, every point of qd is fixed by θ.

Let π be an arbitrary plane through pq. Let π' be the unique plane through b intersecting π in a line. Then $\pi' = \mathsf{proj}_b^{p}(\pi)$. Since $j \in \pi$, the unique plane through j intersecting π is π again. This means that $\mathsf{proj}_j^{b} \circ \mathsf{proj}_b^{p}(\pi) = \pi$. The same token shows $\mathsf{proj}_p^{\ell} \circ \mathsf{proj}_{\ell}^{j}(\pi) = \pi$, and so $\theta(\pi) = \pi$.

Now let π be an arbitrary plane through pd. Set $\pi' = \mathsf{proj}_b^{p}(\pi)$. As in the previous paragraph, since $\ell \in \pi'$, we find $\mathsf{proj}_{\ell}^{j} \circ \mathsf{proj}_j^{b}(\pi') = \pi'$. Projecting π' onto $\mathsf{Res}(p)$ yields π again. Hence $\theta(\pi) = \pi$ again.

Intersecting with Γ, we already see that θ is a root elation fixing all points on dq and all lines through d and q. There remains to show that θ maps un to $u'n'$.

The line pu maps to bu under the first projection and then to ju under the second projection. Since i is the unique point of ju that ℓ is collinear to, the line ju maps to the line $i\ell$ under the third projection and then to ip under the fourth projection. Since i is on pu', we have $pu' = pi$, and with that $\theta(u) = u'$. Projecting preserves incidence, and therefore the unique point on nd that u' is collinear to has to be the point $\theta(n) = n'$. Hence the line nu moves to $n'u'$ under θ. □

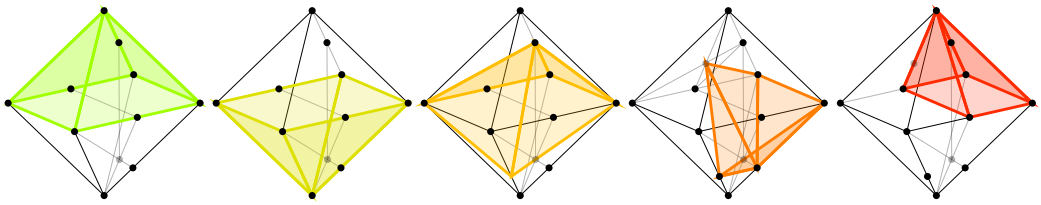


FIG. 8. How the points, lines, and planes map under θ from the proof of Lemma 6.3.2.3.

Observation 6.3.2.4. *If we change the position of l in our construction and define it to be the projection of the intersection point of pv' and jv onto bn, then*

$$\mathsf{proj}_p^l \circ \mathsf{proj}_l^j \circ \mathsf{proj}_j^b \circ \mathsf{proj}_b^p$$

is an automorphism that stabilizes all planes through pq or pn and moves the point $v \neq u$ on uq to the point $v' \neq u$ on uq. For a visualization, see Figure 9.

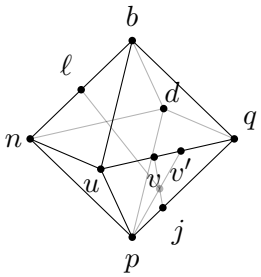


FIG. 9

Observation 6.3.2.5. *If we change the position of l in our construction and define it to be on the line bq, but not equal to q, then*

$$\mathsf{proj}_p^l \circ \mathsf{proj}_l^j \circ \mathsf{proj}_j^b \circ \mathsf{proj}_b^p$$

is an automorphism that fixes every plane containing pq and every line that is contained in either $\langle p, u, q\rangle$ or $\langle p, d, q\rangle$. For a visualization, see Figure 10.

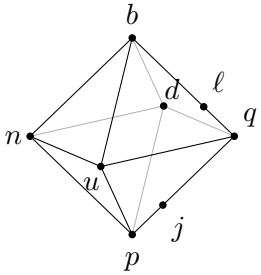


FIG. 10

If $[\mathsf{proj}_{[\mathsf{proj}_{bn}(j),j]}(l), l]$ intersects pn, then θ is the identity. In that case, the lines pn, $[\mathsf{proj}_{bn}(j), j]$, bq, pq, $[\mathsf{proj}_{[\mathsf{proj}_{bn}(j),j]}(l), l]$, and bn form a grid.

6.3.2.2. First kind of root

Lemma 6.3.2.6. *Let Δ be a polar space of rank 3, and let Γ be the generalized quadrangle obtained from two opposite points p, b of Δ by considering $p^\perp \cap b^\perp$. Then all roots of the first kind of Γ are Moufang. Furthermore, every root elation of Γ fixing the interior of a root of the first kind can be written as a self-projectivity of length 4.*

Proof. As in the previous proof, let α be an apartment of Γ defined by a polar frame $\{d, q, u, n\}$, where d is collinear to q and n, and opposite u. Let α' be another apartment of Γ spanned by a polar frame $\{d, q, u', n\}$, where u' is a point collinear to n and q, and opposite d and u. In the following, we will construct an automorphism of Γ that fixes dq and dn pointwise, moves u to u', and fixes all lines through d.

Pick any point on bu not equal to b or u and denote it by j'. Since b and p, and u and u' are opposite, and $u \perp p \perp u' \perp b \perp u$, the lines bu and pu' are opposite. Let j'' be the projection of j' onto pu'. The line bd is opposite $j'j''$, because j' is not collinear to d, and if b were collinear to j'', then b would be collinear to all points on $u'p$, a contradiction. Let ℓ be the projection of d onto $j'j''$. Arguing as above, we see that bu and pd are opposite. Let j be the projection of j' onto pd. Since $j' \perp n$ and $j'' \perp n$, we conclude $\ell \perp n$. In the same way, $\ell \perp q$.

Next, we will show that ℓ is opposite both p and j. The point p is collinear to j'' but not to j', since pd and bu are opposite and p is already collinear to u. In particular, p is not collinear to all points of $j'j''$ and not collinear to ℓ. Suppose that j were collinear to j''. Then j would be collinear to all of pu', and u' would be collinear to p and j, so to all of pd. That contradicts the fact that u' and d are opposite. That means j' is the unique point on $j'j''$ that j is collinear to. In particular, j is not collinear to ℓ. For a visualization, see Figure 11.

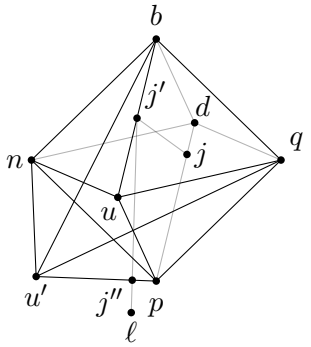


Fig. 11

Since $p \equiv b \equiv j \equiv \ell \equiv p$, we can project in each case from one residue into the other.

We define $\theta \colon \mathsf{Res}(p) \to \mathsf{Res}(p)$ as follows:

$$\theta := \mathsf{proj}_p^\ell \circ \mathsf{proj}_\ell^j \circ \mathsf{proj}_j^b \circ \mathsf{proj}_b^p$$

From the discussion above, it follows that both j and ℓ are collinear to all points of the lines dq and dn. This readily implies that θ fixes every point on $dq \cup dn$.

Also, since j is contained in each plane through the line pd, we find that each plane through pd is stabilized by θ. Hence θ induces a root elation in Γ. We now check that $\theta(u) = u'$.

We have $bu = \mathsf{proj}_b^p(pu)$. Since $j \perp j' \in bu$, we have $jj' = \mathsf{proj}_j^b(bu)$. Since $j' \perp \ell$, we have $j'\ell = \mathsf{proj}_\ell^j(jj')$. Finally, since $p \perp j'' \in j'\ell$, we have $pj'' = \mathsf{proj}_p^\ell(j'\ell)$. But $j'' \in pu'$, hence $pj'' = pu'$. It follows that $\theta(pu) = pu'$, and so $\theta(u) = u'$. This proves the lemma. □

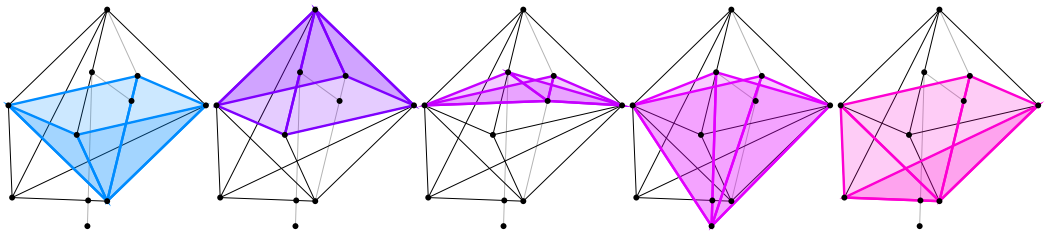


FIG. 12. How the points, lines, and planes map under θ from the proof of Lemma 6.3.2.6.

Corollary 6.3.2.7. *Let Δ be a polar space of rank at least 3, and let Γ be the generalized quadrangle obtained from two opposite singular subspaces U, V of Δ of dimension $n-3$ by considering $U^\perp \cap V^\perp$. Then Γ is a Moufang quadrangle.*

PROOF. The claim follows from Lemmata 6.3.2.3 and 6.3.2.6. □

6.4. Proving the Moufang Property for Polar Spaces of Finite Rank

In the following, we show that the elations we found for generalized quadrangles in Section 6.3.2 extend to elations of the ambient polar space. We will only show this in detail for the case that the latter has rank 3. However, it can be done for arbitrary rank $n \geq 3$, and we will comment on that at the end of this section.

6.4.0.1. Second kind of root

For the rest of the section, let Δ be a polar space of rank 3. We want to show that, for every line dq in Δ through two arbitrary collinear points d and q, we can construct an automorphism $\phi\colon \Delta \to \Delta$ that fixes all planes through dq pointwise, all lines through either d or q linewise, and acts transitively on the points of the lines through either d or q that are not contained in a plane through dq.

Lemma 6.4.0.1. *Let d, q be two collinear points in the polar space Δ. Let m, m' be two collinear points of Δ not collinear to q but such that $d \in mm'$. Then there exists a unique permutation η of the set of points $d^\perp \cup q^\perp$ with the following properties.*

(*i*) $\eta(m) = m'$,
(*ii*) *all points of* $d^\perp \cap q^\perp$ *are fixed,*
(*iii*) *the collineation induced by* η *in each plane* π *containing* d *or* q*, but not both, is a translation of* π *with axis* $\mathsf{proj}_\pi(dq)$ *and center* d *or* q*, respectively,*
(*iv*) η *preserves collinearity.*

PROOF. Let π be a given plane containing dq. Let β be the plane spanned by m and $L := \mathsf{proj}_\pi(m)$. Then β contains d and m' (cf. Fig. 13). Since projective planes inside polar spaces are Moufang, there exists an elation η of β with axis L and center d such that $\eta(m) = m'$. Now let J be a line through q not collinear to d. Then J is opposite every line of β containing d, except for L. For every line $K \subseteq \beta$, with $d \in K \neq L$, we define an action

$$\eta_K \colon J \to J, \quad x \mapsto \mathsf{proj}_J^K(\eta(\mathsf{proj}_K^J(x))).$$

From now on, we will abbreviate the definition of η_K above by the sentence "we copy the action of η on K to J."

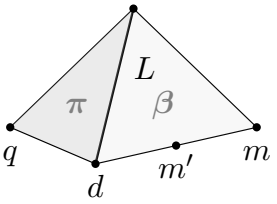


FIG. 13

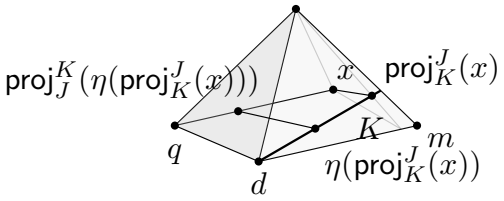


FIG. 14

We claim

(*) The map η_K is independent of K.

Indeed, let K_1 and K_2 be two lines in β through d, distinct from L. Let $y_1 \in K_1 \setminus \{d\}$ be arbitrary. Set $x := \mathsf{proj}_J(y_1)$. If $z = \mathsf{proj}_L(x)$, then $y_1 z = \mathsf{proj}_\beta(x)$. It follows that $\mathsf{proj}_{K_2}(x) = y_1 z \cap K_2 =: y_2$, and hence $z \in \eta(y_1)\eta(y_2)$, as $\eta(z) = z$ (cf. Fig. 15).

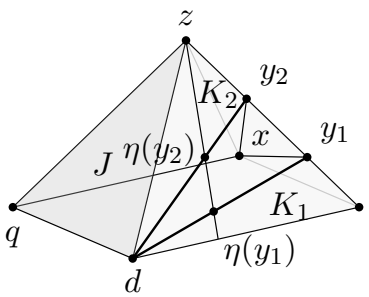


FIG. 15

Since $z = \mathsf{proj}_L(x)$, z is collinear to both x and q, and therefore z is collinear to all points of J. Thus, we have $z = L \cap \mathsf{proj}_\pi(J)$, and as such, z is also equal to $\mathsf{proj}_L(\eta_{K_1}(x)) = \mathsf{proj}_L(\eta_{K_2}(x))$.

It follows that $\mathsf{proj}_\beta(\eta_{K_1}(x)) = \eta(y_1)z$, which, by the above, coincides with $\eta(y_2)z$. Hence, $\eta_{K_1}(x)$ is collinear to $\eta(y_2)$, and with that, we conclude that $\eta_{K_2}(x) = \eta_{K_1}(x)$. The claim is proved.

Thanks to (*), we can denote η_K simply by η. Now let α be any plane through J. We claim that

(**) There exists a unique elation in α with axis $A := \mathsf{proj}_\alpha(dq)$ and center q extending the action of η.

This follows in the same way as (*) by letting mm' play the role of J and α the role of β in the proof of (*).

Let β' be any plane intersecting β in a line through d. Then (**) with the roles of d and q interchanged implies that there is a unique elation in β' with axis $\mathsf{proj}_{\beta'}(dq)$ agreeing with η on the line $\beta \cap \beta'$. By (*), the action of that elation on any line of β' through d (not collinear to q), copied to J, agrees with η. If we keep doing this procedure, then — since in $\mathsf{Res}(d)$ the geometry opposite dq is connected (see [**Van24**]) — the action of η on J, copied on every line through d opposite J, with, additionally, the identity on $d^\perp \cap q^\perp$, defines a permutation η preserving collinearity and inducing a non-trivial elation in every plane through d not containing q.

Note that η, acting on $d^\perp$ as defined in the previous paragraph, is independent of the line J through q in α, as follows from (*) with d and q interchanged. Likewise, we can now extend η from α to $q^\perp$, preserving collinearity, and the action on every line through q not collinear to d is copied from the action of η on any line through d not collinear to q.

Since "copying the action" just means that collinear pairs of points from $d^\perp \times q^\perp$ are mapped to collinear pairs of points, the lemma is proved. □

We now make the connection between η as defined in Lemma 6.4.0.1 and the axial elation defined in Lemma 6.3.2.3.

Lemma 6.4.0.2. *Let d, q be two collinear points in the polar space Δ. Let m, m' be two collinear points of Δ not collinear to q, but such that $d \in mm'$. Let η be the unique permutation of $d^\perp \cup q^\perp$ with the properties mentioned in* Lemma 6.4.0.1. *Let p and b be two opposite points both collinear to d, q, and m. Let θ be the root elation of $p^\perp \cap b^\perp$ with root (q, qd, d) mapping m to m'. Then the action of θ on every line through q or d in $p^\perp \cap b^\perp$ coincides with the action of η on that line.*

Proof. Let j be an arbitrary point on $pq \setminus \{p, q\}$, and let β be the plane spanned by b, m, and d. Define ℓ as the intersection of the line bd with the line joining m with $bm' \cap \mathsf{proj}_\beta(j)$ (cf. Fig. 16).

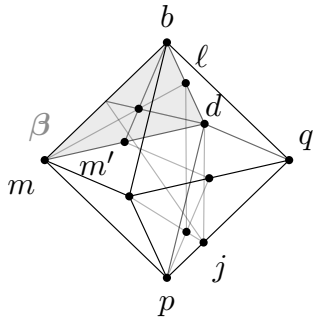


FIG. 16

With the proof of Lemma 6.3.2.3, we find that $p \barwedge \ell \barwedge j \barwedge b \barwedge p$ moves pm to pm'. Hence, this defines θ in $p^\perp \cap b^\perp$. For x on mm', we set $x' := \ell x \cap \mathsf{proj}_\beta(j)$. Then the action of θ on the line mm' is given by $x \mapsto bx' \cap mm'$. It is well known that this is the restriction to mm' of the elation of β with axis bd and center d, mapping m to m'. Hence, $\theta(x) = \eta(x)$ for all $x \in mm'$. Since the actions on the lines of $p^\perp \cap b^\perp$ through q are obtained by copying, for both η and θ, these actions coincide. By copying these to the lines through d, the lemma follows. □

The following is an immediate consequence of the foregoing.

Corollary 6.4.0.3. *Let d, q be two collinear points in the polar space Δ. Let m, m' be two collinear points of Δ not collinear to q, but such that $d \in mm'$. Let η be the unique permutation of $d^\perp \cup q^\perp$ with the properties mentioned in* Lemma 6.4.0.1. *Let p and b be two opposite points both collinear to d, q, and m. Let θ be the root elation of $\mathsf{Res}(p)$ with root $(pq, \langle p, q, d\rangle, pd)$ mapping pm to pm'. Let θ_b be the collineation of $\mathsf{Res}(b)$ defined by $L \mapsto \mathsf{proj}_b(\theta(\mathsf{proj}_p(L)))$. Then θ_b is the unique root elation of $\mathsf{Res}(b)$ for which the action on the lines through b collinear to q or d coincides with the action of η.*

The definition of the mapping $L \mapsto \mathsf{proj}_b(\theta(\mathsf{proj}_p(L)))$ will be abbreviated by "θ is copied from p to b."

The next lemma allows us to extend η to the whole polar space Δ in an unambiguous way.

Lemma 6.4.0.4. *Let q, d, and η be as before. Let x be a point collinear to a unique point y of dq. Let p and b be two opposite points collinear to all of q, d, and x. Then there exists a unique root elation $\eta_{p,b}$ in $p^\perp \cap b^\perp$ corresponding to the root (d, dq, q) having the same action as η on the points collinear to q or d. Moreover, $\eta_{p,b}(x)$ does not depend on p and b.*

PROOF. The existence and uniqueness of $\eta_{p,b}$ follow from Lemma 6.4.0.2. Now let o be another point collinear to all of q, d, x, and assume first that o is opposite both p and b. Let θ_p, θ_b, and θ_o be the root elations of $\mathsf{Res}(p)$, $\mathsf{Res}(b)$, and $\mathsf{Res}(o)$, respectively, for which the action on the lines through p, b, o, respectively, collinear to q (or d) coincides with the action of η. By Corollary 6.4.0.3, θ_p copied from p to b is θ_b; θ_b copied from b to o is θ_o, and θ_p copied from p to o is again θ_o. It follows that $\theta_o(ox)$ intersects both $\theta_p(px)$ and $\theta_b(bx)$ non-trivially. But $\theta_p(px) = p\eta_{p,b}(x)$ and $\theta_b(bx) = b\eta_{p,b}(x)$. Since these two lines are not contained in a plane, the

line $\theta_o(ox)$, which intersects both, contains the intersection point $\eta_{p,b}(x)$. Hence $\eta_{p,o}(x) = \eta_{p,b}(x) = \eta_{o,b}(x)$.

Now assume that o is collinear to b. Note that o is then opposite p. We denote the plane spanned by q, d, b by β. Since Δ has rank 3, o has to be contained in β, and since $\mathsf{proj}_\beta(x) = by$, we have $o \in by$.

Since $\eta_{p,b}(x)$ is also collinear to y, we find that $o \perp \eta_{p,b}(x)$. Hence $\eta_{p,b}(x) = \eta_{p,o}(x)$.

Now let p', b' be two other opposite points collinear to p, b, x. At most one of p, b is collinear to b', say p is opposite b'. By the foregoing, $\eta_{p,b}(x) = \eta_{p,b'}(x) = \eta_{p',b'}(x)$, and the lemma is proved. □

Proposition 6.4.0.5. *Let d, q be two collinear points in the polar space Δ. Let m, m' be two collinear points of Δ not collinear to q, but such that $d \in mm'$. Using the notation of the previous lemmata, the automorphism $\phi\colon \Delta \to \Delta$ defined by*

$$\phi(x) := \begin{cases} x, & \text{if } x \text{ is in a plane with } dq, \\ \eta(x), & \text{if } x \perp q \text{ or } x \perp d, \text{ not in a plane with } dq, \\ \eta_{p,b}(x), & \text{if } x \equiv d, x \equiv q, p \equiv b, x, d, q \in p^\perp \cap b^\perp, \end{cases}$$

is a well-defined collineation on the whole polar space Δ that fixes all planes through dq pointwise, stabilizes all lines through either d or q, and maps m to m'.

Proof. We have already seen that ϕ is well-defined. Since interchanging the roles of m and m' results in a two-sided inverse map, we deduce that ϕ is bijective. It suffices to prove that ϕ maps lines into lines.

This is certainly true for lines intersecting the line dq and for lines collinear to either q or d. There are two possibilities left.

(i) Let L be a line collinear to a unique point $e \in dq \setminus \{d, q\}$. The projection f of q onto L is collinear to dq. Then f is fixed. For each point $x \in L \setminus \{f\}$, Lemma 6.4.0.4 ensures that $\phi(x)$ is contained in the line $\theta_f(L)$ (with the notation of the proof of Lemma 6.4.0.4).

(ii) Let L be opposite dq. We can then pick two opposite points p, b collinear to both L and qd. Then, by Lemma 6.4.0.4 again, $\phi(L) = \eta_{p,b}(L)$ is a line.

The proposition is proved. □

The arbitrariness of m and m' now yields the following consequence.

Corollary 6.4.0.6. *Let Δ be a polar space of rank 3. Then all roots of the second kind are Moufang.*

6.4.0.2. First kind of root

Lemma 6.4.0.7. *Let Δ be a polar space of rank 3. Then all roots of the first kind are Moufang.*

Proof. Let α and β be two planes intersecting in a point o. Let L and M be two opposite lines in α and β, respectively, and let p and p' be two points opposite

o in $L^\perp \cap M^\perp$ (cf. Fig. 17). We aim to construct a collineation ϕ of Δ stabilizing all lines through o, fixing $\alpha \cup \beta$ pointwise, and mapping p to p'.

Let x be a point on L. Let η_x be the root elation of $\mathsf{Res}(x)$ fixing all lines of α through x, fixing all lines through x that intersect β, stabilizing all planes through ox, and mapping xp to xp'. We can copy that action onto every point $y \in \beta$ opposite x; meaning, we define η_y as the unique collineation of $\mathsf{Res}(y)$ mapping any line K through y to $\mathsf{proj}_y(\eta_x(\mathsf{proj}_x K))$, for any $y \in \beta \setminus \mathsf{proj}_\beta(x)$ (cf. Fig. 18). Note that — since η_x fixes all lines through x that intersect β — η_y fixes $\mathsf{proj}_\beta(x)$ pointwise and thus stabilizes all lines through y in β.

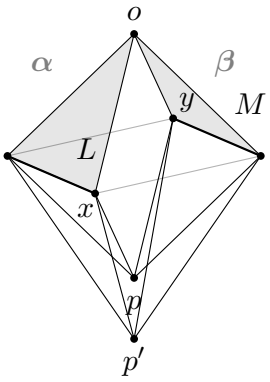


FIG. 17

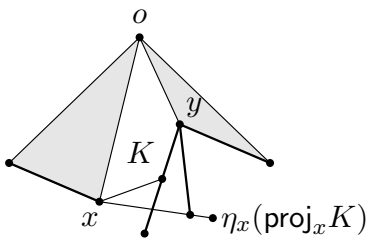


FIG. 18

The collineation η_x defines an action on all lines through x, and the collineations η_y ($y \in \beta \setminus \mathsf{proj}_\beta(x)$) "extend" this action to all lines containing some point of $\beta \setminus \mathsf{proj}_\beta(x)$. We want to show that the action is independent of the choice of $x \in \alpha \setminus \{o\}$.

For that, let $z \in \alpha$ be a point such that $z \neq x$ and such that the line xz does not contain o. For any $y \in \beta$ opposite z, let $\eta_{z,y}$ be the action of η_y copied onto z. Then $\eta_{z,y}$ is always a root elation in $\mathsf{Res}(z)$ with center oz and axes α and $\langle z, \mathsf{proj}_\beta(z)\rangle$. Furthermore, each plane through the line xz has the same image under η_x and under $\eta_{z,y}$. Since a root elation in $\mathsf{Res}(z)$ with center oz is determined by the image of any plane not through oz, we conclude that $\eta_z := \eta_{z,y}$ is independent of y. Interchanging the roles of x and z, we conclude that there exists a unique root elation η_a of $\mathsf{Res}(a)$ for each point a of α, such that the set of all such root elations, together with all η_b for $b \in \beta$, is closed under copying from points to opposite points.

Now let w be an arbitrary point of Δ opposite o. Then $w \notin \alpha \cup \beta$, and we set $A := \mathsf{proj}_\alpha(w)$ and $B := \mathsf{proj}_\beta(w)$ (cf. Fig. 19). Select $a \in A$ and let π be the plane spanned by w and A. Set $\pi' := \eta_a(\pi)$. Select $b \in B$ and let σ be the plane spanned by B and w. By copying, $\sigma' := \eta_b(\sigma)$ intersects π' in a unique point, which we define to be $\phi(w)$. Note that, by copying, $\eta_a(aw) = a\phi(w)$ for all $a \in A$, and $\eta_b(bw) = b\phi(w)$ for all $b \in B$.

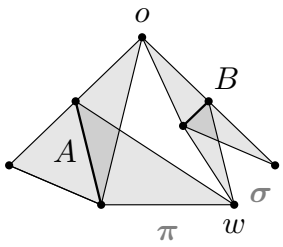


FIG. 19

Now let $w \perp o$. If $w \in \alpha \cup \beta$, we define $\phi(w) = w$. If $w \notin \alpha \cup \beta$, there are two possibilities. First, suppose $A := \mathsf{proj}_\alpha(w)$ and $B := \mathsf{proj}_\beta(w)$ are not coplanar. For an arbitrary point $a \in A \setminus \{o\}$, we define $\phi(w) = \eta_a(aw) \cap \langle w, B\rangle$. Similarly, as in the previous paragraph, we find that this is independent of $a \in A \setminus \{o\}$, and also $\phi(w) = \eta_b(bw) \cap \langle w, A\rangle$ for all $b \in B \setminus \{o\}$.

Secondly, suppose A and B, as defined above, are coplanar. Then for each a in $A \setminus \{o\}$, we have $\eta_a(aw) = aw$, and we define $\phi(w) = w$.

Hence, in all cases, we have that if $c \in \alpha \cup \beta \setminus \{o\}$ is collinear to w, then

$$\eta_c(cw) = c\phi(w).$$

It remains to show that lines are mapped to lines. If a line K is not collinear to o, then it is collinear to unique distinct points $a \in \alpha$ and $b \in \beta$, and the images under ϕ of the points of K are all contained in the line $K' := \eta_a(\langle a, K\rangle) \cap \eta_b(\langle b, K\rangle)$. Applying the inverse mapping (obtained by interchanging the roles of p and p'), we see that $\phi(K)$ coincides with the whole line K'.

Finally, let K be collinear to o. Each point $w \in K$ is mapped onto a point $\phi(w) \in ow$. Consider a plane $\pi \not\ni o$ through K. By the foregoing, the image $\phi(\pi)$ is contained in a plane that intersects $\langle o, K\rangle$ in the set $\{\phi(w) \mid w \in K\}$. Since distinct planes sharing at least two points intersect in lines, we see that $\phi(K)$ is a line.

This completes the proof of the lemma. □

Corollary 6.4.0.8. *Let Δ be a polar space of rank 3. Then Δ is Moufang. Furthermore, every root elation in each point residue can be written as an even self-projectivity of length 4.*

PROOF. The claim follows from Corollary 6.4.0.6 and Lemma 6.4.0.7. □

6.4.0.3. EXTENDING TO RANK $n > 3$

Corollary 6.4.0.9. *Let Δ be a polar space of rank n. Then Δ is Moufang.*

For the induction, suppose we have shown that the elations θ extend in the case where the polar space Δ has rank $n-1$. Suppose $\tilde{\Delta}$ is a polar space of rank n containing Δ. Let $\mathscr{A}$ be some apartment of $\tilde{\Delta}$ spanned by a polar frame

$$\{p_{-n}, p_{-n+1}, \ldots, p_{-1}, p_1, p_2, \ldots, p_n\}.$$

Since we know that the extension works for Δ, we know that θ is determined for all singular subspaces spanned by the points

$$\{p_{-n+1}, p_{-n+2}, \ldots, p_{-1}, p_1, p_2, \ldots, p_{n-1}\}.$$

Now, with the exact same techniques as before, we can copy the action of θ to the surrounding singular subspaces.

Note 6.4.0.10. Corollary 6.4.0.9 also holds for top-thin polar spaces of rank at least 3. Indeed, a rank 3 top-thin polar space is equivalent to a projective space of dimension 3, and the root elations of the latter are just line elations of the former. Then the same inductive arguments as for the thick case above can be used to prove that top-thin polar spaces of rank at least 3 admit all line elations, because line elations are root elations. By the top-thinness, there are no point elations to consider and we conclude that Corollary 6.4.0.9 holds for top-thin polar spaces of rank at least 3.

6.5. Proving the Moufang Property for Polar Spaces of Infinite Rank

We will now use Definition 6.1.2.11 to prove the Moufang property for polar spaces of infinite rank. This section is the only section in which degenerate polar spaces appear. Hence, we explicitly talk of non-degenerate polar spaces, instead of just polar spaces. We refer the reader interested in polar space of infinite rank to [**Pas09**]. For our proof, we will only need the following two facts.

Fact 6.5.0.1. Every non-degenerate polar space of infinite rank is embeddable into some projective space (cf. [**Joh90**, Theorem 5.1]).

Fact 6.5.0.2. Let $\widehat{\Delta}$ be a projective space and let Δ be a non-degenerate polar space embedded in $\widehat{\Delta}$. Then every subspace of $\widehat{\Delta}$ intersects Δ in a (possibly degenerate) polar subspace. This follows straight from the definition.

Lemma 6.5.0.3. *Let Δ be a non-degenerate polar space of infinite rank. Then for every finite set of points P in Δ there exists a finite-rank polar subspace $\Gamma \subset \Delta$ containing all points of P.*

Proof. According to Fact 6.5.0.1, we can embed Δ into some projective space $\widehat{\Delta}$. In $\widehat{\Delta}$ we consider the subspace $\langle P \rangle$ generated by all points of P. Since P is finite, $\langle P \rangle$ has finite dimension in $\widehat{\Delta}$. The intersection of $\langle P \rangle$ with Δ is a polar subspace of Δ of finite rank, but possibly degenerate (Fact 6.5.0.2). Suppose $\langle P \rangle_\Delta := \langle P \rangle \cap \Delta$ is degenerate, and let R be the non-empty radical of $\langle P \rangle_\Delta$. Let S be a singular subspace opposite R. The subspace S is finite-dimensional, because R is finite-dimensional. Now let Γ be the polar subspace of Δ generated by $\langle P \rangle_\Delta$ and S. Then Γ is of finite rank and non-degenerate. □

Lemma 6.5.0.4. *Let Δ be a non-degenerate polar space of infinite rank, and let Γ and Γ' be two distinct polar subspaces of finite rank. Then there exists a polar subspace Δ' of finite rank containing $\Gamma \cup \Gamma'$.*

PROOF. We consider the subspace S of Δ generated by $\Gamma \cup \Gamma'$. Then S is a possibly degenerate polar space of finite rank. Suppose S is degenerate, and let R be the radical of S. Let T be a singular subspace opposite R, and let Δ' be the polar subspace generated by S and T. Then Δ' is a polar subspace of finite rank containing $\Gamma \cup \Gamma'$. □

Theorem 6.5.0.5. *Let Δ be a non-degenerate polar space of infinite rank. Then Δ is Moufang.*

PROOF. **(Line elations)** Let p and q be two collinear points in Δ. Let a be a point in Δ collinear to p but not to q. Let a' be some point on the line pa not equal to p or a. We want to show that we can find a line elation of Δ with center (p, pq, q) which maps a to a'.

Let x be some arbitrary point in Δ. For the set of points $P := \{a, a', p, q, x\}$, we can find a non-degenerate polar subspace $\Gamma \subset \Delta$ of finite rank containing all points of P (Lemma 6.5.0.3). For Γ there exists a line elation θ_Γ with center (p, pq, q) which maps a to a' (Corollary 6.4.0.9).

i) We first want to show that the image of x under θ_Γ does not actually depend on the polar subspace $\Gamma \subset \Delta$. For that, let Γ' be another non-degenerate polar subspace in Δ of finite rank containing all points of P. According to Lemma 6.5.0.4, there exists a non-degenerate polar subspace Λ of finite rank containing $\Gamma \cup \Gamma'$. Now $\theta_\Lambda|_\Gamma$ is θ_Γ, and $\theta_\Lambda|_{\Gamma'}$ is $\theta_{\Gamma'}$, and both are restrictions of θ_Λ, and therefore, for every point $x \in \Gamma \cap \Gamma'$, we have $\theta_\Gamma(x) = \theta_{\Gamma'}(x)$. Thus, the image of x under θ_Γ does not depend on the polar subspace Γ, and we just write θ in the following.

ii) The only thing left to show is that θ maps lines to lines. For that, let L be an arbitrary line in Δ. Then L is contained in a non-degenerate polar subspace Γ of Δ of finite rank. We know that $\theta|_\Gamma(L)$ is a line, and that this image does not depend on $\Gamma \subset \Delta$.

Thus, θ maps lines to lines, is defined for every point $x \in \Delta$, has center (p, pq, q), and maps a to a'.

(Point elations)

For point elations we can use the exact same arguments as for line elations. □

Note 6.5.0.6. Theorem 6.5.0.5 also holds for top-thin infinite rank polar spaces, with the same proof as above, now using Theorem 6.5.0.5.

6.6. Non-existence of Thick Spherical Buildings of Types H_3 and H_4

Now, we would like to give constructions for non-trivial root elations for the thick generalized pentagons contained in thick icosahedral geometries. The goal is to prove that point residues of thick icosahedral geometries are Moufang (cf. Proposition 6.6.2.2), which then implies their non-existence (cf. Corollary 6.6.2.3). Again,

we want to build elations using projections. Thus, our next goal is to define what it means to *project* in an icosahedral geometry.

6.6.1. Projections

Let Δ be a thick icosahedral geometry. Let p and b be two opposite points in Δ. We define proj_p^b (or proj_p for short if it is clear from where we are projecting) as follows.

Projecting a plane through b to the point p: Let π be a plane through b. There exists a unique line in π for which every point has distance 2 from p. There exists a unique point p' in $p^\perp$ that is collinear with that line. We set $\mathsf{proj}_p(\pi) := pp'$.

Projecting a line through b to the point p: Let L be a line through b. There exists a unique point ℓ on L that has distance 2 from p and a unique line L' in $\ell^\perp \cap p^\perp$. We set $\mathsf{proj}_p(L) := \langle p, L' \rangle$.

Notation 6.6.1.1. If we are given points p_i such that p_i is opposite both p_{i-1} and p_{i+1} for $i \in \mathbb{Z}/n\mathbb{Z}$, and we project from p_0 to p_1, …, to p_{n-1}, then we also write $p_0 \mathbin{\overline{\wedge}} p_1 \mathbin{\overline{\wedge}} \dots \mathbin{\overline{\wedge}} p_{n-1}$. If we want to see how a specific point x of $p_0^\perp$ maps from p_0 to p_1, …, to p_{n-1}, and it is clear that we are considering $\theta\colon p_0 \mathbin{\overline{\wedge}} p_1 \mathbin{\overline{\wedge}} \dots \mathbin{\overline{\wedge}} p_{n-1}$, then we also write $p_0 x \mapsto \mathsf{proj}_{p_1}(p_0 x) \mapsto \dots \mapsto x^\theta$.

6.6.2. Constructions of Elations

We will show the following proposition.

Proposition 6.6.2.2. *Let Δ be a thick icosahedral geometry. Then its point residues are Moufang.*

More precisely, we will show the following: Let p and b be two opposite points of Δ. Let Σ be an apartment of the generalized pentagon $b^\perp \cap p^{\perp\perp}$ with points

$$\{b_0, b_1, b_2, b_3, b_4\},$$

such that $b_i \perp b_{i+1}$ for $i \in \mathbb{Z}/5\mathbb{Z}$. Then we can construct an elation that fixes the lines b_0b_1 and b_1b_2 pointwise, stabilizes all lines through b_0 and b_1 in $b^\perp \cap p^{\perp\perp}$, and maps Σ to an apartment Σ' with points $\{b_0, b_1, b_2, b_3', b_4'\}$, such that $b_2 \perp b_3' \perp b_4' \perp b_0$, $b_3 \neq b_3'$, and $b_4 \neq b_4'$. Since p, b, Σ, and Σ' were arbitrary, this shows that point residues of Δ are Moufang.

Proof. Let $p_0, \dots, p_4$ be points that span a pentagon in $p^\perp \cap b^{\perp\perp}$, such that p_i is collinear with $b_{i-1}b_i$ for $i \in \mathbb{Z}/5\mathbb{Z}$. Let b_4' be a point on b_0b_4 not equal to b_0 or b_4, and let b_3' be a point in $b^\perp \cap p^{\perp\perp}$ that is collinear with b_2 and b_4'. We want to define a map $\theta\colon \mathsf{Res}(b) \to \mathsf{Res}(b)$ such that θ fixes b_0b_1 and b_1b_2 pointwise, stabilizes all lines through b_0 and b_1, and moves b_3 to b_3' and b_4 to b_4'. Let d be an arbitrary point on bb_1 not equal to b or b_1.

The point d has distance 2 to all points of a unique line through p_0 in $\langle p, p_0, p_4 \rangle$ that intersects pp_4 in some point p_4'. Every point on bb_1 has distance 2 to p_3. With that, d

has distance 2 to all points of the line $p_3p'_4$ in $\langle p, p_3, p_4\rangle$. We define $d_4 := d^\perp \cap (p_0p'_4)^\perp$ and $d_3 := d^\perp \cap (p_3p'_4)^\perp$. For a visualization, see Figure 6.6.2.

We consider the projectivity $\varphi\colon b \overline{\wedge} p \overline{\wedge} d$ and observe the following, using the notation introduced in 6.6.2:

$$
\begin{aligned}
bb_2 &\mapsto \langle p, p_2, p_3\rangle \mapsto db_2\\
\langle b, b_1, b_2\rangle &\mapsto pp_2 \mapsto \langle d, b_1, b_2\rangle\\
&= \langle b, b_1, b_2\rangle\\
bb_1 &\mapsto \langle p, p_1, p_2\rangle \mapsto db_1\\
\langle b, b_0, b_1\rangle &\mapsto pp_1 \mapsto \langle d, b_0, b_1\rangle\\
&= \langle b, b_0, b_1\rangle\\
bb_0 &\mapsto \langle p, p_0, p_1\rangle \mapsto db_0\\
\langle b, b_0, b_4\rangle &\mapsto pp_0 \mapsto \langle d, b_0, d_4\rangle\\
bb_4 &\mapsto \langle p, p_0, p_4\rangle \mapsto dd_4\\
\langle b, b_3, b_4\rangle &\mapsto pp_4 \mapsto \langle d, d_3, d_4\rangle\\
bb_3 &\mapsto \langle p, p_3, p_4\rangle \mapsto dd_3\\
\langle b, b_2, b_3\rangle &\mapsto pp_3 \mapsto \langle d, b_2, d_3\rangle
\end{aligned}
$$

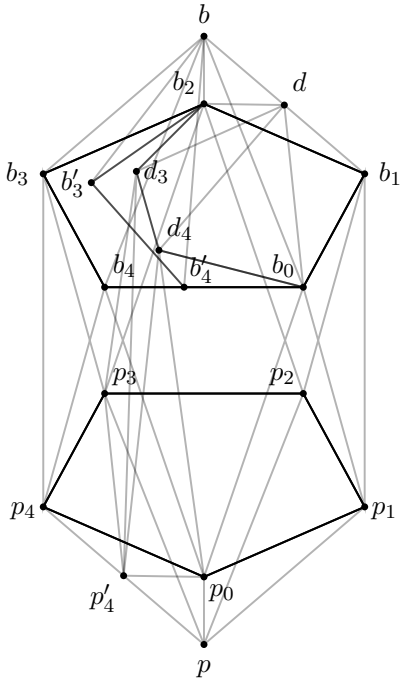


FIG. 20. The points and lines relevant for the construction of a root elation for the generalized pentagon $b^\perp \cap p^{\perp\perp}$.

Let x be an arbitrary point on b_0b_1. Then bx maps to a plane through pp_1 and then to dx. Let y be an arbitrary point on b_1b_2. Then by maps to a plane through pp_2 and then to dy.

We observe that the points b'_3 and d_4 are opposite and consider an apartment Σ' containing b'_3, b, b_2, b'_4, b_0, d, d_3, and d_4. The points b'_4 and d_4 have distance 2 and are both collinear with b_0. Let q_0 be another point on the line $(b'_4)^\perp \cap d_4^\perp$ through b_0. We label the leftover point of the pentagon $d_4^\perp \cap (b'_3)^{\perp\perp}$ in Σ' by q_4. Let q'_4 be the point in Σ' collinear with q_0, q_4, b'_3, and b'_4, and let q_3 be the point collinear with q'_4, q_4, b_2, b'_3, and d'_3. With that, all points of Σ' are labeled. The point b_1 is on the line bd. Let q be the unique point on $q_4q'_4$ with distance 2 to b_1.

We see that q has distance 3 to b and d, distance 2 to b_0, b_1, b_2, b'_3, b'_4, d_3, and d_4, and distance 1 to q_0, q_4, q'_4, q_3, q_2, and q_1. We consider an apartment containing $\langle q, q_0, q'_4\rangle$ and $\langle b, b_1, b_2\rangle$. We denote the point in that apartment that is collinear with q, q_0, b_0, and b_1 by q_1, and the point that is collinear with q, q_3, b_1, and b_2 by q_2 (cf. Fig. 21, Fig. 22).

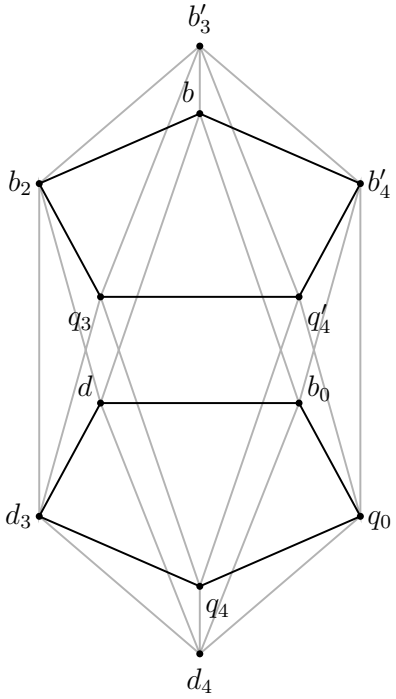


FIG. 21. The apartment containing $\langle b'_3, b, b_2\rangle$ and $\langle d_4, q_0, q_4\rangle$.

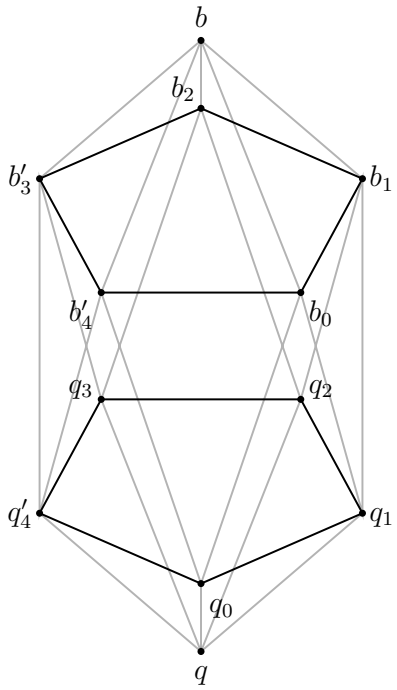


FIG. 22. The apartment containing $\langle b, b_1, b_2\rangle$ and $\langle q, q_0, q'_4\rangle$.

We set $\theta := b \barwedge p \barwedge d \barwedge q \barwedge b$ and claim that θ has the desired properties. Indeed, we observe the following.

$$
\begin{aligned}
&bb_2 \mapsto \langle p, p_2, p_3\rangle \mapsto db_2 \mapsto \langle q, q_2, q_3\rangle \mapsto bb_2\\
&\langle b, b_1, b_2\rangle \mapsto pp_2 \mapsto \langle d, b_1, b_2\rangle = \langle b, b_1, b_2\rangle \mapsto qq_2 \mapsto \langle b, b_1, b_2\rangle\\
&bb_1 \mapsto \langle p, p_1, p_2\rangle \mapsto db_1 \mapsto \langle q, q_1, q_2\rangle \mapsto bb_1\\
&\langle b, b_0, b_1\rangle \mapsto pp_1 \mapsto \langle d, b_0, b_1\rangle = \langle b, b_0, b_1\rangle \mapsto qq_1 \mapsto \langle b, b_0, b_1\rangle\\
&bb_0 \mapsto \langle p, p_0, p_1\rangle \mapsto db_0 \mapsto \langle q, q_0, q_1\rangle \mapsto bb_0\\
&\langle b, b_0, b_4\rangle \mapsto pp_0 \mapsto \langle d, b_0, d_4\rangle \mapsto qp_0 \mapsto \langle b, b_0, b_4\rangle\\
&bb_4 \mapsto \langle p, p_0, p_4\rangle \mapsto dd_4 \mapsto \langle q, p_0, q_4\rangle \mapsto bb'_4\\
&\langle b, b_3, b_4\rangle \mapsto pp_4 \mapsto \langle d, d_3, d_4\rangle \mapsto qq_4 \mapsto \langle b, b'_3, b'_4\rangle\\
&bb_3 \mapsto \langle p, p_3, p_4\rangle \mapsto dd_3 \mapsto \langle q, q_3, q_4\rangle \mapsto bb'_3\\
&\langle b, b_2, b_3\rangle \mapsto pp_3 \mapsto \langle d, b_2, d_3\rangle \mapsto qq_3 \mapsto \langle b, b_2, b'_3\rangle
\end{aligned}
$$

Let x be some arbitrary point on b_0b_1. Then bx maps to a plane through pp_1, to dx, to a plane through qp_1 and back to bx.

Let y be some arbitrary point on b_1b_2. Then by maps to a plane through pp_2, to dy, to a plane through qq_2 and back to by.

To see this in a drawing, we refer to figure 23.

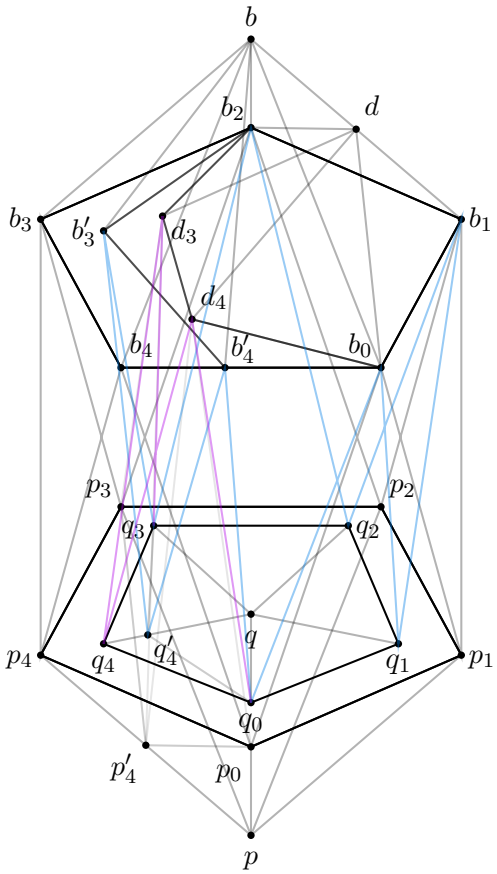


FIG. 23. The established points and lines relevant to determining the images under θ.

The only thing we have to show is that all lines through b_0 and b_1 in $b^\perp \cap p^{\perp\perp}$ are stabilized under θ. For that, we will first consider an arbitrary plane β through bb_1. Since bb_1 first maps to $\langle p, p_1, p_2\rangle$, β will first map to a line in $\langle p, p_1, p_2\rangle$. Since $\mathsf{proj}_d(\langle p, p_1, p_2\rangle) = db_1$ and $d \in \beta$, this line will map back to β. Similarly, $\mathsf{proj}_q(\beta)$ is a line in $\langle q, q_1, q_2\rangle$ that will map back to β when we project to b. This shows that all lines through b_1 in $b^\perp \cap p^{\perp\perp}$ are stabilized under θ.

For the lines through b_0, we first consider the projectivity

$$\psi\colon p \barwedge b \barwedge q \barwedge d \barwedge p.$$

We see that p_1p_2 is fixed pointwise under ψ. Furthermore, a plane γ through pp_1 maps to a line through b in $\langle b, b_0, b_1\rangle$ that intersects b_0b_1 in some point g, to a plane through qq_1, to a line through d in $\langle b, b_0, b_1\rangle$ that intersects b_0b_1 in the same point g, and back to γ. Thus, planes through pp_1 are stabilized under ψ, and in the same way, we can see that planes through pp_2 are stabilized.

We can define a projectivity $\theta' := p \barwedge b \barwedge q' \barwedge d' \barwedge p$, where q' is a point on pp_1 and d' is defined analogously to q previously, opposite p and d', and such that

$$(\mathsf{proj}_p^b)^{-1} \circ \theta' \circ \mathsf{proj}_p^b$$

moves b'_3 to b_3. Because of the given symmetry, θ' will fix p_0p_1 and p_1p_2 pointwise and stabilize all planes through pp_1.

Now we can consider

$$\psi' \colon b \barwedge p \barwedge d' \barwedge q' \barwedge b$$

and see that ψ' fixes $b_0 b_1$ pointwise and stabilizes all planes through bb_0 and bb_1.

We can see that

$$\psi' \circ \theta^{-1} = b \barwedge q \barwedge d \barwedge p \barwedge b \barwedge p \barwedge d' \barwedge q' \barwedge b$$

fixes all planes through bb_1 and all points on $b_0 b_1$, since both ψ' and θ^{-1} do that. But then $\psi' \circ \theta^{-1}$ fixes an apartment in $b^\perp$ and therefore has to be the identity. Thus, θ is a root elation.

□

Corollary 6.6.2.3. *Thick icosahedral geometries do not exist.*

Proof. The assertion now follows directly from [**Tit76**, Théorème 1]. □

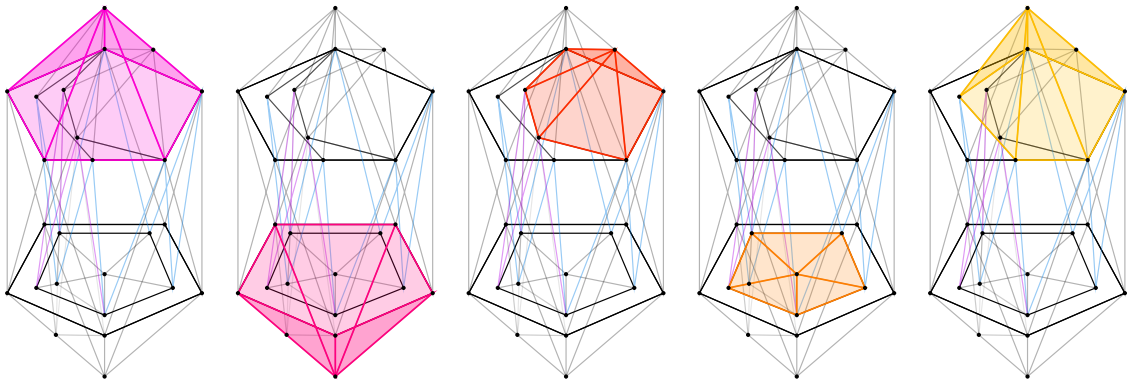


Fig. 24. How the points, lines, and planes map under θ from the proof of Proposition 6.6.2.2.

7. Interlude — About the Why, How and Where

One of the goals of the remaining chapters is to determine all special and general groups of projectivities for simplices with irreducible residues in thick spherical buildings of rank at least three. To this end, it will be useful to understand when a common opposite exists for a given number of simplices of the same type in a thick spherical building. Pursuing this question led to a number of further questions, interesting in their own right. Consequently, the goals of the remaining chapters of this thesis are to

(1) determine all special and general groups of projectivities for simplices with irreducible residues in thick spherical buildings of rank ≥ 3,
(2) classify the *geometric lines* in many spherical buildings of exceptional type,
(3) introduce an analogue of *blocking sets* in finite Lie incidence geometries,
(4) show that, in many cases, lines and geometric lines are precisely the minimal blocking sets in a thick spherical building,
(5) determine all minimal blocking sets of finite Lie incidence geometries of classical type, and many of those of exceptional type.

In this chapter, we will explain goals (2)–(5), describe how they developed, and outline how they are connected.

7.1. The Original Route

In Proposition 3.30 of [**Tit74**], Jacques Tits showed that, given two simplices of the same type in a thick spherical building Δ, there always exists an element of Δ which is opposite to both of them. In this chapter, we would like to generalize this and see how we can make it stronger.

The main reason for this is that it will help us with the determination of the projectivity groups in Chapter 10: determining the general and special projectivity groups of a simplex is more difficult if one has to consider projectivities of arbitrary length. Thus, we would like to be able to decompose an arbitrary sequence of perspectivities into a sequence of perspectivities of a certain prescribed type. That will not always be possible, but in Theorem J and Theorem K, we will show under which circumstances we are allowed to do exactly that. One condition of Theorem J is that we have to find a simplex opposite to four given simplices in the thick spherical

building that we are working with. However, investigating this question for four simplices in buildings of thickness at least 4 is not much simpler than the general case of t simplices in buildings of thickness at least t. Therefore, we will look at this question in full generality. There is also more motivation for this, which we will describe shortly (see Sections 7.2, 7.3, 7.4 & 7.5).

A thick building has thickness at least 3. If we wanted to rephrase Tits's proposition, we could write: given two simplices of the same type in a spherical building Δ of thickness at least 3, there always exists an element of Δ which is opposite to both of them. We will now prove a slight generalization of this (which also only requires a slight modification of Tits's original proof).

Proposition 7.1.0.1. *If a spherical building has thickness at least $t+1$, then there exists a chamber opposite t arbitrarily given chambers. In particular, there exists a vertex opposite t arbitrarily given vertices of the same self-opposite type.*

Proof. We will prove the claim by induction. First consider the case that $t = 2$. Then the condition that every panel is contained in at least $t + 1 = 3$ chambers is equivalent to Δ being a thick building, and the assertion follows with Proposition 3.30 of [**Tit74**]. Now suppose $t > 2$. Suppose we know we can find a chamber opposite $t-1$ given chambers. Let $C_1, \ldots, C_{t-1}$ be $t-1$ different chambers in Δ and let C_t be another chamber in Δ. Among all the chambers in Δ opposite to each C_i, $i \in \{1, \ldots, t-1\}$, let E be a chamber with maximal distance to C_t. Assume that C_t and E are not opposite. Then $\mathsf{dist}(C_t, E) \neq \mathsf{diam}\,\Delta$. Let Σ be an apartment containing both C_t and E. With Proposition 2.41 of [**Tit74**], it follows that there exists a face A of codimension 1 of E such that $E = \mathsf{proj}_A(C_t)$.

Since every panel is contained in $t+1$ chambers, we can find a chamber E' having A as a face that is not equal to E and not equal to $\mathsf{proj}_A(C_i)$ for $i \in \{1, \ldots, t-1\}$. With Proposition 3.19.7 and Lemma 2.30.7 of [**Tit74**], it follows that

$$\begin{cases} \mathsf{dist}(C_i, E') = \mathsf{dist}(\mathsf{proj}_A(C_i), C_i) + 1 \\ \qquad\qquad = \mathsf{dist}(C_i, E) = \mathsf{diam}(\Delta), \text{ for } i \in \{1, \ldots, t-1\}, \\ \mathsf{dist}(C_t, E') = \mathsf{dist}(C_t, E) + 1. \end{cases}$$

That means E' is opposite to each C_i for $i \in \{1, \ldots, t-1\}$ and has a strictly greater distance to C_t than E. That contradicts the fact that E has maximal distance to C_t among the chambers opposite each C_i for $i \in \{1, \ldots, t-1\}$. It follows that C_t and E are opposite. □

A direct consequence of Proposition 7.1.0.1 is that in a spherical building where each panel has thickness at least $t+1$, every set of t vertices of the same type admits a common opposite vertex. However, if the building admits two different thicknesses, say $t_1 + 1 < t_2 + 1$, then Proposition 7.1.0.1 does not help us when we consider t_2 vertices of a type corresponding to panels of thickness $t_2 + 1$. For instance, in a polar space of rank 3 with $q+1$ points on a line and q^2+1 planes through a line, a set of q^2 planes can be placed in q^2 chambers, yet Proposition 7.1.0.1 does not allow us to conclude that they admit a common opposite plane.[1]

[1]We will show, however, that in this particular case q^2 planes do always admit a common opposite. Consequently, refined methods are required when dealing with vertices. These methods will also

Hence the main question is the following:

Question. *If, in a spherical building, panels of cotype i have thickness at least $t+1$, when precisely does there exist a vertex opposite $t+1$ arbitrarily given vertices of type i?*

This question will not only help us in Chapter 10 with the decomposition of perspectivities and the determination of the projectivity groups, but it is also interesting in and of itself, and we will proceed by describing some contexts for which this is relevant.

7.2. Blocking Sets

A notion of great interest in finite geometry is *blocking sets*. They have applications in the theory of communication systems, coding theory, and cryptography (see [**Hol01**]). Originally, blocking sets were defined for projective planes as a set of points in the plane that every line intersects. More generally, a blocking set can be defined as a set of points in a projective space that every hyperplane intersects. In this context, instead of talking about the cardinality of a blocking set, we will talk about its *size*, but we will mean the same. The fundamental starting point to investigate the structure of blocking sets for projective planes is a result of Bose & Burton [**BB66**] that states, in modern terminology, that for a projective plane Γ, in which every line has exactly c points, any blocking set of size at most c in Γ has size exactly c and is a line. We want to generalize the notion of a blocking set to all finite geometries Γ and classify those that contain the same number of points as a line in Γ. We will use the fact that every such geometry corresponds to a spherical building. Using buildings, the result of Bose & Burton can be rephrased as follows: For a projective plane Γ, in which every line has exactly c points, any set of at most c points that does not admit a common opposite contains exactly c points and is a line. We are asking the following question:

Question. *For a finite classical geometry Γ, in which every line has exactly c points – when is it the case that a set of at most c points does not admit a common opposite?*

7.3. Geometric Lines

The central question of [**KV13**] is whether (perhaps partial) knowledge of the opposition relation in spherical buildings determines the lines of any classical geometry. This led the authors of [**KV13**] to the notion of a *geometric line*, which is a set of points in a classical geometry such that

each object of opposite type is opposite either none,

enable us to obtain precise information in the extremal case where we have $t+1$ vertices of a type corresponding to panels of thickness $t+1$.

or all except exactly one point of the set.

Such sets are classified in [**KV13**] for all classical geometries. In the particular case of a projective plane Γ, lines are opposite points, and here a geometric line is a set of points such that each line either (i) contains that set or (ii) has a unique point with it in common.

Suppose Γ is a finite projective plane. Then each line has a constant number of points, say c. Condition (i) implies that a geometric line has at most c points, and (i) and (ii) together imply that each line intersects a geometric line in at least one point, or, in other words, no line is opposite all points of the geometric line. The latter condition is, again, the defining property of blocking sets in projective planes. It turns out that in projective planes (and also in projective spaces) lines and geometric lines are equivalent.

In the finite case, blocking sets of projective planes (and also of projective spaces) are characterized by an ostensibly weaker condition than geometric lines. However, for geometric lines, we can easily find a translation into the language of buildings, using opposition, and with that, we can also investigate them for all point-line geometries associated with spherical buildings. So another question we ask is:

Question. *In a finite Lie incidence geometry Γ with c points on each line, when is a set of at most c points a geometric line, and how do they relate to blocking sets?*

We would also like to mention some work of Cohen & Cooperstein [**CC98**] here. Let Γ be a point-line geometry. A line L of Γ is called a *full line* of $\mathsf{PG}(V)$ if the points of L coincide exactly with all points of some (unique) line of $\mathsf{PG}(V)$. A *full projective embedding* of Γ is a representation of Γ as a set of points of $\mathsf{PG}(V)$, for some vector space V, such that the lines of Γ are full lines of $\mathsf{PG}(V)$. In [**CC98**], Cohen & Cooperstein describe all full embeddings of classical geometries Γ in $\mathsf{PG}(V)$ that contain sets of points that form a line in $\mathsf{PG}(V)$ but do not form a line in Γ. The results of [**KV13**] show that these sets of points are precisely the geometric lines.

7.4. Extremal Problem

It is known that in a polar space with $s+1$ points per line, any set of s points admits an opposite point (this is Exercise 2.13 in [**Van24**]). That means there exists a point that is non-collinear to any given set of at most s points. This is not true for $s+1$, as lines are counterexamples. In this minimal case, it is natural to ask for a classification of all counterexamples. This can be phrased as a Segre-like or extremal problem as follows:

Question. *If a set T of points does not admit a common opposite, then $|T|$ is at least $s+1$; what happens if equality occurs?*

7.5. Opposition Preserving Maps

As mentioned, in [**KV13**], Anna Kasikova and Hendrik Van Maldeghem investigated the interaction between lines in a Lie incidence geometry and the opposition relation in the corresponding building, which exactly led to the introduction of the notion of geometric lines.

In the following chapters, we will classify minimal blocking sets of many finite Lie incidence geometries and uncover their connection to geometric lines. A line is always a blocking set, and it will turn out that, in many cases, lines and geometric lines are exactly the minimal blocking sets that can occur. For many Lie incidence geometries of exceptional type, we will in fact classify their geometric lines first and then use these results to classify their minimal blocking sets.

As an application, we can deduce that certain opposition-preserving maps and transformations of a Lie incidence geometry are bijective collineations. With that, we can characterize automorphisms of exceptional spherical buildings as certain opposition-preserving maps, and we will already describe that here.

Recall that the full automorphism group of a Lie incidence geometry of type $\mathsf{X}_{n,i}$ coincides with the full automorphism group of the corresponding spherical building, restricted to those automorphisms that preserve type i (cf. [**Coh86**, §3]).

Proposition 7.5.0.1. *Let Δ and Δ' be two buildings of the same exceptional type F_4, E_6, E_7 or E_8. Let, with Bourbaki labeling, T_i and T_i' be the set of vertices of type i of Δ and Δ', respectively, where*

- *$i \in \{1,2,3,4\}$ is arbitrary if Δ has type F_4;*
- *$i \in \{1,2,3,4,5,6\}$ is arbitrary if Δ has type E_6;*
- *$i \in \{1,2,6,7\}$ if Δ has type E_7;*
- *$i \in \{7,8\}$ if Δ has type E_8.*

Then any surjective map $\varphi\colon T_i \to T_i'$ preserving opposition and non-opposition is induced by an isomorphism of buildings.

Proof. It suffices to show that φ is a bijective collineation between the corresponding i-Grassmannian geometries. We first show that φ is bijective. Suppose for a contradiction that two vertices $v, u \in T_i$ are mapped onto the same vertex. Lemma 8.2.1.5 yields a vertex $w \in T_i$ opposite v but not opposite u. Then our assumptions imply $\varphi(v) \equiv \varphi(w) \not\equiv \varphi(u) = \varphi(v)$, a contradiction. Hence φ is a bijection. Since opposition and non-opposition is preserved, one deduces that geometric lines are mapped onto geometric lines. If the only geometric lines are the ordinary lines, then this concludes the proof of the proposition.

By [**KV13**, Corollary 6.6] and Proposition 8.3.0.7, we may assume $i = 3$ and Δ has type F_4. It suffices to recognize the planar line pencils of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$ among all geometric lines of $\mathsf{F}_{4,3}(\mathbb{K},\mathbb{K})$. Let Γ be the point-line geometry with point set the points of $\mathsf{F}_{4,3}(\mathbb{K},\mathbb{K})$ and line set the set of ordinary and geometric lines of $\mathsf{F}_{4,3}(\mathbb{K},\mathbb{K})$. We claim that no geometric line different from an ordinary line of $\mathsf{F}_{4,3}(\mathbb{K},\mathbb{K})$ is contained in a maximal subspace of Γ isomorphic to a projective plane. Suppose for a contradiction that the geometric line Z is contained in a maximal singular

subspace α of Γ isomorphic to a projective plane, and that Z is not an ordinary line of $\mathsf{F}_{4,3}(\mathbb{K},\mathbb{K})$. We argue in $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$, where Z is a cone in a symp with vertex p over a hyperbolic line h. Suppose first that α does not contain any ordinary line of $\mathsf{F}_{4,3}(\mathbb{K},\mathbb{K})$. So, the point set of α corresponds to a set Π of lines through p, and the cones over hyperbolic lines correspond to the lines of α. Consequently, any two points on distinct lines of Π are symplectic and the unique hyperbolic line through them is contained in the union of all lines of Π. We select a point q opposite p. Then every line $L \in \Pi$ contains a unique point p_L special to q.

We claim that the set $\beta = \{p_L \mid L \in \Pi\}$, endowed with the hyperbolic lines contained in it, is a projective plane. Indeed, in view of the fact that Π is the point set of a projective plane whose lines are geometric lines of $\mathsf{F}_{4,3}(\mathbb{K},\mathbb{K})$, it suffices to prove that β is closed under taking hyperbolic lines through two arbitrary distinct points y_1 and y_2 of β. Since $p \in \xi(y_1,y_2)$, q is far from $\xi(y_1,y_2)$. Since y_1 and y_2 are collinear to the unique point q' of $\xi(y_1,y_2)$ symplectic to q, all points of the hyperbolic line $h(y_1,y_2)$ defined by y_1,y_2 are collinear to q', by the very definition of hyperbolic line. The claim is proved.

Now [**DSV18**, Lemma 5.21] implies that β is contained in an extended equator geometry $\widehat{E}$. Then [**DSV18**, Proposition 5.24] implies that p is collinear to a set γ of points of $\widehat{E}$ that forms a 3-dimensional projective space when endowed with the hyperbolic lines it contains. Hence the line set $\{px \mid x \in \gamma\}$ forms a projective 3-space in Γ. So, α is not maximal, a contradiction.

Consequently, we may suppose that α contains at least one ordinary line of $\mathsf{F}_{4,3}(\mathbb{K},\mathbb{K})$. Then we have a plane π of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$ through p intersecting h in some point x. Select $y \in h \setminus \{x\}$ and $z \in \pi \setminus px$. Let ξ be the symp containing h; we have $p \in \xi$. Suppose for a contradiction that z is not contained in ξ. Since $z \perp px$ and $y \notin px$, we deduce $z \perp\!\!\!\perp y$. Hence $y \perp px$, a contradiction. Consequently $z \in \xi$ and so $\pi \subseteq \xi$. Now the set of lines of ξ through p forms a 3-dimensional projective space of Γ, contradicting the maximality of α.

Hence Z is not contained in a maximal singular subspace of Γ isomorphic to a projective plane. Evidently, any line of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$ is contained in an ordinary projective plane of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$, which gives rise to a maximal singular subspace of Γ of dimension 2. Hence we can recognize the ordinary lines of $\mathsf{F}_{4,3}(\mathbb{K},\mathbb{K})$ and the proof of the proposition is complete. □

The following is an immediate consequence.

Corollary 7.5.0.2. *Let Δ be a finite building of exceptional type F_4, E_6, E_7 or E_8. Let, with Bourbaki labeling, T_i be the set of vertices of type i of Δ, where*

- *$i \in \{1,2,3,4\}$ is arbitrary if Δ has type F_4;*
- *$i \in \{1,2,3,4,5,6\}$ is arbitrary if Δ has type E_6;*
- *$i \in \{1,2,6,7\}$ if Δ has type E_7;*
- *$i \in \{7,8\}$ if Δ has type E_8.*

Then any map $\varphi\colon T_i \to T_i'$ preserving opposition and non-opposition is induced by an automorphism of Δ.

PROOF. We only have to establish the surjectivity of φ in order to be able to apply Proposition 7.5.0.1. Therefore, we note that the injectivity of φ is proved in a completely similar way as in the first paragraph of the proof of Proposition 7.5.0.1. Now the assertion follows from the trivial fact that an injective transformation of a finite set is always surjective. □

8. Geometric Lines in Exceptional Hexagonic & Minuscule Geometries

As mentioned in the interlude, one of our goals is to classify minimal blocking sets of many finite Lie incidence geometries. As we will see, they are almost always lines or geometric lines in the corresponding geometries (there are exceptions in characteristic 2). In many cases it is therefore beneficial to

(1) classify all geometric lines, and
(2) show the equivalence between them and minimal blocking sets.

We take this approach for the exceptional hexagonic and exceptional *minuscule* geometries; the exceptional *minuscule* geometries are the Lie incidence geometries of types $\mathsf{E}_{6,1}$ and $\mathsf{E}_{7,7}$.

For several geometries, geometric lines have already been classified in [**KV13**], and we summarize the relevant results below. Otherwise, this chapter is dedicated to classifying the geometric lines of the remaining geometries listed above ourselves. This, in turn, will mostly be done by first classifying the *round-up triples of lines* of these geometries. We introduce round-up triples in Section 8.2.

In the first section, we begin by describing all possible mutual positions of lines in exceptional hexagonic geometries, which will turn out to be very useful. After describing round-up triples and establishing some general properties, we classify all round-up triples of lines in exceptional minuscule geometries. In Section 8.2.4, we classify all round-up triples of lines of exceptional hexagonic Lie incidence geometries of rank at least three. Section 8.3 then presents the classification of the geometric lines themselves. Finally, in Section 8.4 we extend this classification to Moufang generalized hexagons.

The results about $\mathsf{E}_{7,7}$ are needed in Chapter 10. Recall that Lie incidence geometries of type $\mathsf{E}_{6,2}$ are exceptional hexagonic geometries of rank at least three. Thus, because of the symmetry of the E_6 diagram, because points of $\mathsf{E}_{6,3}$ correspond to lines of $\mathsf{E}_{6,1}$, and points of $\mathsf{E}_{6,4}$ correspond to lines of $\mathsf{E}_{6,2}$, and because we will have a classification of the round-up triples of points and the round-up triples of lines in both $\mathsf{E}_{6,1}$ and $\mathsf{E}_{6,2}$ at the end of this chapter, we will have a classification of all round-up triples in Lie incidence geometries of types $\mathsf{E}_{6,i}$, for $i \in \{1, 2, 3, 4, 5, 6\}$.

The results in this chapter are based on [**BV25b**].

From [**KV13**] we have the following result for types A_n and D_n.

Proposition 8.0.0.1. *Every geometric line of $\mathsf{A}_{n,j}(\mathbb{K})$, $1 \leq j \leq n$, $n \geq 2$, for $\mathbb{K}$ an arbitrary skew field, is an ordinary line. Hence, every round-up triple of points in such geometry is contained in an ordinary line. The same holds in $\mathsf{D}_{n,j}(\mathbb{K})$, $1 \leq j \leq n$, $n \geq 4$, for $\mathbb{K}$ an arbitrary field.*

For type B_n, again from [**KV13**], we have the following.

Proposition 8.0.0.2. (1) *Let L be a geometric line of a Lie incidence geometry Γ of type $\mathsf{B}_{n,j}$, $1 \leq j \leq n$. Then L is either a line of Γ, or $j \neq n$, Γ is the j-Grassmannian of a symplectic polar space Δ, there exists a singular subspace U of dimension $j-2$ of Δ and L corresponds to a hyperbolic line of $\mathsf{Res}_\Delta(U)$, or $j = n$, all symps of Γ are symplectic polar spaces of rank 2 and L is a hyperbolic line in a symp.*

(2) *Let T be a round-up triple of points of a Lie incidence geometry Γ of type $\mathsf{B}_{n,j}$, $1 \leq j \leq n$, and let Δ be the associated polar space. Then L is either contained in a line of Γ, or $j \neq n$, there exists a singular subspace U of Δ of dimension $j-2$, and T is contained in a hyperbolic line of $\mathsf{Res}_\Delta(U)$, or $j = n$ and L is contained in a hyperbolic line of a symps of Γ.*

The main theorem we will prove in this chapter is the following.

Theorem A. *Let $\mathsf{X}_n \in \{\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8, \mathsf{F}_4, \mathsf{G}_2\}$ and $1 \leq i \leq n$, with $i \notin \{2,4,5\}$ if $n = 7$ and $i \in \{7,8\}$ if $n = 8$.*

Then a geometric line of the Lie incidence geometry of type $\mathsf{X}_{n,i}$ associated to an irreducible thick Moufang spherical building Δ of type X_n, $n \geq 2$, is one of the following.

(1) A line of the Lie incidence geometry.
(2) A hyperbolic line in a symp of the Lie incidence geometry whenever this symp is a symplectic polar space (and this happens (only) in the split case for types $\mathsf{F}_{4,3}$ and $\mathsf{F}_{4,4}$).
(3) A hyperbolic line of a split Cayley hexagon.
(4) An imaginary line of a split Cayley hexagon over a perfect field of characteristic 2.

8.1. Mutual Positions of Lines in Exceptional Hexagonic Geometries

In this section, we describe all possible mutual positions of two lines in exceptional hexagonic geometries. To this aim, we first introduce some notation.

Definition 8.1.0.1 (**Far**)**.** Two objects in a Lie incidence geometry will be called *far (from each other)* if they correspond to vertices in the building, and one vertex is joinable to a vertex opposite the other.

Definition 8.1.0.2 (**Almost far**)**.** In a Lie incidence geometry of type $\mathsf{A}_{5,5}$, $\mathsf{D}_{6,6}$, or $\mathsf{E}_{7,7}$, or a dual polar space of rank 3, we call a point and a line *almost far* if every point of the line is symplectic to the point (however, this does not exist in dual polar spaces).

Definition 8.1.0.3 (**Closest point**, **projection point**)**.** Let L and M be two lines. We will call a point $p \in L$ *closest to* M

- if $p \in M$, and no other point of L is contained in M,
- if $p \perp M$, and no other point of L is collinear to M,
- if $p^\perp \cap M \neq \varnothing$, and no other point of L is collinear to a point of M,
- if p is symplectic to all points of M, and no other point of L is symplectic to all points of M,
- if p is symplectic to a point of M, and no other point of L is symplectic to a point of M,
- if p is special to all points of M, and no other point of L is special to all points of M.

Such a closest point does not always exist. If it exist, then we also sometimes call it the *projection point* of M onto L.

Definition 8.1.0.4 (**Symplectic lines**, **special lines**, **opposite lines**)**.** Let L and M be two lines contained in a polar space. We call L and M

- *symplectic* if they are contained in a common 3-space or if they intersect and are not contained in a common plane.
- *special* if they do not intersect and there exists a unique point on each line, collinear to all points of the other line.
- *opposite* if every point on each line is opposite some point of the other line.

The line-Grassmannian of a polar space is a hexagonic geometry. Thus, in other words, we call two lines symplectic/special/opposite in a polar space if that is their relation in the corresponding hexagonic geometry.

Notation 8.1.0.5. Let L and M be two lines. We describe the mutual position of two lines with four parameters (a, b, c, d) as follows.

(x) If there is a unique point on L closest to M, call it x; otherwise, let x be an arbitrary point on L.
(y) If there is a unique point on M closest to L, call it y; if not, but there is a unique point on M closest to x, call this point y; otherwise, let y be an arbitrary point on M.
(x') Let x' be any point on L distinct from x.
(y') If there is a unique point on M closest to x', and it is different from y, call it y'; otherwise, let y' be any point on M distinct from y.

The paramenters a, b, c and d are elements of $\{=, \perp, \perp\!\!\!\perp, \bowtie, \equiv\}$, and defined as follows.

(a) a is the relation between x and y,
(b) b is the relation between x and y',
(c) c is the relation between x' and y,
(d) d is the relation between x' and y'.

We will find that such a 4-tuple unambiguously determines the mutual position of two lines.

Notation 8.1.0.6. In a shorthand notation, we write

$$0 \text{ for } =, \qquad 1 \text{ for } \perp, \qquad \tfrac{3}{2} \text{ for } \perp\!\!\!\perp, \qquad 2 \text{ for } \bowtie, \qquad \text{and} \qquad 3 \text{ for } \equiv,$$

where $\perp$, $\perp\!\!\!\perp$, $\bowtie$, and $\equiv$ are as in Notation 4.9.0.2. We also denote the mutual position of a pair of lines (L, M) as $\delta(L, M)$.

- The *inverse* of $\delta(L, M)$ is $\delta(M, L)$.
- The *dual* of $\delta(L, M)$ is $\delta(M, K)$, where K is the line opposite L in some apartment containing L and M. This is independet of the choice of apartment.

One the level of point distances, 0 is dual to 3, 1 to 2 and $\frac{3}{2}$ is self-dual.

There are four classes of positions, taking the homogeneity with respect to the points of the lines into account, as we will describe now. For that, let L and M be two lines.

Class I — Completely homogeneous

In this class, every point of L has the same distance to each point of M. All positions here are equal to their inverse. The cases are:

(1111) $(\perp, \perp, \perp, \perp)$: the two lines span a singular 3-space.
($\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{3}{2}$) $(\perp\!\!\!\perp, \perp\!\!\!\perp, \perp\!\!\!\perp, \perp\!\!\!\perp)$: each point of L is symplectic to each point of M.

The dual of (1111) is the following:

(2222) $(\bowtie, \bowtie, \bowtie, \bowtie)$: each point of L is special to each point of M.

Class II — Projection homogeneous

In this class, each point of L has a unique projection onto M. For each point p on L, let p' be its unique projection onto M. Then the distances between p and p' are the same for all $p \in L$. Furthermore, the distances between p and q' are all the same for all $p \in L$, $q' \in M$, $q' \neq p'$. All distances between lines are their own inverse. The cases are:

(0110) $(=, \perp, \perp, =)$: the lines are equal.
($1\frac{3}{2}\frac{3}{2}1$) $(\perp, \perp\!\!\!\perp, \perp\!\!\!\perp, \perp)$: the lines are opposite in a symp.
($\frac{3}{2}22\frac{3}{2}$) $(\perp\!\!\!\perp, \bowtie, \bowtie, \perp\!\!\!\perp)$: the lines are each other's projection from a symp to an opposite symp.
(2332) $(\bowtie, \equiv, \equiv, \bowtie)$: the lines are opposite.

Class III — Symmetric non-homogeneous

In this class, both lines contain a unique closest point with respect to the other line. Moreover, all the positions are their own inverse again. The cases are:

(0111) $(=, \perp, \perp, \perp)$: the lines are coplanar.
($011\frac{3}{2}$) $(=, \perp, \perp, \perp\!\!\!\perp)$: the lines intersect and determine a unique symp.
(0112) $(=, \perp, \perp, \bowtie)$: the lines intersect and are locally opposite.
($111\frac{3}{2}$) $(\perp, \perp, \perp, \perp\!\!\!\perp)$: the lines are special in (the line-Grassmannian of) a symp.

- $(1\frac{3}{2}\frac{3}{2}\frac{3}{2})$ $(\perp, \perp\!\!\perp, \perp\!\!\perp, \perp\!\!\perp)$: the projection point on L is contained in a symp with M, while L is locally close to the symp (at the projection point).
- $(1\frac{3}{2}\frac{3}{2}2)$ $(\perp, \perp\!\!\perp, \perp\!\!\perp, \bowtie)$: the projection point on L is contained in a symp with M, while L is locally opposite the line connecting the two projection points, and hence far from the symp (at the projection point).
- $(\frac{3}{2}\frac{3}{2}\frac{3}{2}2)$ $(\perp\!\!\perp, \perp\!\!\perp, \perp\!\!\perp, \bowtie)$: the lines are locally close to the symp determined by the projection points, and the projections of the lines onto the symp intersect in a unique point.
- $(\frac{3}{2}222)$ $(\perp\!\!\perp, \bowtie, \bowtie, \bowtie)$: the lines are locally far from the symp determined by the projection points, and the projections of the lines onto the symp are special in that symp.
- (1223) $(\perp, \bowtie, \bowtie, \equiv)$: the lines are each other's projection from a point to an opposite point.
- $(\frac{3}{2}223)$ $(\perp\!\!\perp, \bowtie, \bowtie, \equiv)$: the lines are locally far from the symp determined by the projection points, and the projections of the lines onto the symp are opposite lines in that symp.
- (2223) $(\bowtie, \bowtie, \bowtie, \equiv)$: There is a point $p \in L$ and a point $q \in M$ such that p and q are opposite. The projection of L onto q is locally symplectic to M. Equivalently, L and M lie in opposite symps ξ_L and ξ_M and the projection of L onto ξ_M is special to M in ξ_M.

Class IV — Asymmetric positions

Up to now, in all 4-tuples, the second and third entries coincided. This is going to change now. The final class that we consider in this paragraph contains the asymmetric mutual positions, that is, those that do not coincide with their inverse. There are four cases with two projection points. We consider $\delta(L, M)$. The cases are:

- $(1\frac{3}{2}12)$ $(\perp, \perp\!\!\perp, \perp, \bowtie)$: L is coplanar with the projection point of L on M; M and this plane are locally far.
- $(11\frac{3}{2}2)$ $(\perp, \perp, \perp\!\!\perp, \bowtie)$: M is coplanar with the projection point of M on L; L and this plane are locally far.
- $(1\frac{3}{2}22)$ $(\perp, \perp\!\!\perp, \bowtie, \bowtie)$: M and the projection point of M on L are in a symp; L is locally far from this symp and locally opposite the line through the projection points.
- $(12\frac{3}{2}2)$ $(\perp, \bowtie, \perp\!\!\perp, \bowtie)$: L and the projection point of L on M are in a symp; M is locally far from this symp and locally opposite the line through the projection points.

At last, there are four cases where only one line has a projection point. Hence, there will be only two distinct distances around, and the corresponding 4-tuples have the shape (a, a, b, b) or (a, b, a, b) (where these are each other's inverse).

- $(1\frac{3}{2}1\frac{3}{2})$ $(\perp, \perp\!\!\perp, \perp, \perp\!\!\perp)$: L is coplanar with the projection point of L on M; M and this plane are locally almost far.
- $(11\frac{3}{2}\frac{3}{2})$ $(\perp, \perp, \perp\!\!\perp, \perp\!\!\perp)$: M is coplanar with the projection point of M on L; L and this plane are locally almost far.

($\frac{3}{2}\frac{3}{2}22$) ($\perp\!\!\!\perp, \perp\!\!\!\perp, \bowtie, \bowtie$): M is contained in a symp close to a point on L, which is collinear to a line of the symp opposite M in that symp.

($\frac{3}{2}2\frac{3}{2}2$) ($\perp\!\!\!\perp, \bowtie, \perp\!\!\!\perp, \bowtie$): L is contained in a symp close to a point on M, which is collinear to a line of the symp opposite L in that symp.

We note that dual distances are obtained from each other by interchanging and dualizing the first and fourth entries, and dualizing the second and third entries.

We now have the following result.

Lemma 8.1.0.7. *Let L and M be two arbitrary lines of an exceptional hexagonic Lie incidence geometry Δ of uniform symplectic rank r. Then $\delta(L, M)$ is one of the 4-tuples enumerated above in* **Class I** *up to* **Class IV**. *All cases occur, except for metasymplectic spaces, where the following positions cannot occur:*

$$(1111),\ (\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}),\ (2222),\ (1\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}),\ (\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}2),\ (1\tfrac{3}{2}1\tfrac{3}{2}),\ (11\tfrac{3}{2}\tfrac{3}{2}),\ (\tfrac{3}{2}\tfrac{3}{2}22),\ (\tfrac{3}{2}2\tfrac{3}{2}2).$$

Proof. We note that the existence of a given mutual position is equivalent to the existence of its dual.

Part I. It is convenient to first consider the case where all points of L have the same distance to all points of M. Then we have one of the three completely homogeneous cases. For the existence of (1111), consider two lines in a common singular subspace of dimension at least 3. Conversely, if $\delta(L, M) = (1111)$, then L and M span a singular subspace of dimension 3. Since these do not exist in metasymplectic spaces, this mutual distance occurs if and only if Δ is not a metasymplectic space. By duality, the same holds for (2222).

Let $\delta(L, M) = (\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{3}{2})$. Pick $p, p' \in L$ and $q, q' \in M$, $p \neq p'$, $q \neq q'$. If q' were collinear to only a line of $\xi(p, q)$, then, by Lemma 5.4.1.8(ii), p and q' would be special; a contradiction (and it follows that Δ does not have type F_4). That means $p^\perp \cap M^\perp$ must be a singular subspace U of dimension $r - 2$.

We claim that $p'^\perp \cap U$ is $(r - 5)$-dimensional.

To fix the ideas, suppose Δ has type $\mathsf{E}_{8,8}$. Then $r = 7$ and $\dim(U) = 5$. Since $p'^\perp \cap \xi(p, q)$ is a 6′-space and $\langle p, U\rangle$ is a 6-space, we have that $p'^\perp \cap \langle p, U\rangle$ contains a line K. In $\mathsf{Res}_\Delta(K)$, we have a point z' (corresponding to p') close to each symp through a given 4-space W (corresponding to $\langle p, U\rangle$). If $z'^\perp \cap W = \varnothing$, then there exist non-collinear points a and b in different symps through W, both collinear to z'. The symp $\xi(a, b)$ contains z' and a plane α of W; hence z' is collinear to a line of W after all, a contradiction. So $z'^\perp \cap W \neq \varnothing$ and by parity it is a line. The claim follows.

We observe that,

(i) in case of $\mathsf{E}_{8,8}$, L and M arise from opposite lines in the residue of a plane (which is a parapolar space of type $\mathsf{D}_{5,5}$),

(ii) in case of $\mathsf{E}_{7,1}$, L and M arise from opposite lines in a given para of type $\mathsf{D}_{5,5}$ in a given point residual,

(iii) in case of $\mathsf{E}_{6,2}$, L and M are opposite lines in a given para of type $\mathsf{D}_{5,5}$.

The last claim might be the least straightforward, as in this case $L^\perp \cap M^\perp = \varnothing$. So let us prove this case as an example. The other cases are then easier because the parapolar spaces in the respective residues are simpler; they have types $\mathsf{D}_{5,5}$ and $\mathsf{D}_{6,6}$, respectively.

Translated to type $\mathsf{E}_{6,1}$, we have to prove that, if each 5-space through a given plane α intersects each 5-space through another given plane β in exactly a point, then

- either α meets β in a point at which they are locally opposite,
- or α and β are contained in a symp in which they are (locally) opposite.

So suppose α and β do not intersect. Then one checks that, if U_1 and U_2 are two distinct 5-spaces through α and W_1 and W_2 two distinct 5-spaces through β, the points

$$p_{ij} = U_i \cap W_j, \quad i, j \in \{1, 2\},$$

form a quadrangle. That quadrangle is contained in a unique symp ξ that contains both α and β. If the latter are not ξ-opposite, then, arguing in the polar space $\mathsf{D}_{5,1}(\mathbb{K})$, we find $4'$-spaces through them that intersect in a plane. This yields adjacent 5-spaces through them; a contradiction. This concludes the completely homogeneous case. We obtain all members of Class I.

Part II. Next we consider the case where each point of $L \cup M$ has a unique closest point on the other line. One can deduce that the distances between such closest points are always the same. A point $p \in L$ and its closest point $q \in M$ can be

equal, collinear, symplectic, or special,

in which case p and the points of $M \setminus \{q\}$ are

collinear, symplectic, special, or opposite,

respectively (use Lemma 5.4.1.7, for instance). Then the lines are

equal, opposite in a symp, the projections of each other from opposite symps, or opposite,

respectively. Hence we obtain precisely all cases of Class II.

Part III. Having done the more homogeneous cases separately, we proceed by considering the different distances that can occur as the smallest distance between points of L and M. In order to do so, we let $p \in L$ and $q \in M$ be points at minimal distance.

Case 1: $p = q$. Here the lines L and M meet in $p = q$, and we only have the three possibilities (0111), (011$\frac{3}{2}$), and (0112) of Class III. Existence is trivial in these cases.

Case 2: $p \perp q$. Set $K = pq$. We consider the different possible mutual positions of L and K, and of K and M. First, suppose that K and M are coplanar; say they span the plane α. Then α and L are contained in a common symp if and only if one of the following two possibilities occurs:

(1) $L \perp M$; and we are in case (1111),
(2) $|L^\perp \cap M| = 1$; and we are in case (111$\frac{3}{2}$).

So we may assume that no point of $L \setminus \{p\}$ is collinear to any point of $\alpha \setminus \{p\}$.

Let ξ be a symp containing α. There are again two possibilities:

(1) $L^\perp \cap \xi$ is a line N.
Then N is not contained in α, so that there is a unique point $q' \in M$ collinear to all points of N. We have $q' \perp\!\!\!\perp p'$, for all $p' \in L \setminus \{p\}$, and $q'' \bowtie p'$, for all $q'' \in M \setminus \{q'\}$ and all $p' \in L \setminus \{p\}$. Thus, we are in case $(11\frac{3}{2}2)$.
(2) $L^\perp \cap \xi$ is a maximal singular subspace U.
Then $U \cap M = \varnothing$, and each point of M is symplectic to each point of $L \setminus \{p\}$. We obtain $(11\frac{3}{2}\frac{3}{2})$.

If L and K are coplanar, then we obtain the inverse distances (1111), $(111\frac{3}{2})$, $(1\frac{3}{2}12)$, and $(1\frac{3}{2}1\frac{3}{2})$. Hence we may assume that K is not coplanar with either L or M. Pick $p' \in L \setminus \{p\}$ and $q' \in M \setminus \{q\}$. If both pairs $\{p', q\}$ and $\{p, q'\}$ are special, then we have (1223). So we may assume $\{p, q'\}$ is symplectic. Again, there are some possibilities:

(1) $L^\perp \cap \xi(p, q')$ is a line N not $\xi(p, q')$-opposite M.
Then our assumptions imply that $q \perp N$. — Then we have $(1\frac{3}{2}\frac{3}{2}2)$.
(2) $L^\perp \cap \xi(p, q')$ is a line N which is $\xi(p, q')$-opposite M. — Then we have $(1\frac{3}{2}22)$.
(3) $L^\perp \cap \xi(p, q')$ is a maximal singular subspace U. — Then we have $(1\frac{3}{2}\frac{3}{2}\frac{3}{2})$.

The case where L and K are contained in a common symp gives additionally rise to the inverse $(12\frac{3}{2}2)$ of $(1\frac{3}{2}22)$. This takes care of all distances beginning with "collinear".

Case 3: $p \perp\!\!\!\perp q$**.** Let ξ be the symp containing both p and q. Note that none of L and M are contained in ξ, as otherwise we are back in the previous case. There are a few possibilities:

(1) Both $L^\perp \cap \xi$ and $M^\perp \cap \xi$ are maximal singular subspaces, and they intersect in a subspace of dimension at least 1. Then we are in the homogeneous case $(\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{3}{2})$, which we already discussed in detail in Part I.
(2) Both $L^\perp \cap \xi$ and $M^\perp \cap \xi$ are maximal singular subspaces, and they intersect in exactly a point. By Part I, we are not in case $(\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{3}{2})$, hence we have distance $(\frac{3}{2}\frac{3}{2}\frac{3}{2}2)$.
(3) Both $L^\perp \cap \xi$ and $M^\perp \cap \xi$ are maximal singular subspaces, and they are disjoint. Select $p' \in L \setminus \{p\}$ and $q' \in M \setminus \{q\}$. We claim that $p' \perp\!\!\!\perp q'$. Indeed, the only alternative is $p' \bowtie q'$. If so, let $p' \perp r \perp q'$. Then, by considering symps through points of $U := L^\perp \cap \xi$ and q', we see that $r \perp U$; likewise, r is collinear to each point of $M^\perp \cap \xi$, a contradiction. Hence this leads to $(\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{3}{2})$, which we discussed in Part I.
(4) Suppose $L^\perp \cap \xi$ is a maximal singular subspace U of ξ and $M^\perp \cap \xi$ is a line K. If $K \cap U$ is a point x, then $\langle x, L\rangle$ and $\langle x, M\rangle$ are planes intersecting in x, and it follows from Lemma 5.4.1.10 that each point of L is symplectic to either a unique point of M, or all points of M. This now leads to $(\frac{3}{2}\frac{3}{2}\frac{3}{2}2)$ again (and replacing p with the unique point on L symplectic to all points of M, we are back to (2)). Hence we may assume that $K \cap U = \varnothing$. If some point p' of $L \setminus \{p\}$ were symplectic to some point of $M \setminus \{q\}$, then we

could replace p by p' and are back to situation (2). If no point of $L \setminus \{p\}$ is symplectic to any point of $M \setminus \{q\}$, then we have distance $(\frac{3}{2}2\frac{3}{2}2)$.

(5) Similarly, $L^\perp \cap \xi$ a line and $M^\perp \cap \xi$ a maximal singular subspace leads to $(\frac{3}{2}\frac{3}{2}\frac{3}{2}2)$ or $(\frac{3}{2}\frac{3}{2}22)$.

(6) Finally, suppose $L^\perp \cap \xi$ is a line K and $M^\perp \cap \xi$ is a line N. If K and N intersect, then we are back to a previous case already handled, namely $(\frac{3}{2}22\frac{3}{2})$. If K and N are ξ-special, then we claim that we have distance $(\frac{3}{2}222)$. Indeed, the alternative would be that some point $q' \in M \setminus \{q\}$ is symplectic to some point $p' \in L \setminus \{p\}$. This would imply that p' is locally close to the symp determined by q' and $N^\perp \cap K$. But this implies that q is symplectic to p', which contradicts the fact that q is not noncollinear to all points of K. If K and N are ξ-opposite, then, using similar arguments, we have distance $(\frac{3}{2}223)$.

This takes care of the case $p \perp\!\!\!\perp q$.

Case 4: $p \bowtie q$**.** Since we may assume we are not in the completely homogeneous case, there are opposite pairs of points on $L \cup M$. Since we may also assume that we are not in the projection homogeneous case, we may assume that no point of $L \setminus \{p\}$ is special to any point of $M \setminus \{q\}$. But then every point of L is special to q, and every point of M is special to p, whereas each point of $L \setminus \{p\}$ is opposite each point of $M \setminus \{q\}$. This is (2223).

One checks that the cases involving a singular subspace of dimension at least 3 (this includes each case where some point is collinear to a maximal singular subspace of a symp) are precisely the positions

$$(1111),\ (\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}),\ (2222),\ (1\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}),\ (\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}2),\ (1\tfrac{3}{2}1\tfrac{3}{2}),\ (11\tfrac{3}{2}\tfrac{3}{2}),\ (\tfrac{3}{2}\tfrac{3}{2}22),\ \text{and}\ (\tfrac{3}{2}2\tfrac{3}{2}2).$$

The lemma is completely proved. □

Definition 8.1.0.8 (**Free points**)**.** For any ordered pair of lines (L, M), we call each point of L distinct from the projection point, if it exists, a *free point (for* (L, M)*)*; hence if there is no projection point, every point is free.

We define the following order:

$$0 < 1 < \frac{3}{2} < 2 < 3,$$

and use it to state the next lemma.

Lemma 8.1.0.9. *Let $L, M \in \mathscr{L}$ be two lines. Let $x \in L$ be a free point for (L, M). Then there exists a line K through x, not locally opposite L at x such that*

- *if $\delta(L, M) = (abcd)$, then whenever $L' \ni x$ is locally opposite K, we have $\delta(L', M) = (cdef)$ for some e, f, if $c \leq d$;*
- *otherwise, $\delta(L', M) = (dcef)$, where $(cdef)$ or $(dcef)$ is determined according to* Table 1*.*

{1}	(0110)	$\to$	(0112)	{14}	$(\frac{3}{2}222)$	$\to$	(2223)
{2}	(0111)	$\to$	$(11\frac{3}{2}2)$	{15}	$(\frac{3}{2}223)$	$\to$	(2332)
{3}	$(011\frac{3}{2})$	$\to$	$(1\frac{3}{2}22)$	{16}	(2223)	$\to$	(2332)
{4}	$(111\frac{3}{2})$	$\to$	$(1\frac{3}{2}22)$	{17}	(2332)	$\to$	(2332)
{5}	$(1\frac{3}{2}\frac{3}{2}1)$	$\to$	$(1\frac{3}{2}22)$	{18}	(1111)	$\to$	$(11\frac{3}{2}2)$
{6}	(0112)	$\to$	(1223)	{19}	$(\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{3}{2})$	$\to$	$(\frac{3}{2}\frac{3}{2}22)$
{7}	$(1\frac{3}{2}12)$	$\to$	(1223)	{20}	(2222)	$\to$	(2223)
{8}	$(11\frac{3}{2}2)$	$\to$	$(\frac{3}{2}223)$	{21}	$(1\frac{3}{2}1\frac{3}{2})$	$\to$	$(1\frac{3}{2}22)$
{9}	$(1\frac{3}{2}\frac{3}{2}2)$	$\to$	$(\frac{3}{2}223)$	{22}	$(11\frac{3}{2}\frac{3}{2})$	$\to$	$(\frac{3}{2}\frac{3}{2}22)$
{10}	$(12\frac{3}{2}2)$	$\to$	$(\frac{3}{2}223)$	{23}	$(1\frac{3}{2}\frac{3}{2}\frac{3}{2})$	$\to$	$(\frac{3}{2}\frac{3}{2}22)$
{11}	$(\frac{3}{2}22\frac{3}{2})$	$\to$	$(\frac{3}{2}223)$	{24}	$(\frac{3}{2}\frac{3}{2}22)$	$\to$	(2223)
{12}	(1223)	$\to$	(2332)	{25}	$(\frac{3}{2}2\frac{3}{2}2)$	$\to$	$(\frac{3}{2}223)$
{13}	$(1\frac{3}{2}22)$	$\to$	(2223)	{26}	$(\frac{3}{2}\frac{3}{2}\frac{3}{2}2)$	$\to$	$(\frac{3}{2}223)$

TABLE 1

PROOF. We have to treat the 26 cases one by one. However, some cases are immediate, or at least one can check them without needing creativity, and we skip those. This concerns many of the cases where $K = L$, namely {1}, {3}, {6}, {12}. Other cases can be settled by routine considerations, after determining K, and in these cases, we only give a description of K. Basically, K can always be thought of as a kind of projection of M onto x. In cases {12}, {15}, {16}, and {17}, the point x is opposite some point of M and then K is really that projection, so these cases are straightforward and we skip them too. More tricky cases are treated in full detail. This concerns, in particular, the cases that cannot occur in type F_4.

{2} $(=, \perp, \perp, \perp) \to (\perp, \perp, \perp\!\!\perp, \bowtie)$.
The line K is any line through x in the plane spanned by L and M.

{4} $(\perp, \perp, \perp, \perp\!\!\perp) \to (\perp, \perp\!\!\perp, \bowtie, \bowtie)$.
The line K is the line joining x with the unique point of M collinear to each point of L.

{5} $(\perp, \perp\!\!\perp, \perp\!\!\perp, \perp) \to (\perp, \perp\!\!\perp, \bowtie, \bowtie)$.
Here, K is the line joining x with the unique point $y \in M$ collinear to x. Then let N be a line through x locally opposite K and pick $z \in N \setminus \{x\}$. Let ξ be the symp containing L and M and set $z^\perp \cap \xi = M'$. Then M' and M are ξ-opposite since, if they were not, either x would be collinear to M (which contradicts the fact that L and M are ξ-opposite), or y would be collinear to all points of M' (which contradicts the fact that $z \bowtie y$ by the choice of N locally opposite K).

{7} $(\perp, \perp\!\!\perp, \perp, \bowtie) \to (\perp, \bowtie, \bowtie, \equiv)$.
Let $y \in M$ be collinear to each point of L. Then $K = xy$ and the rest follows from Lemma 5.4.1.7.

{8} $(\perp, \perp, \perp\!\!\perp, \bowtie) \to (\perp\!\!\perp, \bowtie, \bowtie, \equiv)$.
Here $K = L$. Set $p \in L$ as the unique point collinear to each point of M

and $q \in M$ as the unique point of M symplectic to x. By Lemma 5.4.1.7, every other point of M is opposite each point of $L' \setminus \{x\}$. It follows that the distance from L' to M is $(\frac{3}{2}223)$.

{9} $(\perp, \perp\!\!\!\perp, \perp\!\!\!\perp, \bowtie) \to (\perp\!\!\!\perp, \bowtie, \bowtie, \equiv)$.
Let $p \in L$ and $q \in M$ be collinear. Let ξ be the symp through M and p and let z be the unique point of ξ collinear to M and collinear to x. Set $K = xz$. Then, precisely like in the previous case {8}, we conclude that the distance from L' to M is $(\frac{3}{2}223)$.

{10} $(\perp, \bowtie, \perp\!\!\!\perp, \bowtie) \to (\perp\!\!\!\perp, \bowtie, \bowtie, \equiv)$.
Let ξ be the symp containing L and the unique point $q \in M$. Then there is a unique point $z \in \xi$ collinear to M and x. Putting $K = xz$, the rest of the argument is the same as for the previous cases {8} and {9}.

{11} $(\perp\!\!\!\perp, \bowtie, \bowtie, \perp\!\!\!\perp) \to (\perp\!\!\!\perp, \bowtie, \bowtie, \equiv)$.
Lemma 5.4.1.13 yields a point z collinear to each point of $L \cup M$. Then $K = xz$ and the same arguments as in the three previous cases imply that the distance from L' to M is $(\frac{3}{2}223)$.

{13} $(\perp, \perp\!\!\!\perp, \bowtie, \bowtie) \to (\bowtie, \bowtie, \bowtie, \equiv)$.
Let $p \in L$ be the point of L contained in a common symp ξ with M. Then $x \neq p$ is collinear to a line N of ξ. This line N is ξ-opposite M. Then K is any line through x in the plane spanned by x and N. Set $K \cap N = \{p'\}$ and $p'^{\perp} \cap M = \{q'\}$. Then the position (2223) between L' and M follows from the facts that L' is locally opposite K at x; K is locally opposite $p'q'$ at p' and $p'q'$ is locally symplectic to M at q'.

{14} $(\perp\!\!\!\perp, \bowtie, \bowtie, \bowtie) \to (\bowtie, \bowtie, \bowtie, \equiv)$.
Let ξ be the symp through the unique points $p \in L$ and $q \in M$. Set $N := x^{\perp} \cap \xi$ and let z be the point on N collinear to q. Lemma 5.4.1.10 yields a line $N' \ni z$ consisting of all points collinear to a point of M and x. Then K is any line through x in the plane $\langle x, N' \rangle$. Similar arguments as in the previous case {13} show that the mutual position of L' and M is (2223).

{18} $(\perp, \perp, \perp, \perp) \to (\perp, \perp, \perp\!\!\!\perp, \bowtie)$.
Let K be any line through x intersecting M. Then, by Lemma 5.4.1.10, there is a unique point on M symplectic to the points of $L' \setminus \{x\}$. This implies that the mutual position between L' and M is $(11\frac{3}{2}2)$.

{19} $(\perp\!\!\!\perp, \perp\!\!\!\perp, \perp\!\!\!\perp, \perp\!\!\!\perp) \to (\perp\!\!\!\perp, \perp\!\!\!\perp, \bowtie, \bowtie)$.
Pick two points y_1, y_2 on M. By Lemma 5.4.1.8, the point y_2 is collinear to a maximal singular subspace U of $\xi(x, y_1)$. Then x is collinear to a hyperplane W of U, and K is any line joining x with a point z of W. Let $u \in L' \setminus \{x\}$ be arbitrary. Then u and z are special.

Suppose u were symplectic to some point of M, and we may without loss of generality assume that point is y_1. Then u would have to be collinear to a line N of $\xi(x, y_1)$ including x and y_1. But x is not collinear to any point of M. It follows that u is special to every point of M. With that, L' contains exactly one point, which is x, symplectic to all points of L_2 and otherwise, all remaining points of L' are special to all points of L_2, since $u \in M$ was arbitrary.

{20} $(\bowtie, \bowtie, \bowtie, \bowtie) \to (\bowtie, \bowtie, \bowtie, \equiv)$.
Lemma 5.4.1.10 yields a line N consisting of all points collinear to x and some point of M. Then K is any line through x in the plane $\langle x, N\rangle$. Note that K is locally symplectic to L at x. Let $u \in L' \setminus \{x\}$. Then there is a unique point on N symplectic to u; all other points of N are special to u. Lemma 5.4.1.7 implies that u is opposite all but exactly one point of M. Since x is special to all points of M, the mutual distance between L' and M is (2223).

{21} $(\perp, \perp\!\!\!\perp, \perp, \perp\!\!\!\perp) \to (\perp, \perp\!\!\!\perp, \bowtie, \bowtie)$.
Let q be the unique point of M collinear to all points of L. Then K is the line qx. Let q' be any point of $M \setminus \{q\}$. Then M is contained in the symp $\xi(q', x)$. Let u be an arbitrary point of L' not equal to x. Then u and q are special, and consequently u can only be collinear to a line of the symp $\xi(q', x)$ through x, which implies that u and q' are special. In summary, q is collinear to x and special to every other point of L', and every point in $M \setminus \{q\}$ is symplectic to x and special to every point in $L' \setminus \{x\}$.

{22} $(\perp, \perp, \perp\!\!\!\perp, \perp\!\!\!\perp) \to (\perp\!\!\!\perp, \perp\!\!\!\perp, \bowtie, \bowtie)$.
Let p be the point of L collinear to every point of M. Then $x \neq p$. Here, $K = L$. Let u be some arbitrary point of L' not equal to x. Then u and p are special. The point x is symplectic to every point of M. Considering the symp ξ containing x and an arbitrary point $y \in M$, we see that $u^\perp \cap \xi$ is a line N through x (indeed, $p \in \xi$ and $u \bowtie p$). Now, y is only collinear to a unique point of L, since y is not collinear to x. It follows with Lemma 5.4.1.8 that u is special to y. We obtain $(\frac{3}{2}\frac{3}{2}22)$.

{23} $(\perp, \perp\!\!\!\perp, \perp\!\!\!\perp, \perp\!\!\!\perp) \to (\perp\!\!\!\perp, \perp\!\!\!\perp, \bowtie, \bowtie)$.
Let ξ be the symp through M and the unique point p of L collinear to some point q of M. One can see that $x^\perp \cap \xi$ is a maximal singular subspace U of ξ. Then K is any line through x and a point of $M^\perp \cap U$. Considering the respective symps through x and the points of M, we see that the points of M and $L' \setminus \{x\}$ are special. Hence we obtain $\left(\frac{3}{2}\frac{3}{2}22\right)$.

{24} $(\perp\!\!\!\perp, \perp\!\!\!\perp, \bowtie, \bowtie) \to (\bowtie, \bowtie, \bowtie, \equiv)$.
Let $p \in L$ be symplectic to all points of M. Lemma 5.4.1.11 yields a singular subspace $U = p^\perp \cap M^\perp$ contained in each symplecton containing p and a point of M, and such that $\langle p, U\rangle$ is a maximal singular subspace of each such symp. Combining this with Lemma 5.4.1.10, we find a line $N \subseteq U$ consisting of all points collinear to x and some point of M. Then K is an arbitrary line through x in the plane $\langle x, N\rangle$. Taking into account that N and M are contained in a common symp ξ in which they are ξ-opposite, we arrive at (2223) for the distance from L' to M.

{25} $(\perp\!\!\!\perp, \bowtie, \perp\!\!\!\perp, \bowtie) \to (\perp\!\!\!\perp, \bowtie, \bowtie, \equiv)$.
Here, there is a unique point $q \in M$ symplectic to each point of L, in particular to x. The line M is collinear to a unique line N of $\xi(q, x)$ and x is collinear to a unique point z of N. Then K is the line xz and it is immediate that L' and M are at mutual distance $\left(\frac{3}{2}223\right)$.

{26} $(\perp\!\!\!\perp, \perp\!\!\!\perp, \perp\!\!\!\perp, \bowtie) \to (\perp\!\!\!\perp, \bowtie, \bowtie, \equiv)$.
Let $p \in L$ be the point symplectic to all points of M and let $q \in M$ be the point symplectic to all points of L. Then M is collinear to a unique line N

of $\xi(q, x)$, and x is collinear to a unique point z of N. We define $K = xz$. Then Lemma 5.4.1.7 implies that each point of $L' \setminus \{x\}$ is opposite each point of $M \setminus \{q\}$. Since $x \perp\!\!\!\perp q$ and x is special to all points of $M \setminus \{q\}$, we conclude that the mutual distance of L' and M is given by $(\frac{3}{2}223)$.

This completes the proof of the lemma. □

Combing

Applying Lemma 8.1.0.9 over and over again, we eventually arrive at an opposite pair. In *Table 3*, we list the consecutive distances when we apply this algorithm. The penultimate column lists the local mutual position of L and K (with respect to the first arrow in the row), and the last column mentions when K is not unique.

Definition 8.1.0.10 (**Combing** and **combing distances**)**.**
Applying Lemma 8.1.0.9 will be called *combing*. We say that we *combed* once (or twice, or 3-times), after we applied Lemma 8.1.0.9 once (or twice, or 3-times). The possible distances we can obtain after combing will be called *combing distances*.

Definition 8.1.0.11 (**Level of a line with respect to another line**)**.** We define the *level of a line* M with respect to a line L as

- 0, if M and L are opposite,
- the number of times we have to apply Lemma 8.1.0.9 to arrive at an opposite pair.

It is the length of the sequence in Table 3.

{1}	(0110)	→	(0112)	→	(1223)	→	(2332)		equal	
{2}	(0111)	→	$(11\frac{3}{2}2)$	→	$(\frac{3}{2}223)$	→	(2332)		eq or coll	not unique
{3}	$(011\frac{3}{2})$	→	$(1\frac{3}{2}22)$	→	(2223)	→	(2332)		equal	
{4}	$(111\frac{3}{2})$	→	$(1\frac{3}{2}22)$	→	(2223)	→	(2332)		collinear	
{5}	$(1\frac{3}{2}\frac{3}{2}1)$	→	$(1\frac{3}{2}22)$	→	(2223)	→	(2332)		symplectic	
{6}	(0112)	→	(1223)	→	(2332)				equal	
{7}	$(1\frac{3}{2}12)$	→	(1223)	→	(2332)				collinear	
{8}	$(11\frac{3}{2}2)$	→	$(\frac{3}{2}223)$	→	(2332)				equal	
{9}	$(1\frac{3}{2}\frac{3}{2}2)$	→	$(\frac{3}{2}223)$	→	(2332)				collinear	
{10}	$(12\frac{3}{2}2)$	→	$(\frac{3}{2}223)$	→	(2332)				symplectic	
{11}	$(\frac{3}{2}22\frac{3}{2})$	→	$(\frac{3}{2}223)$	→	(2332)				collinear	
{12}	(1223)	→	(2332)						equal	
{13}	$(1\frac{3}{2}22)$	→	(2223)	→	(2332)				eq or coll	not unique
{14}	$(\frac{3}{2}222)$	→	(2223)	→	(2332)				coll or sympl	not unique
{15}	$(\frac{3}{2}223)$	→	(2332)						collinear	
{16}	(2223)	→	(2332)						symplectic	

{17}	(2332)	→	(2332)						special	
{18}	(1111)	→	(11$\frac{3}{2}$2)	→	($\frac{3}{2}$223)	→	(2332)		collinear	not unique
{19}	($\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{3}{2}$)	→	($\frac{3}{2}\frac{3}{2}$22)	→	(2223)	→	(2332)		collinear	not unique
{20}	(2222)	→	(2223)	→	(2332)				symplectic	not unique
{21}	(1$\frac{3}{2}$1$\frac{3}{2}$)	→	(1$\frac{3}{2}$22)	→	(2223)	→	(2332)		collinear	
{22}	(11$\frac{3}{2}\frac{3}{2}$)	→	($\frac{3}{2}\frac{3}{2}$22)	→	(2223)	→	(2332)		equal	
{23}	(1$\frac{3}{2}\frac{3}{2}\frac{3}{2}$)	→	($\frac{3}{2}\frac{3}{2}$22)	→	(2223)	→	(2332)		collinear	not unique
{24}	($\frac{3}{2}\frac{3}{2}$22)	→	(2223)	→	(2332)				collinear	not unique
{25}	($\frac{3}{2}$2$\frac{3}{2}$2)	→	($\frac{3}{2}$223)	→	(2332)				symplectic	
{26}	($\frac{3}{2}\frac{3}{2}\frac{3}{2}$2)	→	($\frac{3}{2}$223)	→	(2332)				collinear	

TABLE 3. Combing distances between lines.

8.2. Round-up Triples of Exceptional Hexagonic & Minuscule Geometries of Rank ≥ 3

In this section, we classify all *round-up triples* of lines in the Lie incidence geometries of types $\mathsf{E}_{6,1}$, $\mathsf{E}_{7,7}$, and in exceptional hexagonic Lie incidence geometries of rank at least three. For that, we first define round-up triples and prove some of their general properties.

8.2.1. Definition and General Properties

Definition 8.2.1.1 (**Round-up triples**)**.** Let Δ be any spherical building, and let v_1, v_2, v_3 be three vertices of the same type. We call $\{v_1, v_2, v_3\}$ a *round-up triple* if no vertex of Δ is opposite exactly one of v_1, v_2, v_3 (cf. [**KV13**]). By definition, round-up triples in Lie incidence geometries correspond to round-up triples in the corresponding building.

Lemma 8.2.1.2. *Let $\{v_1, v_2, v_3\}$ be a round-up triple of vertices of some common type in a spherical building Δ. If v_1 and v_2 are joinable to a common vertex v, then also v_3 is joinable to v. Hence, in this case, $\{v_1, v_2, v_3\}$ is a round-up triple in the residue of v.*

PROOF. Consider an apartment Σ of Δ containing v and v_3 and let w be the unique vertex of Σ opposite v_3. By the definition of round-up triple, we may assume that w is opposite v_1. Let Σ' be an apartment containing w and the simplex $\{v, v_1\}$. Let $\varphi\colon \Sigma' \to \Sigma$ be an isomorphism of complexes fixing both w and v (as required to

exist by the definition of a building). Then $\varphi(v_1) = v_3$ as opposition is preserved. Hence v is joinable to v_3, as φ preserves the simplicial structure. □

Lemma 8.2.1.3. *Let $\{v_1, v_2, v_3\}$ be a round-up triple of vertices of some common type in a spherical building Δ. Suppose there are simplices $\{v, w_1\}$ and $\{v, w_2\}$ such that w_i is joinable to v_i, $i = 1, 2$. Suppose also that v_1 and v_2 are not joinable to a common vertex. Then there exists a vertex w_3 joinable to both v and v_3, and the type of w_3 can be chosen equal to the type of either w_1 or w_2.*

PROOF. Consider an apartment Σ of Δ containing $\{v, w_1\}$ and v_3 and let w be the unique vertex of Σ opposite v_3. Assume for contradiction that v_1 is opposite w. Let Σ^* be an apartment containing $\{v_1, w_1\}$ and w, and let $\varphi^*\colon \Sigma^* \to \Sigma$ be an isomorphism fixing w and w_1. Then $\varphi^*(v_1) = v_3$ by uniqueness of opposites in apartments. Hence w_1 is joinable to v_3, and Lemma 8.2.1.2 implies that it is also joinable to v_2, contradicting our assumptions. We conclude that w is opposite v_2. Let Σ' be an apartment containing $\{v_2, w_2\}$ and w and let Σ'' be an apartment containing $\{v, w_2\}$ and w.

Let $\varphi''\colon \Sigma'' \to \Sigma$ be an isomorphism fixing w and v and denote $\varphi''(w_2)$ briefly as w_3. Note that, by the definition of buildings, we may assume that φ'' is type-preserving, implying that w_3 has the same type as w_2. Also, w_3 is joinable to v, as w_2 is. Let $\varphi'\colon \Sigma' \to \Sigma''$ be an isomorphism fixing w_2 and w. Then $\varphi = \varphi'' \circ \varphi'\colon \Sigma' \to \Sigma$ is an isomorphism fixing w, hence mapping v_2 to v_3. But $\varphi(w_2) = w_3$. Hence w_3 is joinable to both v_3 and v. Interchanging the roles of v_1 and v_2, we can also choose the type of w_3 to be the same as the one of w_1.

This completes the proof of the lemma. □

Setting $v_2 = w_2$ in the previous lemma, noting that the only vertex of the same type as v_2 joinable to v_3 is v_3 itself, and then also noting the symmetry of the situation, we obtain the following consequence.

Corollary 8.2.1.4. *Let $\{v_1, v_2, v_3\}$ be a round-up triple of vertices of some common type in a spherical building Δ. Suppose there exists a simplex $\{w_1, w_2\}$ such that w_i is joinable to v_i, $i = 1, 2$. Suppose also that v_1 and v_2 are not joinable to a common vertex. Then v_3 is joinable to both w_1 and w_2.*

The next two properties will be used later, but we include them here to avoid revisiting a buildings-theoretic setting.

Lemma 8.2.1.5. *If S_1 and S_2 are two distinct simplices of a spherical building, then there exists at least one simplex opposite S_1, but not opposite S_2.*

PROOF. Noting that in any apartment, each vertex has a unique opposite, the lemma follows immediately by considering an apartment through S_1 and S_2. □

Corollary 8.2.1.6. *Let F be a simplex of a spherical building Ω. Then a collection T of vertices in $\mathsf{Res}(F)$ admits an opposite in $\mathsf{Res}(F)$ (viewed as a spherical building on its own) if and only if it admits an opposite vertex in Ω. Also, a set of vertices in $\mathsf{Res}(F)$ is a geometric line of $\mathsf{Res}(F)$ if and only if it is a geometric line of Ω. A triplet of vertices is a round-up triple in $\mathsf{Res}(F)$ if and only if it is a round-up triple in Ω.*

PROOF. If v is a vertex in $\mathsf{Res}(F)$ opposite each vertex of T in $\mathsf{Res}(F)$, then, for any simplex F' opposite F, the vertex $\mathsf{proj}_{F'}^{F}$ is opposite every member of T by Proposition 2.5.0.4. Conversely, let v' be a vertex of Ω opposite each member of T. Select $t \in T$. Applying Proposition 2.5.0.4, we find a simplex F' in $\mathsf{proj}_{v'}^{t}(F)$ opposite F. Then, again by Proposition 2.5.0.4, $\mathsf{proj}_{F}^{F'}(v')$ is, in $\mathsf{Res}(F)$, opposite every member of T. The second and third assertions now also follow. □

8.2.2. Round-up Triples of Lines of $\mathsf{E}_{6,1}(\mathbb{K})$

Lemma 8.2.2.7. *Let $\{L_1, L_2, L_3\}$ be a round-up triple of lines in the exceptional Lie incidence geometry $\mathsf{E}_{6,1}(\mathbb{K})$ such that L_1 and L_2 have at least one point in common. Then exactly one of the following holds.*

(*i*) $L_1 = L_2 = L_3$,
(*ii*) *L_1, L_2, L_3 are three lines in a common planar line pencil.*

PROOF. If $L_1 = L_2 \neq L_3$, then a 4-space opposite L_1 but not opposite L_3 (which exists by Lemma 8.2.1.5) violates the defining property of a round-up triple. Hence, if $L_1 = L_2$, then (*i*) holds.

Now assume $L_1 \cap L_2 = \{x\}$. By Lemma 8.2.1.2, $x \in L_3$. Then Corollary 8.2.1.6 and Proposition 8.0.0.1 imply that L_1, L_2, L_3 are contained in a plane, and (*ii*) follows. □

Lemma 8.2.2.8. *Let $\{L_1, L_2, L_3\}$ be a round-up triple of pairwise disjoint lines in the exceptional Lie incidence geometry $\mathsf{E}_{6,1}(\mathbb{K})$. Let y_1 be an arbitrary point of L_1. If y_1 is symplectic to some point of L_2, then it is collinear to some point of L_3.*

PROOF. Suppose $y_i \in L_i$, $i = 1, 2$, with $y_1 \perp\!\!\!\perp y_2$. We claim that there exists a point $u \in y_1^{\perp} \cap y_2^{\perp}$ not collinear to any point of L_3. Indeed, suppose each point of $y_1^{\perp} \cap y_2^{\perp}$ is collinear to some point of L_3. If that point of L_3 is unique, then, since $\{y_1, y_2\}^{\perp\perp} = \{y_1, y_2\}$ in each hyperbolic quadric, either y_1 or y_2 belongs to L_3, a contradiction. If L_3 were contained in $\xi(y_1, y_2)$, then y_1 would be collinear to a point of L_3, and the assertion would follow. If L_3 intersected $\xi(y_1, y_2)$ in a point z, then for each point $z_3 \in L_3$ we would have $z_3^{\perp} \cap \xi(y_1, y_2) \subseteq z^{\perp} \cap \xi(y_1, y_2)$, and this only contains $y_1^{\perp} \cap y_2^{\perp}$ if $z \in \{y_1, y_2\}$, a contradiction again. Hence L_3 is disjoint from $\xi(y_1, y_2)$. Now Lemma 4.5.0.18 implies that L_3 is collinear to a plane π of $\xi(y_1, y_2)$, and all points of $\xi(y_1, y_2)$ collinear to some point of L_3 are collinear to all points of π, a contradiction since $y_1^{\perp} \cap y_2^{\perp}$ is only collinear to $\{y_1, y_2\}$. The claim is proved.

Now, by Proposition 2.5.0.4, a 4-space U through u locally opposite the projection of L_3 onto u is opposite L_3, but not opposite either L_1 or L_2 as u is collinear to both $y_1 \in L_1$ and $y_2 \in L_2$. This final contradiction proves the lemma. □

Lemma 8.2.2.9. *Let $\{L_1, L_2, L_3\}$ be a round-up triple of lines in the exceptional Lie incidence geometry $\mathsf{E}_{6,1}(\mathbb{K})$. Then exactly one of the following holds.*

(*i*) $L_1 = L_2 = L_3$,
(*ii*) *L_1, L_2, L_3 are three lines in a common planar line pencil.*

PROOF. By Lemma 8.2.2.7, the statement is true if for some $i, j \in \{1, 2, 3\}$, $i \neq j$, the lines L_i and L_j have a point in common. So we may assume that L_1, L_2, L_3 are pairwise disjoint.

By Lemma 8.2.2.8, we may assume that some point $x_1 \in L_1$ is collinear to some point $x_2 \in L_2$. Set $M := x_1x_2$. Lemma 8.2.1.3 implies that M also intersects L_3. So $M \cap L_3$ is also a point x_3. Now select $y_1 \in L_1 \setminus \{x_1\}$. By Lemma 8.2.2.8, we may again assume that y_1 is collinear to some point y_2 of L_2. Then, as in the previous paragraph, the line L_3 has a point y_3 in common with y_1y_2. We have $y_3 \neq x_3$ as otherwise L_1, L_2 are contained in the plane generated by the lines M and y_1y_2, contradicting our assumption that $L_1 \cap L_2 = \varnothing$. Similarly, $y_2 \neq x_2$. If all of x_1, x_2, x_3 are collinear to all of y_1, y_2, y_3, then all of L_1, L_2, L_3 are contained in a singular 3-space, which is a residue of a simplex of type $\{2, 5, 6\}$ in the corresponding building. Then Corollary 8.2.1.6 and Proposition 8.0.0.1 lead to the contradiction that L_1, L_2, L_3 are coplanar and hence not disjoint. So we may assume that x_1 is not collinear to y_2. But then all of L_1, L_2, L_3 are contained in the symp $\xi(x_1, y_2)$, which is again a residue, and so Corollary 8.2.1.6 and Proposition 8.0.0.1 again lead to a contradiction.

The lemma is proved. □

8.2.3. Round-up Triples of Lines of $\mathsf{E}_{7,7}(\mathbb{K})$

Lemma 8.2.3.10. *Let $\{L_1, L_2, L_3\}$ be a round-up triple of lines in the exceptional Lie incidence geometry Γ of type $\mathsf{E}_{7,7}$ such that L_1 and L_2 intersect. Then exactly one of the following holds.*

(*i*) $L_1 = L_2 = L_3$,
(*ii*) *L_1, L_2, L_3 are three lines in a common planar line pencil.*

PROOF. If $L_1 = L_2$, then also $L_3 = L_1$ since otherwise there exists a line opposite L_3 and not opposite L_1. So we may assume $L_1 \cap L_2 = \{x\}$. By Lemma 8.2.1.2, we see that $x \in L_3$ and Corollary 8.2.1.6, in combination with Proposition 8.0.0.1, implies (*ii*). □

Lemma 8.2.3.11. *Let $\{L_1, L_2, L_3\}$ be a round-up triple of pairwise disjoint lines in an exceptional Lie incidence geometry of type $\mathsf{E}_{7,7}$. Then no point of L_2 is collinear to any point of L_1.*

PROOF. Let, for a contradiction, M be a line joining a point $x_1 \in L_1$ to a point $x_2 \in L_2$. Note that $L_1 \neq M \neq L_2$. Applying Lemma 8.2.1.3, we find that M intersects L_3. Set $x_i := M \cap L_i$, $i = 1, 2, 3$.

Assume, for a contradiction, that x_1 is symplectic to some point $y_3 \in L_3$. Set

$$\xi := \xi(x_1, y_3).$$

Noting that $M \subseteq \xi$, Corollary 8.2.1.4 yields $x_1 \in L_2 \subseteq \xi$, a contradiction. Hence, x_1 is collinear to each point of L_3. But then again, every line through x_1 intersecting L_3 meets L_2, and so L_2 and L_3 are contained in a common plane, hence intersecting, contradicting our assumptions. The lemma is proved. □

Proposition 8.2.3.12. *Let $\{L_1, L_2, L_3\}$ be a round-up triple of lines in the exceptional Lie incidence geometry Γ of type $\mathsf{E}_{7,7}$. Then exactly one of the following holds.*

(*i*) $L_1 = L_2 = L_3$,
(*ii*) L_1, L_2, L_3 *are three lines in a common planar line pencil.*

PROOF. In view of Lemma 8.2.3.10 and Lemma 8.2.3.11, it suffices to show that no round-up triple $\{L_1, L_2, L_3\}$ exists for which no point of L_i coincides with or is collinear to any point of L_j, $i, j \in \{1, 2, 3\}$, $i \neq j$. So suppose, for a contradiction, such a triple does exist. Select $x_1 \in L_1$. Then there exists $x_2 \in L_2$ symplectic to x_1. Set $\xi := \xi(x_1, x_2)$. Suppose, for a contradiction, that some point $x_3 \in L_3$ is opposite some point $y_{12} \in x_1^\perp \cap x_2^\perp$. Then we can find a line through y_{12} opposite L_3, but that line is certainly not opposite either L_1 or L_2 as it contains a point y_{12} collinear to points of L_1 and L_2. Hence no point of L_3 is opposite any point of $x_1^\perp \cap x_2^\perp$. If some point $x_3 \in L_3$ were far from ξ, this would imply that the unique point x_3' of ξ collinear to x_3 is collinear to all of $x_1^\perp \cap x_2^\perp$, forcing $x_3' \in \{x_1, x_2\}$, contradicting our assumption that no point of L_3 is collinear or equal to any point of $L_1 \cup L_2$. Hence all points of L_3 are close to ξ. Note also that, by Lemma 8.2.1.3, some point $x_3 \in L_3$ belongs to ξ. By interchanging the roles of L_3 and L_i, $i = 1, 2$, we see that each point of $L_1 \cup L_2$ is close to ξ. Since ξ is hyperbolic, there exists a point $y \in L_2^\perp \cap x_3^\perp \setminus x_1^\perp$. Let M be a line through y locally opposite ξ and select $z \in M \setminus \{y\}$. However, if y is not collinear to all points of L_3, then we (re)choose M locally not opposite the symp through y and L_3. In any case, z is not opposite any point of $L_2 \cup L_3$. Since it is opposite x_1, we find a line K through z opposite L_1. But K is not opposite either L_2 or L_3 by the properties of z, a contradiction.

The proposition is proved. □

8.2.4. Round-up Triples of Lines of Exceptional Hexagonic Geometries of Rank at least 3

After having considered the cases of $\mathsf{E}_{6,1}$ and $\mathsf{E}_{7,7}$, we now classify round-up triples in all exceptional hexagonic geometries of rank at least three.

Lemma 8.2.4.13. *Let $\{L_1, L_2, L_3\}$ be a round-up triple of lines in an exceptional hexagonic Lie incidence geometry Δ of rank at least 3 such that L_1 and L_2 intersect. Then exactly one of the following holds.*

(*i*) $L_1 = L_2 = L_3$,
(*ii*) L_1, L_2, L_3 *are three lines in a common planar line pencil,*
(*iii*) L_1, L_2, L_3 *are three lines in a common symp ξ containing a common point p and contained in a common hyperbolic line of $\mathsf{Res}_\xi(p)$. This only happens if Δ corresponds to a building of type F_4.*

PROOF. If $L_1 = L_2$, then also $L_3 = L_1$ since otherwise there exists a line opposite L_3 and not opposite L_1. So we may assume $L_1 \cap L_2 = \{x\}$. By Lemma 8.2.1.2, also $x \in L_3$. By Corollary 8.2.1.6, $\{L_1, L_2, L_3\}$ is a round-up triple in $\mathsf{Res}(x)$. The result

now follows from Proposition 8.0.0.1 for types E_6 and E_7, from Proposition 8.0.0.2 for type F_4, and from [**KV13**, Corollary 5.5] for type E_8. □

Lemma 8.2.4.14. *Let $\{L_1, L_2, L_3\}$ be a round-up triple of disjoint lines in an exceptional hexagonic Lie incidence geometry of rank at least 3. Then no point of L_2 is collinear to any point of L_1.*

PROOF. Let, for a contradiction, M be a line joining a point $x_1 \in L_1$ to a point $x_2 \in L_2$. Note that $L_1 \neq M \neq L_2$. Lemma 8.2.1.3 shows that M intersects L_3, say in the point x_3. Assume first that M and L_i are locally opposite at x_i, for every $i \in \{1, 2, 3\}$. Let π be any plane containing M. Let K_i be the line in π through x_i not locally opposite L_i at x_i, guaranteed to exist by Lemma 5.4.1.10. Suppose first that $z := K_1 \cap K_2$ does not belong to K_3. Let N be a line locally opposite zx_3 at z. Then any point $u \in N \setminus \{z\}$ is opposite some point of L_3, but is not opposite any point of $L_1 \cup L_2$. It follows that there exists a line through u opposite L_3, but not opposite either L_1 or L_2, a contradiction. Hence we may assume that there exists some line $K_3' \subseteq \pi$ through x_3 intersecting K_2 in some point $y_2 \notin K_1$, with y_2 special to every point of $L_3 \setminus \{x_3\}$. Then we pick a line N' through y_2 locally opposite K_3' at y_2, but not locally opposite x_1y_2 at y_2. Then no point w on N' is opposite some point of $L_1 \cup L_2$ since the pair $\{w, x_1\}$ is collinear or symplectic, and the pair $\{y_2, y_2'\}$, with $y_2' \in L_2 \setminus \{x_2\}$, is symplectic. As above, there exists a line through w opposite L_3, but not opposite either L_1 or L_2.

So, we may assume without loss of generality that M and L_3 are contained in a symp ξ. Then Corollary 8.2.1.4 implies $x_1 \in L_2 \subseteq \xi$, contrary to our assumptions. The lemma is proved. □

Lemma 8.2.4.15. *Let $\{L_1, L_2, L_3\}$ be a round-up triple of lines in an exceptional hexagonic Lie incidence geometry Δ of rank at least 3 such that no point of L_i is collinear to any point of L_j, for all $i, j \in \{1, 2, 3\}$, $i \neq j$. Then no point of L_1 is symplectic to any point of L_2.*

PROOF. Suppose for a contradiction that some point $x_1 \in L_1$ is symplectic to some point $x_2 \in L_2$. Let ξ be the corresponding symp. By Lemma 8.2.1.3, L_3 shares a point x_3 with ξ.

We claim that L_3 is collinear to a maximal singular subspace of ξ. Indeed, suppose not. Then $L_3^\perp \cap \xi$ is a line L_3^*. There are two cases.

(1) *Suppose every point of $x_1^\perp \cap x_2^\perp$ is collinear to x_3.* Then $x_3 \in \{x_1, x_2\}^{\perp\perp}$ and Δ corresponds to type F_4. Let ξ_3 be an arbitrary symp not containing L_3 and locally opposite ξ at x_3. Select $z_3 \in \xi_3 \setminus x_3^\perp$. Since $x_1 \equiv z_3 \equiv x_2$, we can define $M_i := \mathsf{proj}_{z_3}^{x_i}(L_i)$, $i = 1, 2$. If $M_1 \neq M_2$, we can take a line K through z_3 locally opposite M_1 at z_3, but not locally opposite M_2 at z_3, and then K is opposite L_1, and not opposite either L_2 or L_3 (the latter because z_3 is symplectic to every point of L_3), a contradiction. Hence, we may assume that $M_1 = M_2$. Set $u_i := M_i \cap x_i^\bowtie$, $i = 1, 2$. If $u_1 \neq u_2$, then we may replace z_3 with any point in $(M_1^\perp \cap \xi_3) \setminus \{z_3\}$ and apply the previous argument. So, we may assume $u_1 = u_2$. Let N_i be the line through u_i intersecting L_i, say in the point w_i, $i = 1, 2$. Then, since by Lemma 5.4.1.7, x_1 and w_2 are not opposite, the same Lemma 5.4.1.7

implies that N_1 and N_2 are not locally opposite at u_1. Hence w_1 and w_2 are symplectic (as we may assume that they are not collinear by Lemma 8.2.4.14).

Hence, by Lemma 8.2.1.3, the line L_3 intersects $\xi(w_1, w_2)$ in a point w_3, which we may assume to belong to $\{w_1, w_2\}^{\perp\perp}$ (as otherwise we are in case (2) below). Hence $u_1 \perp w_3$ and so $u_1 = [w_3, u_1]$, which, however, is contained in ξ_3 and coincides with $L_3^{\perp} \cap \xi_3 \cap u_1^{\perp}$. Hence $u_1 \perp x_3$. But then, similarly, $w_1 = [u_1, x_1] \in \xi$, implying $w_1 = x_1$; a contradiction.

(2) *Suppose some point* $y \in x_1^{\perp} \cap x_2^{\perp}$ *is not collinear to* x_3. Select $y_3 \in L_3 \setminus \{x_3\}$. Then, $y_3 \bowtie y$ and $u = [y, y_3] \in L_3^* \setminus \{x_3\}$. Let M be some line through y locally opposite yu at y. Let m be some arbitrary point on M not equal to y. Let L_3' be the projection of L_3 onto m. Note that $L_3' \neq M$ as $L_3 \neq uy_3$. Hence there exists a line K through m locally opposite L_3' but not locally opposite M. Then K is opposite L_3 but not opposite L_1 and L_2, because, by Lemma 5.4.1.7, no point of K is opposite x_1 or x_2.

Since both cases lead to contradictions, we conclude that L_3 is collinear to a maximal singular subspace U_3 of ξ. Likewise, L_1 and L_2 are also collinear to respective maximal singular subspaces U_1 and U_2 of ξ. Note that this implies that ξ is hyperbolic.

It follows that, since x_1 is not collinear to $x_2 \in U_2$, the set $x_3^{\perp} \cap U_2$ contains some point z_2 that is not collinear to x_1. Let y_3 be an arbitrary point of $L_3 \setminus \{x_3\}$. If $y_3 \perp\!\!\!\perp z_2$, let ξ_3 be the symp through y_3 and z_2. If $y_3 \perp z_2$, then let ξ_3 be a symp containing L_3 and z_2. Let ξ_2 be a symp through z_2 locally opposite ξ but not locally opposite ξ_3. Let $w_2 \in \xi_2$ be symplectic to z_2. Then, since z_2 is collinear to each point of L_2, Lemma 5.4.1.7 implies that w_2 is not opposite any point of L_2. Also, since ξ_2 is not locally opposite ξ_3, the point w_2 is not opposite any point of L_3. But w_2 is opposite x_1 and so there is a line K through w_2 opposite L_1, and K is not opposite either L_2 or L_3, a contradiction.

This completes the proof of the lemma. □

Lemma 8.2.4.16. *Let* $\{L_1, L_2, L_3\}$ *be a round-up triple of lines in an exceptional hexagonic Lie incidence geometry of rank at least* 3. *Let* $x_1 \in L_1$ *and* $x_2 \in L_2$ *be collinear to a common point* y. *Then* y *is collinear to a point of* L_3.

Proof. Suppose y is not collinear to any point of L_3. Let Σ be an apartment containing L_3 and y. Since y is not collinear to any point of L_3, it is not special to and not opposite at least two points of the line L_3^* that is opposite L_3 in Σ. But then y is equal, collinear or symplectic with each point of L_3^*, implying that no point of L_3^* is opposite either x_1 or x_2. Hence L_3^* is opposite L_3, but not opposite either L_1 or L_2, a contradiction. □

Lemma 8.2.4.17. *Let* $\{L_1, L_2, L_3\}$ *be a round-up triple of lines in an exceptional hexagonic Lie incidence geometry* Δ *of rank at least* 3 *such that each point of* L_1 *is special to each point of* L_2. *Then some point of* L_3 *is symplectic, collinear or equal to some point of* $L_1 \cup L_2$.

Proof. Using Lemma 5.4.1.10, we see that the set of points collinear to a point of L_1 and to a point of L_2 is a hyperbolic quadric Q_{12} (of rank 2). Each line of Q_{12} is collinear to a unique point of $L_1 \cup L_2$. We claim that non-collinear points on

Q_{12} are also non-collinear in Δ. Indeed, one checks that in that case Q_{12} generates a 3-space U. Picking non-collinear respective points in Q_{12} and L_1, we see that they are symplectic and the corresponding symps contain L_1 and at least a plane of U. The intersection of two such symps (with different planes in U) contains a 3-space. Hence the symps coincide. Now the symp through L_1 and U has a 3-space in common with the symp through L_2 and U and hence L_1 and L_2 are contained in a common symp, a contradiction.

By Lemma 8.2.4.16, each point of Q_{12} is also collinear to a unique point of L_3 (unique indeed since, if not, then Lemma 5.4.1.10 would yield a point of L_3 collinear or symplectic to some point of L_1, contradicting Lemma 8.2.4.14 and Lemma 8.2.4.15. It follows that, for any line L in Q_{12}, collinearity defines either a bijection between L and L_3, or a constant transformation from L to L_3. In the latter case, the unique points of L_3 and $L_1 \cup L_2$ collinear with all points of L are symplectic, contradicting Lemma 8.2.4.15. In the former case, pick $p \in L$ and let L' be the unique line of Q_{12} through p distinct from L. Then again, collinearity defines a bijection between L' and L_3. Hence there is a point of L_3 collinear to two non-collinear points of Q_{12}, and since these points are also non-collinear in Δ, this contradicts Lemma 5.4.1.10.

The lemma is proved. □

Lemma 8.2.4.18. *Let L_1, L_2, L_3 be three lines of an exceptional hexagonic Lie incidence geometry of rank at least 3 such that each point of L_1 is special to or opposite each point of L_2. Then $\{L_1, L_2, L_3\}$ is not a round-up triple.*

Proof. By Lemmata 8.2.4.13, 8.2.4.14 and 8.2.4.15, we may assume that no point of L_3 is equal, collinear or symplectic to any point of $L_1 \cup L_2$. Moreover, by Lemma 8.2.4.17, we may also assume that some point of L_3 is opposite L_1 and some point of L_3 is opposite some point of L_2. This implies that the mutual positions of L_i and L_j, $i, j \in \{1, 2, 3\}$, $i \neq j$, are given by either (2223) or (2332).

We may assume that $\{L_1, L_2, L_3\}$ is a round-up triple. By the nature of (2223) and (2332), there exists at most one point of L_3 that is not opposite all points of L_1, and at most one point of L_3 that is not opposite all points of L_2. Hence we find a point $x_3 \in L_3$ that is opposite at least one point x_1 of L_1 and at least one point x_2 of L_2. We can then project L_i, $i = 1, 2$, from x_i onto x_3 and obtain lines L_i' and points $y_i \in L_i'$ collinear to a point of L_i. Lemma 8.2.4.16 and the uniqueness of the projections yield $L_1' = L_2' =: M_3$ and $y_1 = y_2 =: y$

Let M_i, $i = 1, 2$, be the lines through y intersecting L_i. Our assumptions imply that these lines are pairwise locally opposite at y. By Proposition 8.0.0.1, Proposition 8.0.0.2 and [**KV13**, Corollary 5.6], $\{M_1, M_2, M_3\}$ is not a round-up triple in the point residual $\mathsf{Res}(y)$. Hence, up to renumbering, we find a line $M \ni y$ locally opposite M_1 , and not locally opposite either M_2 or M_3 at y. Pick $z \in M \setminus \{y\}$. Then Lemma 5.4.1.7 implies that z is opposite x_1, but not opposite any point of $L_2 \cup L_3$. It follows that each line K through z opposite L_1 (which exists) is not opposite either L_2 or L_3, a contradiction.

This proves the lemma completely. □

We summarize the previous lemmata into the following proposition.

Proposition 8.2.4.19. *Let $\{L_1, L_2, L_3\}$ be a round-up triple of lines in an exceptional hexagonic Lie incidence geometry of rank at least 3. Then exactly one of the following holds.*

(*i*) $L_1 = L_2 = L_3$,
(*ii*) L_1, L_2, L_3 *are three lines in a common planar line pencil,*
(*iii*) L_1, L_2, L_3 *are three lines in a common symp ξ isomorphic to a symplectic polar space of rank 3, and they are contained in a geometric line of ξ but not in a plane.*

PROOF. Suppose $\{L_1, L_2, L_3\}$ is a round-up triple of lines in an exceptional hexagonic Lie incidence geometry of rank at least 3 such that none of the situations (*i*), (*ii*), or (*iii*) holds. By Lemma 8.2.4.18, we may assume that some point of L_i is symplectic, collinear, or equal to some point of L_j, for each distinct pair i, j in $\{1, 2, 3\}$. By Lemma 8.2.4.13, we may assume that L_1, L_2, L_3 are pairwise disjoint. Then, Lemma 8.2.4.14 implies that each point of L_i, $i \in \{1, 2, 3\}$, is symplectic, special, or opposite any point of L_j, $j \in \{1, 2, 3\} \setminus \{i\}$. With that, Lemma 8.2.4.15 implies that each point of L_i, $i \in \{1, 2, 3\}$, is special or opposite any point of L_j, $j \in \{1, 2, 3\} \setminus \{i\}$, which contradicts the assumption. □

8.3. Geometric Lines in Exceptional Hexagonic & Minuscule Geometries of Rank ≥ 3

We now turn to the classification of geometric lines in exceptional minuscule and exceptional hexagonic geometries of rank at least three.

Corollary 5.6 of [**KV13**] states that geometric lines of Lie incidence geometries of type $\mathsf{E}_{6,1}$ and $\mathsf{E}_{7,7}$ are exactly the lines. For hexagonic Lie incidence geometries, there is a partial classification given in [**KV13**]. Our intention is to extend this. We first briefly repeat the results of [**KV13**].

Proposition 8.3.0.1 (Theorem 6.5 in [**KV13**])**.** *Let L be a geometric line in an exceptional hexagonic geometry Γ. Then exactly one of the following cases occurs.*

(1) *L is an ordinary line of Γ,*
(2) *L is a hyperbolic line in a symplecton of Γ isomorphic to a symplectic polar space (and this only occurs in hexagonic geometries of type $\mathsf{F}_{4,4}$ arising from a split building of type F_4),*
(3) *L consists of mutually opposite points.*

In view of the results of Kasikova and Van Maldeghem, it only remains to classify geometric lines consisting of mutually opposite points in exceptional hexagonic geometries of rank at least three. The following lemma will be very useful for that, as it reduces the problem to equator geometries.

Lemma 8.3.0.2. *Each geometric line L containing opposite points of any (exceptional) hexagonic geometry Γ is a geometric line of any equator geometry of Γ containing at least two points of L.*

PROOF. Let x, y be two points of the geometric line L, consisting of mutually opposite points of Γ. We claim that no point of L is special to any point of $E(x, y)$, the equator geometry with poles x and y. Indeed, suppose $z \in L$ is special to $u \in E(x, y)$. Extend the unique path $z \perp [u, z] \perp u$ to a path $z \perp [u, z] \perp u \perp v$, with $v \bowtie [u, z]$. Then v is opposite $z \in L$, but since $x \perp\!\!\!\perp u \perp v$, Lemma 5.4.1.7 implies that v is not opposite x. Similarly, v is not opposite y, a contradiction to L being a geometric line. The claim is proved.

It immediately follows from the previous claim that no point of L is either collinear to any point of $E(x, y)$, or belongs to $E(x, y)$. Indeed, if $z \in L$ were collinear to $u \in E(x, y)$, then we can consider the unique point v on the line uz not opposite some point $w \in E(x, y)$ opposite u. The claim in the previous paragraph implies $v \neq z$. But then $w \equiv z$, while w is not opposite either x or y, a contradiction. If $z \in L \cap E(x, y)$, then a point $u \in E(x, y)$ opposite z is not opposite both x and y, again a contradiction. For the same reason, no point of L is opposite any point of $E(x, y)$.

Hence we have shown that each point of L is symplectic to each point of $E(x, y)$. Taking two opposite points $a, b \in E(x, y)$, this implies $L \subseteq E(a, b)$. Since opposition in $E(x, y)$ as a Lie incidence geometry coincides with the opposition inherited from Γ, the assertion follows. □

Counterexamples to the converse of Lemma 8.3.0.2 will be given in type F_4 (see the proof of the next theorem).

We can now prove the announced classification.

Theorem 8.3.0.3. *Let L be a geometric line in an exceptional hexagonic geometry Γ. Then L does not consist of mutually opposite points.*

PROOF. By Lemma 8.3.0.2, the non-existence of geometric lines consisting of mutually opposite points in hexagonic geometries of types E_6, E_7 and E_8 follows from the non-existence of such geometric lines in the Lie incidence geometries of types $\mathsf{D}_{6,2}$ and $\mathsf{A}_{5,\{1,5\}}$. The former case is taken care of by Proposition 8.0.0.1. In the latter case, by taking again equator geometries, see Note 5.5.0.3, we reduce the question to the case $\mathsf{A}_{3,\{1,3\}}$. Then by Lemma 8.3.0.2, we see that L consists of incident point-plane pairs in $\mathsf{PG}(3, \mathbb{K})$ with the point ranging over a given line K and the plane determined by the point and a given line K' skew to K. Consider a point $x \notin K \cup K'$ and a plane α through x intersecting both K and K' in respective unique points, say y and y', respectively. Then $\{x, \alpha\}$ is not opposite $\{\langle K', x\rangle \cap K, \langle K', x\rangle\}$ and not opposite $\{y, \langle y, K'\rangle\}$, but opposite every other member of L, a contradiction.

This shows that no geometric line of Γ consists of opposite points, for Γ of types $\mathsf{E}_{6,2}$, $\mathsf{E}_{7,1}$ or $\mathsf{E}_{8,8}$.

Hence we may suppose that Γ has type F_4. In that case, a geometric line T consisting of mutually opposite points is a geometric line of the polar space Δ of rank 3 corresponding to a point residual of Γ. It follows from [**KV13**, Lemma 4.8] that Δ is a symplectic polar space (hence Γ is split — but possibly over a non-perfect field) and T is a hyperbolic line. By the transitivity of the automorphism group on the set of hyperbolic lines, we may assume that every hyperbolic line of each equator geometry is a geometric line. Now let x, y be two points and $E(x, y)$ the

corresponding equator geometry. Let ξ be a symp corresponding to a line L of $E(x,y)$. Assume first that the underlying field is not perfect of characteristic 2. Then, again by [**KV13**, Lemma 4.8], there exists a point $a \in \xi$ either collinear to at least two points of L but not all, or not collinear to any point of L. Also, more precisely, since Δ is split, ξ is a polar space corresponding to a quadric in $\mathsf{PG}(6,\mathbb{K})$, L is the intersection of the perps of two opposite lines, and so, a is collinear to either 0 or 2 points of L. Let b be a point of Γ far from ξ and symplectic to a. Then $b^{\neq}$ intersects every hyperbolic line of $E(x,y)$ in one or all its points (as these are all assumed to be geometric lines), L in 0 or 2 points, and, by the above argument for a applied to other appropriate points, intersecting every line in either 0, 1, 2 or all points. We view $E(x,y)$ in its natural embedding in $\mathsf{PG}(5,\mathbb{K})$. Let π be a non-singular plane of $\mathsf{PG}(5,\mathbb{K})$ (with respect to the underlying non-degenerate alternating form) containing L. We can choose π such that the unique point $x_L \in L$ for which $\pi \subseteq p_L^{\perp}$ is opposite b. Let $\ell \neq x_L$ be another point on L opposite b in Γ (which exists by our assumptions above and the fact that there are at least four points on a line — indeed, $\mathbb{F}_2$ is perfect of characteristic 2, and so $|\mathbb{K}| \geq 3$) and let h_1, h_2, h_3 be three distinct hyperbolic lines of $E(x,y)$ in π through ℓ. Then there are unique points $a_1 \in h_1$, $a_2 \in h_2$ and $a_3 \in h_3$ not opposite ℓ. Suppose first that a_1, a_2, a_3 lie on the same line L' of π. Then the whole line L' belongs to $b^{\neq}$, and hence at least one point of L does, namely $L \cap L'$. But then two points of L do, and connecting that second point, which does not coincide with x_L, with all points of L' leads to $\pi \subseteq b^{\neq}$, a contradiction. Hence we may assume that a_1, a_2, a_3 are not contained in the same line of π. One can see that joining with hyperbolic lines yields all points of π, except for x_L. Since $|\mathbb{K}| > 2$, this is again a contradiction to $|b^{\neq} \cap L| \in \{0, 2\}$.

Now assume that $\mathbb{K}$ is perfect of characteristic 2. We may embed every equator geometry in an extended equator geometry $\widehat{E}$, which is then isomorphic to a symplectic polar space of rank 4 (see Definition 5.5.0.13). By [**DSV18**, Corollary 5.38], there exists a point b of Γ such that $H := b^{\neq} \cap \widehat{E}$ is a (hyperbolic) polar subspace of type $\mathsf{D}_{4,1}$. Note that we still may assume that every hyperbolic line is a geometric line. Then H is also a geometric hyperplane of the ambient projective space $\mathsf{PG}(7,\mathbb{K})$ of $\widehat{E}$, and hence coincides with $p^{\perp}$ for some point $p \in \widehat{E}$ (where the perp $\perp$ is now with respect to the symplectic polar space). This is a contradiction.

The theorem is proved. □

Proposition 8.3.0.4. *Suppose every round-up triple of points in a Lie incidence geometry Δ is contained in a line. Then each geometric line of Δ is an ordinary line.*

Proof. Let L be a geometric line of Δ. By Proposition 7.1.0.1, L contains at least three elements. Pick $x_1, x_2 \in L$. Every triple of elements of L containing x_1, x_2 is a round-up triple and hence contained in a line M, which coincides with the unique line through x_1 and x_2. Hence $L \subseteq M$. If $L \neq M$, then each point opposite x_1 and not opposite some point of $M \setminus L$ would be opposite all members of L, a contradiction. The proposition is proved. □

The next statement about geometric lines follows from the previous considerations about round-up triples.

Corollary 8.3.0.5. *Let L be a geometric line in any of the exceptional Lie incidence geometries of the following types: $\mathsf{E}_{6,j}$, $j \in \{1,2,\ldots,6\}$, $\mathsf{E}_{7,j}$, $j \in \{1,2,6,7\}$, $\mathsf{E}_{8,8}$, $\mathsf{E}_{8,7}$. Then L is an ordinary line. Conversely, every line is a geometric line.*

PROOF. It is clear that each line is a geometric line. If L is a geometric line, then each triple of points of L is a round-up triple. Our previous results about round-up triples imply that L is contained in a line L^*. If there were a point $x \in L^* \setminus L$, then we could find a point opposite each member of L by considering a line M opposite L^* and taking the unique point of M not opposite x. This contradiction shows $L = L^*$, and the proposition is proved. □

We can also summarize the situation for type F_4 as follows.

Proposition 8.3.0.6. *Let L be a geometric line in a Lie incidence geometry Γ of type $\mathsf{F}_{4,i}$, $i \in \{1,2,3,4\}$. Then one of the following occurs.*

(1) *L is an ordinary line,*
(2) *L is a hyperbolic line in a symplecton of Γ isomorphic to a symplectic polar space (and this only occurs in Lie incidence geometries of type $\mathsf{F}_{4,4}$ arising from a split building of type F_4),*
(3) *L is a hyperbolic line in a symp isomorphic to a symplectic polar space of rank 2 (and this only occurs in Lie incidence geometries of type $\mathsf{F}_{4,3}$ arising from a split building of type F_4).*

PROOF. Similarly to the proof of Corollary 8.3.0.5, this follows from Proposition 8.2.4.19, Proposition 8.3.0.1, and Theorem 8.3.0.3. □

So far we have considered the point-Grassmannians, now we turn to line-Grassmannians. The basic work for the following proposition was done in Proposition 8.2.4.19.

Proposition 8.3.0.7. *Let T be a geometric line of the line-Grassmannian of an exceptional hexagonic Lie incidence geometry Δ of rank at least 3. Then exactly one of the following holds.*

(i) T is an ordinary line of the corresponding line-Grassmannian parapolar space, that is, a planar line pencil of Δ;
(ii) Δ is $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$ and T is a cone over a hyperbolic line in a symplectic symp.

PROOF. If Δ is not $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$, then Proposition 8.3.0.4, together with Proposition 8.2.4.19, imply (i).

Suppose now $\Delta \cong \mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$. If some pair of elements of T is contained in an ordinary line K of the line-Grassmannian of Δ, then, again by the previous lemmata and the fact that every triple of members of T is a round-up triple, all elements are contained in that line; hence all triples are, and Proposition 8.3.0.4 implies that $T = K$.

Next suppose that two elements L, M of T are not coplanar. Then, again by Lemmata 8.2.4.13, 8.2.4.14, 8.2.4.15, 8.2.4.17 and 8.2.4.18, they are contained in a hyperbolic line H of the point residual of a symp ξ. The same lemmata now imply that $T \subseteq H$ and Proposition 8.0.0.2(ii) yields $T = H$, and ξ is a symplectic polar space. The proposition is proved. □

8.4. Geometric Lines in Moufang Hexagons

We now classify geometric lines in Moufang hexagons. We first consider the case of generalized hexagons and then specify to Moufang hexagons. As before, we first consider round-up triples.

Lemma 8.4.0.1. *Let $\Gamma = (X, \mathscr{L})$ be a generalized hexagon and $\{x_1, x_2, x_3\}$ a round-up triple of points. Suppose $x_1 \perp x_2$. Then x_1, x_2, x_3 are contained in a common line.*

Proof. Let L be the line containing x_1 and x_2. If $x_3 \notin L$, then some point u of L is special to x_3. Let z be a point collinear to u but special to $[x_3, u]$. Then $z \equiv x_3$ whereas $x_1 \not\equiv z \not\equiv x_2$, a contradiction. □

Lemma 8.4.0.2. *Let $\Gamma = (X, \mathscr{L})$ be a generalized hexagon and $\{x_1, x_2, x_3\}$ a round-up triple of points. Suppose $x_1 \bowtie x_2$. Then x_1, x_2, x_3 are contained in $[x_1, x_2]^\perp \cap y^\bowtie$, for every point $y \in [x_1, x_2]^\equiv \cap x_1^\bowtie \cap x_2^\bowtie$.*

Proof. By Lemma 8.4.0.1 we have $x_3 \neq [x_1, x_2]$. If $x_3 \perp [x_1, x_2]$, then the assertion follows directly from the definition of round-up triple. If $x_3 \bowtie [x_1, x_2]$, then any point $z \in [x_1, x_2]^\perp \setminus \langle [x_1, x_2], [[x_1, x_2], x_3] \rangle$ is opposite x_3, but not opposite either x_1 or x_2, a contradiction. Finally, if $x_3 \equiv [x_1, x_2]$, then $[x_1, x_2]$ violates the defining property of $\{x_1, x_2, x_3\}$ being a round-up triple. □

Lemma 8.4.0.3. *Let $\Gamma = (X, \mathscr{L})$ be a generalized hexagon and $\{x_1, x_2, x_3\}$ a round-up triple of points. Suppose $x_1 \equiv x_2$. Then x_1, x_2, x_3 are contained in every distance-3 trace containing at least two of them.*

Proof. Let L be an arbitrary line of Γ containing a point x_1' collinear to x_1 and also a point x_2' collinear to x_2. By Lemma 8.4.0.2, $x_3 \notin L$. Assume x_3 is special to some point $y \in L$ with $[x_3, y] \notin L$. Then any point of $L \setminus \{y\}$ is opposite x_3 and not opposite both x_1, x_2. So, x_3 is collinear to some point of L, distinct from both x_1' and x_2' (use Lemma 8.4.0.2 again). Now the assertion follows. □

Proposition 8.4.0.4. *Let $\Gamma = (X, \mathscr{L})$ be a generalized hexagon and let T be a geometric line of Γ. Then T is either a line, a hyperbolic line, or a regular distance-3 trace.*

Proof. Since every triple of points of a geometric line is a round-up triple, the previous three lemmata imply that T is contained in either a line, or a hyperbolic line, or a distance-3 trace. But if T would not be equal to one of these objects, then, in each case, one can find a point opposite every member of T, a contradiction. □

We can now specify to the Moufang case.

Proposition 8.4.0.5. *Let $\Gamma = (X, \mathscr{L})$ be a Moufang generalized hexagon and let T be a geometric line. Then T is either*

(1) *an ordinary line, or*
(2) *a hyperbolic line in a split Cayley hexagon, or*

(3) *a distance-*3 *trace (or imaginary line) in a split Cayley hexagon over a perfect field in characteristic* 2.

PROOF. By Proposition 8.4.0.4 there are three possibilities for T. The first one is a line, which leads to (1). The second one is a hyperbolic line. Let T be collinear to the unique point c. Since T is a geometric line, every hyperbolic line in $c^\perp$ intersects T in exactly one point. By transitivity of the automorphism group on paths $x_1 \perp x_2 \perp x_3$, with $x_1 \bowtie x_3$, which follows readily from the Moufang condition, we see that every pair of hyperbolic lines in $c^\perp$ intersects. Then [**Van98**, Corollary 5.14] implies that Γ is a split Cayley hexagon.

The third possibility is that T is a regular distance-3 trace. Let $x, y \in T$. Then we can write $T = \{x, y\}^{\neq\neq}$, as follows from the proof of Lemma 8.4.0.3. Let $L \in \mathscr{L}$ be such that it contains unique points x' and y' special to x and y, respectively, and $x' \neq y'$. That such a line exists is easily seen. Now every point of L is special to precisely one point of T, since T is a geometric line. This means that, in the terminology of [**Van98**, Definitions 6.5.5], the set T is a long imaginary line, and [**Van98**, Theorem 6.5.6] now implies that Γ is a split Cayley hexagon over a perfect field in characteristic 2.

The theorem is proved. □

Note 8.4.0.6. For every natural number $n \geq 5$, there exists an (obvious) analogue of Proposition 8.4.0.4 for the class of (thick) generalized n-gons. This requires defining a "regular" distance i-trace, $2 \leq i \leq \frac{n}{2}$, similarly to a hyperbolic line (which would be a regular distance-2 trace) and a regular distance 3-trace for a generalized hexagon. Proofs are straightforward generalizations of the above proofs for hexagons. Restricting to Moufang octagons (the only class of Moufang buildings not yet considered in this chapter), one obtains that, using the results of John van Bon, Hans Cuypers, and Hendrik Van Maldeghem [**vCV96**] (see also [**Van98**, Section 6.5]), the only geometric lines in Moufang octagons are the ordinary lines.

9. Characterisation of Lines in Finite Lie Incidence Geometries

In this chapter we will answer all questions posed in the interlude (Chapter 7) for all finite thick irreducible spherical buildings Δ which are either of classical type, or which are Moufang and correspond to hexagonic or minuscule Lie incidence geometries. That means, most importantly, we have to answer the following:

If Δ has thickness at least $s+1$, does there exist a vertex opposite to $s+1$ arbitrarily given vertices of the same type?

For that, we now first present a definition for *blocking sets* for finite thick irreducible spherical buildings.

Definition 9.0.0.1 (Blocking sets). If in an irreducible thick finite spherical building Δ of type X_n, for $n \geq 2$, a set T of vertices of type i of Δ admits no common opposite vertex, then we call T a *blocking set* of Δ.

As we know, if Δ has thickness $s+1$, we can always find an opposite vertex to s arbitrarily given vertices of the same type. Thus, sets containing only s vertices of the same type cannot be blocking sets. On the other hand, it is easy to construct a blocking set, if one is allowed to take much more than $s+1$ vertices, and it is less interesting to do an investigation in that direction. Our main question can be rephrased as:

What are the minimal blocking sets of Δ?

In Section 9.1, we state our main results. Then we present our proofs, beginning with the classical cases: first for buildings of type A_n, i.e., projective spaces, in Section 9.2, and then for buildings of type $\mathsf{B}_n = \mathsf{C}_n$ and D_n, i.e., polar spaces, in Section 9.4. We provide an easier alternative proof for all polar spaces associated to quadrics in Section 9.3. After that, we use our results of Chapter 8, to prove our results for the exceptional cases. We start with the minuscule geometries $\mathsf{E}_{6,1}$ and $\mathsf{E}_{7,7}$, and for both, we first show the result for points (Section 9.5.1, Section 9.6.1), and then for lines (Section 9.5.2, Section 9.6.2). After that, we provide a uniform proof for points of exceptional hexagonic geometries of rank at least three (Section 9.7), and then turn to the lines of these geometries (Section 9.8). For the lines, we introduce *Combing Algorithms*, which will essentially take care of the proof (Section 9.8.2); only in the case of type $\mathsf{F}_{4,4}$ do we need to pay some extra attention (Section 9.8.3). At the end, we take a look at rank-2 geometries and prove our results for generalized hexagons (Section 9.9).

9.1. Main Results

We first phrase the assumptions of our main results uniformly in building-theoretic terms, and only afterwards we specify to the various Lie incidence geometries. We use Bourbaki labeling of the types.

The results in this chapter are based on [**BV26**] and [**BV25b**].

9.1.1. The Classical Cases

Theorem B. *If in an irreducible thick finite classical spherical building Δ of type X_n, for $n \geq 2$, the panels of cotype $\{i\}$ are s-thick, then every set T of $s+1$ vertices of type i of Δ admits a common opposite vertex except in precisely the following six cases.*

(1) *The set T is a line in the corresponding Lie incidence geometry of type $\mathsf{X}_{n,i}$.*
(2) *Δ is a generalized quadrangle of order (s,t) and T is an ovoid in a subquadrangle of order $(s/t,t)$, or Δ is a generalized quadrangle of order (t,s) and T is a spread in a subquadrangle of order $(t,s/t)$.*
(3) *Δ is the building corresponding to a symplectic polar space of rank at least 3 and T is a hyperbolic line in the residue of a simplex of type $\{1,2,\ldots,i-1\}$.*
(4) *Δ is the building corresponding to a parabolic quadric Γ and T is a set of generators of Γ such that*
   - *all members of T, viewed as subspaces of Γ, contain a common singular subspace U of codimension 2,*
   - *in the residue of U, which is a generalized quadrangle associated to a quadric, the set T is a regulus.*
(5) *Δ is the building corresponding to a small Hermitian polar space of rank at least 3 in characteristic 2, $i = n-1$, and T is an ovoid in a symplectic subquadrangle of order (t,t) of the residue of a simplex of type $\{1,2,\ldots,n-2\}$.*
(6) *Δ is the building corresponding to an elliptic quadric of rank at least 3 in characteristic 2, $i = n$, and T is a spread in a symplectic subquadrangle of order (s,s) of the residue of a simplex of type $\{1,2,\ldots,n-2\}$.*

In Case (2), if $s = t$, then T is either a hyperbolic line in Δ, or in the dual of Δ; the latter is the generalized quadrangle obtained from Δ by interchanging the points and the lines.

For projective spaces, this theorem can be stated as follows, with the convention that the empty projective subspace has dimension -1.

Corollary 9.1.1.1. *Let $0 \leq k < n$ be integers, and let q be a prime power. Let T be a set of $q+1$ k-dimensional subspaces of $\mathsf{PG}(n,q)$. If no $(n-k-1)$-space is disjoint from each member of T, then there exist a $(k-1)$-space U and a $(k+1)$-space W such that T coincides with the set of k-spaces containing U and contained in W.*

For polar spaces we have the following formulation (excluding the rather trivial case of a grid, which corresponds to a reducible building of type $\mathsf{A}_1 \times \mathsf{A}_1$).

Corollary 9.1.1.2. *Let Γ be a polar space of rank r at least 2 and order (s,t), with $t > 1$ if $r = 2$. Let T be either*

- *a set of $s+1$ singular subspaces of Γ of dimension $k \leq r-2$, or*
- *only if $t > 1$, a set of $t+1$ maximal singular subspaces (and we set $k = r-1$), or*
- *only if $t = 1$, a set of $s+1$ maximal singular subspaces of the same natural class (and again $k = r-1$).*

Then there exists a singular subspace of dimension k opposite each member of T, except if

(1) *$k \leq r-2$ and all members of T contain a given $(k-1)$-dimensional subspace and are contained in a given $(k+1)$-dimensional singular subspace,*
(2) *$k = r-1$, Γ is not hyperbolic, and all members of T contain a given $(r-2)$-dimensional subspace,*
(3) *$k \leq r-2$, Γ is symplectic, and all members of T contain a given $(k-1)$-dimensional subspace in the residue of which they form a hyperbolic line,*
(4) *$k = r-1$, Γ is either parabolic or hyperbolic, and all members of T contain a given $(r-3)$-dimensional subspace in the residue of which they form a regulus,*
(5) *$k = r-2$, Γ is a small Hermitian polar space over a field of characteristic 2, and all members of T contain a given $(r-3)$-dimensional subspace in the residue of which they form an ovoid in a symplectic subquadrangle of order (t,t),*
(6) *$k = r-1$, Γ is elliptic over a field of characteristic 2, and all members of T contain a given $(r-3)$-dimensional subspace in the residue of which they form a spread in a symplectic subquadrangle of order (s,s),*
(7) *$r = 2$ and T is either an ovoid in a subquadrangle of order $(s/t, t)$, or a spread in a subquadrangle of order $(s, t/s)$.*

In the last case, if $s = t$, then T is half of a subquadrangle of order $(1,s)$ or $(s,1)$, respectively. If $s \neq t$, then the situation in (vii) leads to a blocking set with Lemma 9.4.0.3.

The definition of a blocking set with the size of a line in a classical geometry is much weaker than that of a geometric line. However, considering buildings of rank at least three, we have the following connection to geometric lines.

Theorem C. *If in an irreducible thick finite classical spherical building Δ of type X_n ($n \geq 3$) the panels of cotype $\{i\}$ are s-thick (s odd), then a set T of vertices of type i of Δ is a geometric line in the i-Grassmannian geometry (of type $\mathsf{X}_{n,i}$) if and only if it has size $s+1$ and does not admit a common opposite vertex. The condition on s can be lifted if Δ does not correspond to either a small Hermitian polar space if $i = n-1$, or an elliptic quadric if $i = n$.*

Note that our proofs regarding blocking sets in the classical cases do not use the notion of geometric lines.

Remark 9.1.1.3. Recall from Definition 5.0.0.2 the notion of *split building*. In the finite classical case, this concerns the projective spaces, the symplectic polar spaces (which are then said to be of type C_n), the parabolic polar spaces (type B_n), and the oriflamme complexes of hyperbolic polar spaces (type D_n). These have not just a Coxeter diagram attached, but a Dynkin diagram, where nodes correspond to fundamental roots of a root system. Then the cases in the conclusions of Theorem B, where the set T is not a line in the Lie incidence geometry of type $\mathsf{X}_{n,i}$, occur precisely when i represents a short root in the root system corresponding to the Dynkin diagram. This behavior will sustain in the exceptional case.

We summarize part of our results in the language of point–line geometries.

Summary 9.1.1.4. For a finite classical geometry Δ, in which every line has exactly $s+1$ points, any set of at most $s+1$ points that does not admit a common opposite, contains exactly $s+1$ points and is either

- a line,
- a hyperbolic line in a symplectic residue,
- an ovoid in a proper ideal symplectic subquadrangle in characteristic 2 of a residue.

If $s+1$ is even, a set of at most $s+1$ points is a geometric line if and only if no object of the opposite type of a point is opposite each point of the set.

Another corollary of our results is the following.

Corollary 9.1.1.5. *If in an irreducible thick finite classical spherical building Δ of type X_n, $n \geq 2$, the panels of cotype $\{i\}$ are s-thick, then every set T' of at most s vertices of type i of Δ admits a common opposite vertex.*

Proof. We can always complete T' to a set T of $s+1$ vertices by adding a vertex such that T is not a set as in the conclusion of *Theorem B*. □

Note that, if a building has different thicknesses for different types of vertices, then this Corollary does not follow from Proposition 7.1.0.1, for the panels of the type of the biggest thickness; it only follows if s is the smallest thickness of panels.

9.1.2. The Exceptional Cases

With regards to blocking sets, we have the following theorem.

Theorem D. *Let $\mathsf{X}_n \in \{\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8, \mathsf{F}_4, \mathsf{G}_2\}$, and $1 \leq i \leq n$ with $i \notin \{2,4,5\}$ if $n = 7$, and $i \in \{7,8\}$ if $n = 8$. If, in an irreducible thick finite Moufang spherical building Δ of type X_n, $n \geq 2$, the panels of cotype $\{i\}$ are s-thick, then every set of $s+1$ vertices of type i of Δ admits a common opposite vertex except precisely in the following four cases.*

(1) *The $s+1$ vertices form a line in the corresponding Lie incidence geometry of type $\mathsf{X}_{n,i}$.*

(2) Δ *is a building of type* F_4*, and the* $s+1$ *vertices correspond to an ovoid in a subquadrangle* Γ' *of order* $(s/t, t)$ *in a rank* 2 *residue isomorphic to (the building corresponding to) a Moufang quadrangle* Γ *of order* (s, t) *(and either* Γ *is a symplectic quadrangle and the set of* $s+1$ *vertices corresponds to a hyperbolic line in* Γ*, or* Γ *is a Hermitian quadrangle of order* $(s, \sqrt{s})$*,* s *even, and* Γ' *is a symplectic quadrangle of order* $(\sqrt{s}, \sqrt{s})$*).*

(3) Δ *is a split building of type* G_2 *and the* $s+1$ *vertices form a hyperbolic (or ideal) line in the corresponding Lie incidence geometry of type* $\mathsf{G}_{2,2}$*, which is a split Cayley hexagon.*

(4) Δ *is a building of type* G_2 *in characteristic* 2 *and the* $s+1$ *vertices form a distance-*3 *trace in the corresponding Lie incidence geometry of type* $\mathsf{G}_{2,2}$*, which is either a split Cayley hexagon, or a twisted triality hexagon of order* $(s, \sqrt[3]{s})$*.*

Theorem D will follow

- from Proposition 9.5.1.1 for types $\mathsf{E}_{6,1}$ and $\mathsf{E}_{6,6}$,
- from Proposition 9.5.2.4 for types $\mathsf{E}_{6,3}$ and $\mathsf{E}_{6,5}$,
- from Proposition 9.6.1.1 for type $\mathsf{E}_{7,7}$,
- from Proposition 9.6.2.2 for type $\mathsf{E}_{7,6}$,
- from Proposition 9.7.1.1, Proposition 8.3.0.1 and Theorem 8.3.0.3 for types $\mathsf{E}_{6,2}, \mathsf{E}_{7,1}, \mathsf{E}_{8,8}, \mathsf{F}_{4,1}$ and $\mathsf{F}_{4,4}$,
- from Proposition 9.8.2.5, Theorem 9.8.3.9 and Proposition 8.3.0.7 for types $\mathsf{E}_{6,4}, \mathsf{E}_{7,2}, \mathsf{E}_{8,7}, \mathsf{F}_{4,2}$ and $\mathsf{F}_{4,3}$,
- from Corollary 9.9.0.3 for types $\mathsf{G}_{2,1}$ and $\mathsf{G}_{2,2}$.

Next, we state the connection to geometric lines.

Theorem E. *Let* $\mathsf{X}_n \in \{\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8, \mathsf{F}_4, \mathsf{G}_2\}$ *and* $1 \leq i \leq n$*, with* $i \notin \{2, 4, 5\}$ *if* $n = 7$ *and* $i \in \{7, 8\}$ *if* $n = 8$*. Then a geometric line of the Lie incidence geometry of type* $\mathsf{X}_{n,i}$ *associated to an irreducible thick Moufang spherical building* Δ *of type* X_n*,* $n \geq 2$*, is one of the following.*

(1) *A line of the Lie incidence geometry.*
(2) *A hyperbolic line in a symp of the Lie incidence geometry whenever this symp is a symplectic polar space.*
(3) *A hyperbolic (or ideal) line of a split Cayley hexagon.*
(4) *An imaginary line in the Lie incidence geometry of type* $\mathsf{G}_{2,2}$*, in the split case over a perfect field of characteristic* 2*.*

Theorem E follows from [**KV13**, Corollary 5.6] for the types $\mathsf{E}_{6,1}, \mathsf{E}_{6,6}$ and $\mathsf{E}_{7,7}$. It will follow

- from Corollary 8.3.0.5 for types $\mathsf{E}_{6,3}$ and $\mathsf{E}_{6,5}$,
- from Corollary 8.3.0.5 for type $\mathsf{E}_{7,6}$,
- from Proposition 8.3.0.1 and Theorem 8.3.0.3 for types $\mathsf{E}_{6,2}, \mathsf{E}_{7,1}, \mathsf{E}_{8,8}, \mathsf{F}_{4,1}$ and $\mathsf{F}_{4,4}$,
- from Proposition 8.3.0.7 for types $\mathsf{E}_{6,4}, \mathsf{E}_{7,2}, \mathsf{E}_{8,7}, \mathsf{F}_{4,2}$ and $\mathsf{F}_{4,3}$,
- from Proposition 8.4.0.5 for types $\mathsf{G}_{2,1}$ and $\mathsf{G}_{2,2}$.

9.2. Projective Spaces—Type A_n

The following lemma proves Theorem B for buildings of type A_n, that is, for projective spaces and their Grassmannians.

Lemma 9.2.0.1. *If no $(n-k-1)$-space is disjoint from each member of a set T of $q+1$ projective k-spaces of $\mathsf{PG}(n,q)$, then there exist a $(k-1)$-space U and a $(k+1)$-space W such that T coincides with the set of k-spaces containing U and contained in W.*

Proof. We first do the cases $n \leq 3$ and then proceed by induction on n. If $k = 0$, then the assertion follows directly from the main result of [**BB66**]. Dually, the case $k = n-1$ follows. This takes care of the case $n = 2$, because in this case $k \in \{0, n-1\}$. Now suppose $n = 3$ and $k = 1$. Suppose at least two members of T intersect in a point, say, $L_1 \cap L_2 = \{p\}$, $L_1, L_2 \in T$. Since we may suppose that T is not a planar line pencil, there is some point $x \in \langle L_1, L_2\rangle$ not contained in any member of T. In the residue of x, the lines obtained from T form a set of at most q members (since L_1 and L_2 define the same line), and hence, by [**BB66**] again, we find a line K through x disjoint from all members of T. Now suppose every pair of members of T is disjoint. Pick a point x not on any member of T (that is possible since there are q^3+q^2+q+1 points and only $(q+1)^2$ on members of T). Pick $L_1, L_2 \in T$ arbitrarily. Then there exists a unique line K through x intersecting both L_1 and L_2 non-trivially. Since $K \setminus \{x\}$ contains q points, there exists a member $L_3 \in T$ not meeting K. Therefore, in the residue of x, not all lines corresponding to the members of T go through a common point. With that, again, there exists a line through x not intersecting any member of T. This shows the case $n = 3$.

We proceed by induction. By duality, we may assume $2k+1 \leq n$, and we consider two different cases.

(1) *Suppose each pair of members of T intersects in a $(k-1)$-space.* Then one can see that there are again two cases.
   (*i*) *The members of T contain a common $(k-1)$-space U.* Then we may assume they are not all contained in a common $(k+1)$-space. Intersecting the situation with a hyperplane H not containing U, we obtain $q+1$ $(k-1)$-spaces in $\mathsf{PG}(n-1,q)$ all going through the same $(k-2)$-space, but not contained in a common k-space. Applying induction, we obtain an $((n-1)-(k-1)-1)$-space $Z \subseteq H$ disjoint from each member of T.
   (*ii*) *The members of T are contained in a common $(k+1)$-space W.* Here, we may assume that they do not contain a common $(k-1)$-space. Since $k+1 \leq n-1$, we can apply induction and find a point $p \in K$ not contained in any member of T. Let C be an $(n-2-k)$-space complementary to W. Then $\langle p, W\rangle$ is an $(n-1-k)$-space disjoint from all members of T.

(2) *Some pair $\{A_1, A_2\}$ of members of T intersect in at most a $(k-2)$-space.* If either $2k+1 \leq n-1$, or not all pairs in T are disjoint, then we can find a point x outside the span $\langle A_1, A_2\rangle$ of two members $A_1, A_2 \in T$, with $k+2 \leq \dim\langle A_1, A_2\rangle \leq n-1$, and not lying in any member of T (use a simple count). It follows that we can

apply induction in the residue of x and obtain an $(n-k-1)$-space through x disjoint from each member of T.

So we may assume that $2k+1 = n$ and all members of T are pairwise disjoint. Then the proof is similar to the last arguments of the case $(k,n) = (1,3)$ above. □

9.3. Oriflamme geometries—Type D_n

The main purpose of this section is to prove Theorem B for the extremal type n for buildings of type D_n. We first prove Theorem B for points of the corresponding oriflamme geometries (cf. Section 3.3.1), and provide a general proof for points of polar spaces of any type below (see Lemma 9.4.0.1). We prove this separately for oriflamme geometries nonetheless, because this proof does not use the notion of geometric lines and it is simpler in nature. It holds for all polar spaces associated to quadrics, but we will not need this.

Lemma 9.3.0.1. *If every line of a hyperbolic quadric Q of rank at least 3 contains exactly $s+1$ points, then there exists a point non-collinear to each point of an arbitrary set T of $s+1$ (distinct) points, except if these points are contained in a single line.*

Proof. Let $T = \{p_0, \ldots, p_s\}$ be a set of $s+1$ distinct points of Q, and suppose first that p_0 is collinear to p_1. Since not all points $p_0, p_1, \ldots, p_s$ are contained in one line, we find a point $b \in p_0p_1$ not contained in T. Suppose now that p_0 and p_1 are not collinear. Then, since $\{p_0,p_1\}^{\perp\perp} = \{p_0,p_1\}$, we find a point $b \in \{p_0,p_1\}^{\perp}$ not collinear to p_2.

In any case, the point b has the property that the number of lines joining b to a collinear point in T is at most s. Applying Proposition 7.1.0.1, we find a line L through b such that no point of T that is collinear to b is collinear to all points of T. Consequently, every point p_i of T is collinear to a unique point p_i' of L. Since $p_0' = p_1'$ and $|L| = s+1$, there is at least one point $q \in L$ not collinear to any member of T. □

Lemma 9.3.0.2. *If every line of a hyperbolic quadric Q with Witt index $d \geq 3$ contains exactly $s+1$ points, then there exists a maximal singular subspace opposite each member of an arbitrary set T of $s+1$ (distinct) maximal singular subspaces of common type, except if these maximal singular subspaces contain a common singular subspace of codimension 2 in each.*

Proof. For $d = 3$, this is Lemma 9.2.0.1, Case $\mathsf{PG}(3,s)$. For $d = 4$, triality yields the result using Lemma 9.3.0.1. So suppose $d \geq 5$. We argue by induction on d and consider various possibilities.

(*i*) *All members of T contain a common point p.* In this case, we apply induction in the residue $\mathsf{Res}_\Delta(p)$ at p and obtain a maximal singular subspace M through p intersecting every member of T in just p. Let p' be a point of Δ opposite p and let M' be the unique maximal singular subspace of Δ containing p' and

adjacent to M, that is, intersecting M in a subspace of dimension $n-2$. Then M' is disjoint from each member of T.

(*ii*) *At least two members of T intersect, but they do not all share a common point.* Set $T = \{V_0, V_1, \ldots, V_s\}$. Let p be a point of Δ contained in at least two members of T, say V_0, V_1, but not in all of them, say $p \notin V_2$. Let T' be the set of maximal singular subspaces through p obtained by taking those of T containing p, and taking for each member V_i not containing p an arbitrary maximal singular subspace V_i' containing p and intersecting V_i in a subspace of codimension 2. If all members of T' intersected in a common subspace of dimension $d-3$, then by replacing V_2' with another maximal singular subspace through p intersecting V_2 in a subspace of dimension $d-3$ (distinct from $V_2 \cap V_2'$), we obtain a new set T' not having that property. Hence the induction hypothesis applied to $\mathsf{Res}_\Delta(p)$ yields a maximal singular subspace W through p intersecting each member of T' exactly in p, and hence, since T_i and T_i' have a subspace of dimension $d-3$ in common and W intersects each member of T in a subspace of even dimension (which cannot contain a plane), W intersects each member V_i of T in exactly one point p_i. Since $p_0 = p_1$, the number of such points is at most s, and so we find a hyperplane H of W disjoint from all $V_i \in T$. The unique maximal singular subspace U distinct from W and containing H has the opposite type of the V_i and hence is disjoint from all of them (since it cannot intersect any of them in at least a line, as this would imply that H intersects a member of T non-trivially).

(*iii*) *Each pair of members of T is opposite.* A simple count yields a point b not contained in any member of T. Let T' be the set of maximal singular subspaces through b intersecting a given member of T in a hyperplane. Suppose two members V_0', V_1' of T' intersect in a subspace U of dimension $d-3$. Then the corresponding members V_0, V_1, respectively, of T intersect U in a subspace of dimension $d-4$, and those have mutually a subspace in common of dimension at least $d-5 \geq 0$, a contradiction to the disjointness of V_0 and V_1. Hence induction yields a maximal singular subspace M through b intersecting each member of T' in just $\{b\}$. Since all points collinear to b of the union of the members of T are contained in the members of T', and all points of M are collinear to b, we conclude that M is disjoint from each member of T. □

9.4. Polar Spaces—Types B_n, C_n and D_n

We first prove Theorem B for type 1 vertices, that is, for points of the corresponding polar space. We distinguish between the cases of rank at least 3 and rank 2, since in rank 2 there are additional examples that do not occur in higher rank.

Lemma 9.4.0.1. *If every line of a polar space Δ of rank at least 3 contains exactly $s+1$ points, then there exists a point non-collinear to each point of an arbitrary set T of $s+1$ (distinct) points, except if these points are contained in a single line, or if they form a hyperbolic line in case Δ is a symplectic polar space.*

PROOF. Suppose no point of Δ is non-collinear to each point of T, that is, each point of Δ is collinear to some member of T. We prove that T is a geometric line in the sense of [**KV13**], and then the result follows from Lemmata 4.7 and 4.8 of [**KV13**].

So we have to prove that every point of Δ is collinear to exactly one or all points of T. Since we assume that each point is collinear to at least one point of T, it suffices to show that if some point is collinear to at least two points of T, then it is collinear to all points of T. Suppose, for a contradiction, that some point p is collinear to at least two points x, y of T, and opposite the point $z \in T$. We claim that we may assume that $p \notin T$. Indeed, suppose it is. Then we may as well assume $p = y$. Let L be the line through x and y. Then there are at most s points of T contained in L, hence there is a point $q \in L \setminus T$. The number of lines through q containing a line that also contains a point of T is at most s, hence Exercise 2.11(*iii*) of [**Van24**] in $\mathsf{Res}_\Delta(q)$ yields a line $K \ni q$ not collinear to any point of T. Since x and y project to the same point on K, there is some point on K opposite each point of T, a contradiction. The claim is proved. But now exactly the same argument replacing q with p (and using the fact that z is not collinear to p to obtain the assertion that there are at most s lines through p containing a point of T) again leads to a contradiction. Hence the lemma is proved. □

Lemma 9.4.0.2. *If every line of a polar space Δ of rank 2 contains exactly $s+1$ points, and every point is contained in exactly $t+1$ lines, $t > 1$, then there exists a point non-collinear to each point of an arbitrary set T of $s+1$ (distinct) points, except if either these points are contained in a single line, or they form an ovoid in a subquadrangle of order $(s/t, t)$. In particular, if $s = t$, they form either a line or a large hyperbolic line, that is, $T = T^{\perp\perp} = \{y, z\}^\perp$, for every pair of distinct points y, z of $T^\perp$.*

PROOF. Suppose that no point is opposite all points of T. Suppose first that T contains (at least) two collinear points, say x_1, x_2. Let the line L through x_1 and x_2 contain ℓ points of T. If $\ell = s+1$, then we are done, so assume $2 \leq \ell \leq s$. Let p be a point on $L \setminus T$. If there were some line K through p containing no point of T, then, since x_1 and x_2 project onto the same point p of K, there exists a point on K opposite each member of T, a contradiction. Hence

$$\ell + t(s + 1 - \ell) \leq |T| = s + 1,$$

implying $t \leq 1$, a contradiction.

Now suppose every pair of points in T is opposite. Pick $x_1, x_2 \in T$ arbitrarily. Let $z \in \{x_1, x_2\}^\perp$ be arbitrary and note $z \notin T$. Then, as in the previous paragraph, every line through z contains at least (and hence exactly, by our assumption that no pair of points of T is collinear) one point of T. Hence $s \geq t$. Set

$$X = T \cup \{x_1^\perp \cap x_2^\perp \mid x_1, x_2 \in T\}$$

and let $\mathscr{L}$ be the set of intersections of X with a line intersecting X in at least two points. A moment's thought reveals that X, together with the line set $\mathscr{L}$, is an ideal subquadrangle Γ, say with order (s', t), and T is an ovoid in Γ. This implies that $s't = s$, so $s' = s/t$.

If, in the last case, $s = t$, then Γ has order $(1, t)$, implying that

$$X = (x_1^\perp \cap x_2^\perp) \cup \{x_1, x_2\}^{\perp\perp},$$

with x_1, x_2 two arbitrary members of T. Any ovoid of X containing x_1, x_2 coincides with $\{x_1, x_2\}^{\perp\perp}$. □

We now show that the ovoid case in the previous lemma really leads to examples of minimal blocking sets.

Lemma 9.4.0.3. *Suppose Δ is a generalized quadrangle of order (s, t) with $s, t \geq 2$. Suppose Γ is a subquadrangle of order $(s/t, t)$ and suppose also that Γ has an ovoid O. Then $|O| = 1+s$ and no point of Δ is opposite all members of O, and each point of Δ not in Γ is collinear to a unique member of O.*

Proof. By 1.8.1 of [**PT84**], we have $|O| = 1 + (s/t)t = 1 + s$. Each point of Γ is collinear to $1 + t$ members of O. Let x be a point of Δ not in Γ. By 2.2.1 of [**PT84**], x is collinear to $1 + s/t$, hence at least one, points of Γ. Since Γ is ideal in Δ, we readily deduce that x is contained in a unique line of Δ that contains a line L of Γ, and the points of Γ collinear to x are precisely the points of L. Since O is an ovoid of Γ, we have $|O \cap x^\perp| = |O \cap L| = 1$, and the lemma is proved. □

Remark 9.4.0.4. Suppose in Lemma 9.4.0.2, Δ is Moufang, and T is not a line. Then Δ has order (q, q), (q^2, q), or (q^3, q^2). In the latter case, T is an ovoid in a subquadrangle of order (q, q^2), which contradicts 1.8.3 of [**PT84**]. In the case the order is (q, q), we note that the only Moufang quadrangles of order (q, q) with large hyperbolic lines are the symplectic quadrangles, as follows from Theorem 1.4 of [**vCV96**]. Finally, let Δ be a Moufang quadrangle of order (q^2, q). Then any subquadrangle of order (q, q) is a symplectic polar space (see [**PT84**, §3.5]) and, by 3.4.1(i) of [**PT84**], admits ovoids if and only if q is even.

For proving Theorem B for maximal singular subspaces, the parabolic quadrics (which correspond to the split buildings of type B_n) play an exceptional role. That is why we treat them first.

Lemma 9.4.0.5. *If in a parabolic quadric Q of Witt index n, $n \geq 2$, every line contains precisely $s + 1$ points, then every set T of $s + 1$ maximal singular subspaces admits an opposite maximal singular subspace, except if they form a line or a hyperbolic line in the corresponding dual polar space $\Gamma(Q)$.*

Proof. Embed Q naturally in a hyperbolic quadric Q' of Witt index $n+1$. Each member M of T is contained in a unique maximal singular subspace M' of Q' of a given type. The set of all such subspaces M' admits an opposite maximal singular subspace W', if they do not form a line in the corresponding half-spin geometry, that is, by [**KV13**], if T is not a line or a hyperbolic line in $\Gamma(Q)$. Then $W = W' \cap Q$ is a maximal singular subspace of Q disjoint from every member of T. □

Now we can treat the other cases. It is convenient to first collect the rank-2 case from what we already showed above.

Lemma 9.4.0.6. *If every line of a polar space Δ of rank 2 contains exactly $s+1$ points, and every point is contained in exactly $t+1$ lines, $t>1$, then there exists a line disjoint from each member of an arbitrary set T of $t+1$ (distinct) lines, except if these lines either contain a common point, or form a spread in a subquadrangle of order $(s,t/s)$.*

PROOF. This is just the dual of Lemma 9.4.0.2. □

We can now show the general case.

Lemma 9.4.0.7. *Let Δ be a finite polar space of rank $n\geq 3$ and of order (s,t), with both s and t at least 2. Suppose Δ does not correspond to a parabolic quadric. Then every set $T=\{V_0,V_1,\ldots,V_t\}$ of $t+1$ maximal singular subspaces admits an opposite maximal singular subspace, except if they contain a common singular subspace U either of codimension 1 (that is, they form a line in the corresponding dual polar space $\Gamma(Q)$), or of codimension 2, $t=s^2$, and they form a spread in a symplectic subquadrangle of order (s,s), s even, of $\mathsf{Res}_\Delta(U)$ (and then Δ corresponds to an elliptic quadric).*

PROOF. We prove this by induction on the rank n of Δ, where Lemma 9.4.0.6 serves as our base, noting that a parabolic quadric is characterized by $s=t$ and admitting a full embedded grid. In particular, if $s=t$, then the case in Lemma 9.4.0.6 of a spread in a subquadrangle does occur.

Counting yields a point x not in any of the subspaces V_i, $i=0,1,\ldots,t$. In $\mathsf{Res}_\Delta(x)$ we can now consider the maximal singular subspaces $V_i':=\langle x,x^\perp\cap V_i\rangle$, $i=0,1,\ldots,t$. A maximal singular subspace through x locally opposite all of those at x is opposite every member of T. Hence, by induction, there are two possibilities.

(1) *The intersection of all V_i' is a subspace W of dimension $n-2$.*
Set $U_i:=V_i\cap W$. If $U_0=U_1=\cdots=U_t$, then we can apply induction in $\mathsf{Res}_\Delta(U_0)$. So, without loss of generality, we may assume for a contradiction that $U_1\neq U_2$. Then we select an arbitrary point $y\in V_0'\setminus W$. The subspaces $\langle y,y^\perp\cap V_0\rangle=V_0'$ and $\langle y,y^\perp\cap V_1\rangle\supseteq\langle y,U_1\rangle$ intersect in the $(n-2)$-dimensional singular subspace $\langle y,U_1\rangle$ and hence induction implies that $\langle y,U_2\rangle=\langle y,U_1\rangle$. Consequently, $U_1=\langle y,U_1\rangle\cap W=\langle y,U_2\rangle\cap W=U_2$, a contradiction.

(2) *The intersection of all V_i' is a subspace W of dimension $n-3$, and W is also the pairwise intersection of the V_i.*
Here, the same arguments as in the previous paragraph (1) apply, except when $n=3$, since in this case $W=x$. Thus, the case $n=3$ must be treated separately and explicitly. In this case, Δ arises from an elliptic quadric, $t=s^2$, s is even, all planes V_i' are locally opposite each other and form a spread in a subquadrangle of order (s,s) of $\mathsf{Res}_\Delta(x)$. Suppose for a contradiction that the members of T are not pairwise opposite and let V_0 not be opposite V_1. Then counting yields a point $x'\perp V_0\cap V_1$ not contained in any member of T. Switching the roles of x and x' brings us back to Case (1). Hence all members of T are pairwise opposite, that is, disjoint. Let L_0 be any line of V_0 and let $\{p_1\}=L_0^\perp\cap V_1$. If some point $x''\in\langle p_1,L_0\rangle$ does not belong to any member of T, then we can switch the roles of x and x'' and this brings us back to Case (1). Hence every point of $\langle p_1,L_0\rangle$ belongs to some member of T. Varying L_0 in V_0, we obtain $(s^2-1)(s^2+s+1)$

points, which then must cover all points of $V_2 \cup V_3 \cup \cdots \cup V_t$. If we view Δ as an elliptic quadric in $\mathsf{PG}(7,s)$, then T is contained in the 5-dimensional subspace Y of $\mathsf{PG}(7,s)$ generated by the planes V_0 and V_1. Since Y intersects Δ in a hyperbolic polar space of rank 3, and such a polar space does not contain three pairwise opposite planes, this leads to a contradiction.

The proof is complete. □

We leave it to the reader to check that the second case of the previous lemma does occur, that is, if Δ corresponds to an elliptic quadric, a set T of $t+1$ generators containing a common singular subspace U of dimension $n-3$ forms a spread in a symplectic subquadrangle of order (s,s), s even, of $\mathsf{Res}_\Delta(U)$, then no generator of Δ is opposite each member of T. Likewise, the second case of the next lemma also really occurs.

Lemma 9.4.0.8. *Let Δ be a finite polar space of rank $n \geq 3$ and order (s,t), and suppose that Δ is not symplectic. Let $T = \{\alpha_0, \ldots, \alpha_s\}$ be a set of $s+1$ distinct singular subspaces of common dimension $\ell \leq n-2$ in Δ, with $\ell \neq n-2$ if $t = 1$ and $\ell \geq 1$, such that they do not form a pencil in any singular subspace of dimension $\ell+1$. Assume furthermore that, if Δ is a small Hermitian polar space and $\ell = n-2$, then T is not an ovoid in a symplectic subquadrangle of order (t,t) in the residue of an $(\ell-1)$-dimensional subspace of Δ contained in each element of T. Then there exists a singular subspace of Δ that is opposite to all of $\alpha_0, \ldots, \alpha_s$.*

Proof. Let, for given order (s,t) and rank n, the polar space Δ and $\alpha_0, \ldots, \alpha_s$ be a smallest (with respect to $n-\ell$) counterexample to the lemma.

We claim that there exists a singular subspace β of dimension $\ell+1$ such that $\mathsf{proj}_\beta(\alpha_i)$ is a point p_i, for all $i \in \{0,1,\ldots,s\}$. Indeed, first assume $\ell \leq n-3$. We select arbitrarily singular subspaces β_i of dimension $\ell+1$ containing α_i, for all i. We can easily choose them in such a way that they do not have a subspace of dimension $\ell-1$ in common. The induction hypothesis implies that we find a singular subspace β of dimension ℓ opposite each β_i, $i = 0,1,\ldots,s$. Then β satisfies the claim. Now suppose $\ell = n-2$, and note that in this case we assume $t > 1$ (as submaximal singular subspaces do not belong to the oriflamme geometry). We proceed by induction on the rank n, including the case of rank 2. If $n = 2$, then the number of lines not disjoint from T is at most $(s+1)(t+1)$. Since there are $(1+t)(1+st)$ lines in total, and $t > 1$, the claim follows. Now let $n \geq 3$. We can find a point x not contained in any member of T. Applying induction in $\mathsf{Res}(x)$ proves the claim.

Thus, for each α_i there is a unique point p_i in β collinear to all points of α_i. If $\{p_0, p_1, \ldots, p_s\}$ is not a line, then by Lemma 9.2.0.1 there exists a subspace $\alpha \subseteq \beta$ of dimension ℓ that contains none of the points $p_0, p_1, \ldots, p_s$. In this case, α is opposite to every member of T, and we are done.

Hence we may assume that $\{p_0, p_1, \ldots, p_s\}$ is a line L. The set of points of β_i collinear to all points of L forms a subhyperplane H_i contained in α_i, as a hyperplane of the latter.

Step I. Suppose $\ell \leq n-3$. We claim that all points of $\alpha_i \setminus H_i$ are collinear to all points of $\alpha_j \setminus H_j$, for distinct $i,j \in \{0,1,\ldots,s\}$. Indeed, we may assume

$(i,j) = (0,1)$. Let $x_0 \in \alpha_0 \setminus H_0$ and $x_1 \in \alpha_1 \setminus H_1$. Then $x_0^\perp \cap \beta =: K_0$ and $x_1^\perp \cap \beta =: K_1$ are two hyperplanes of β none of which contain L, but containing p_0 and p_1, respectively. It follows that $N := K_0 \cap K_1$ is a subspace of dimension $\ell - 1 \geq 0$ disjoint from L. The line $N^\perp \cap \beta_i$ intersects α_i in a unique point x_i, as otherwise N belongs to the perp of a point of H_i, contradicting the facts that also L belongs to the perp, that L and N are complementary, and β and β_i are opposite (and note that the notation x_i is in conformity with the definitions of x_0 and x_1 above). We may hence view $x_0, x_1, \ldots, x_s$ as points of $\mathsf{Res}_\Delta(N)$. If they do not constitute a line in $\mathsf{Res}_\Delta(N)$, then by Lemma 9.4.0.1 we can find a point opposite all of them, meaning using Lemma 3.6.2.21, we can find a subspace α of dimension ℓ containing N opposite each of α_i, $i \in \{0, 1, \ldots, s\}$. Hence all of $x_0, x_1, \ldots, x_s$ are collinear and the claim is proved.

The previous claim now easily implies that all members of T are contained in a common singular subspace, say M, which we may assume to be maximal. Since by assumption, they do not form a Grassmann line in U, Lemma 9.2.0.1 yields a subspace $U \subseteq M$ of dimension $n - 2 - \ell$ disjoint from all α_i. Let M' be a maximal singular subspace of Δ opposite M. Then $U^\perp \cap M$ has dimension ℓ and is opposite each member of T (use Lemma 3.6.2.21 again).

Step II. Now suppose $\ell = 1$ and $n = 3$. Note that, if all elements of S intersect in a common point p, our assumptions yield a line M locally opposite every element of S in $\mathsf{Res}(p)$. Then on M we can choose a point $q \neq p$. Let N be a line locally opposite M in $\mathsf{Res}(q)$. Then the line N is opposite each element of S.

Recall that the points $p_0, \ldots, p_s$ form a line L in the plane β. Let b be an arbitrary point in $\beta \setminus L$. The projection of b onto α_i is a point b_i. We can find a line N locally opposite all lines bb_i at b, except if these lines form a line or an ovoid in a subquadrangle of order $(s/t, t)$ of $\mathsf{Res}(b)$. Then, in Δ, the line N is opposite all L_i.

Hence, for all $b \in \beta \setminus L$, we may suppose that the lines bb_i form a line or an ovoid in a subquadrangle of order $(s/t, t)$ of $\mathsf{Res}(b)$. In the former case, we say that b has type (L), in the latter case we say that b has type (O). We consider four possibilities.

(*i*) If every point of $\beta \setminus L$ has type (L), then, as in Step I, all points of $\alpha_i \setminus H_i$ are collinear to all points of $\alpha_j \setminus H_j$, for all $i, j \in \{0, 1, \ldots, s\}$. This implies that all members of T are contained in a unique plane γ. Each member of T contains $L^\perp \cap \gamma$, leading to T being a line pencil in γ, a contradiction.

(*ii*) If each point of $\beta \setminus L$ has type (O), then no point of $L_i \setminus H_i$ is collinear to any point of $L_j \setminus H_j$, $i \neq j$, and so each point H_i is collinear to each line L_j. In particular, all H_i are collinear. Since they are collinear to L, they lie in the same plane δ with L.

Since L_i is collinear to p_i, H_i and H_j, and not to p_j, $i \neq j$, we deduce that $H_j \in \langle p_i, H_i \rangle$. Switching the roles of i and j, we find $H_i = \langle p_i, H_i \rangle \cap \langle p_j, H_j \rangle = H_j$. Thus, all members of T share a common point, and the first paragraph of Step II concludes this case.

(*iii*) Next, suppose that every point in $\beta \setminus L$, except one (say b), has type (O). Note that by Remark 9.4.0.4 $s = t^2 \geq 4$. We consider L_0 and L_1 and set $b_i = b^\perp \cap L_i$, $i = 0, 1$. Then $b_0 \perp b_1$. As in (*ii*), all other points of $L_0 \setminus H_0$ have to be collinear to H_1, implying, since $s \geq 4$, that $H_1 \perp L_0$. In particular, $H_1 \perp b_0$, and so $b_0 \perp L_1$. Likewise, $H_0 \perp L_1$ and $b_1 \perp L_0$. Thus, $L_0 \perp L_1$,

contradicting the existence of points of type (O). Therefore, this case cannot occur.

(*iv*) Finally, we consider the case where at least one point $d \in \beta \setminus L$ has type (O) and at least two points $b, c \in \beta \setminus L$ have type (L). Again, $s = t^2 \geq 4$. Set

$$b_i = b^\perp \cap L_i, \qquad c_i = c^\perp \cap L_i, \qquad d_i = d^\perp \cap L_i, \qquad i = 0, 1, \ldots, s.$$

Suppose, without loss of generality, that $bc \cap L = p_0$. Then $b_0 = c_0$ and $b_0 \perp L_i$ for $i = 1, 2, \ldots, s$. Consequently, every point of $bc \setminus \{p_0\}$ has type (L). The same argument shows that every point of $\beta \setminus (bc \cup L)$ has type (O).

It follows that each point of $L_0 \setminus \{b_0\}$ is collinear only with the point H_i of L_i. Hence $H_i \perp L_0$, and therefore $H_0 \perp H_i$ for all $i = 1, 2, \ldots, s$. Thus all points H_i lie in a common plane δ containing L. Since $b_i \perp b_j$ but b_i is not collinear with c_j, we have $H_i \neq H_j$ for all distinct $i, j \in \{1, 2, \ldots, s\}$. Moreover, as $L_0 \perp p_0, H_i$ for all i, all H_i lie on a single line H of δ. Since $H \setminus \{p_0\}$ contains s points, we may assume without loss of generality that $H_0 = H_1$. However, this implies that $L_0 \perp L_1$, a contradiction.

Hence this case does not occur, and the proof of Step II is complete.

Step III. Finally, suppose $\ell = n - 2$ and $n \geq 4$. Let U be an $(n-4)$-dimensional subspace of β disjoint from L, and let $W \supseteq U$ be an $(n-3)$-dimensional subspace of β, also disjoint from L. If $A := \{\langle U, U^\perp \cap \alpha_i \rangle \mid i = 0, 1, \ldots, s\}$, considered as a set of lines of $\mathsf{Res}_\Delta(U)$, is neither a planar line pencil nor a set of lines through some point x forming an ovoid in a symplectic ideal subquadrangle of the point residual at x, then we can select an $(n-2)$-dimensional subspace α^* through U locally opposite each member of A. Then α^* is opposite each member of T (using Lemma 3.6.2.21 again).

Hence we may assume that A is either a planar line pencil — we say A is of *type (PLP)* — or a set of lines through some point x forming an ovoid in a symplectic ideal subquadrangle of the point residual at x — and we say A is of *type (OSS)*.

In the former case, the set $B := \{\langle W, W^\perp \cap \alpha_i \rangle \mid i = 0, 1, \ldots, s\}$ is a line in the generalized quadrangle $\mathsf{Res}_\Delta(W)$ — and we say that W is of *type (L)*. in the latter case, B is an ovoid in a symplectic ideal subquadrangle of $\mathsf{Res}_\Delta(W)$ — and we say W is of *type (O)*.

It follows that, in the former case, all subspaces of W of dimension $n-4$ are of type (PLP), and in the latter case, all such subspaces are of type (OSS). Now varying W, we easily deduce that all $(n-2)$-dimensional subspaces of W disjoint from L are either of type (L) or of type (O). Now we continue just like in Step II and arrive at contradictions.

This concludes the proof of the lemma. □

It remains to consider symplectic polar spaces. It turns out that type-2 elements, that is, lines, cannot be included in the general proof. We will treat them separately. However, the final proof is inductive, and the result for lines in rank 3 is needed to prove the general case, which is then used to prove the result for lines in higher rank. This explains the rather peculiar conditions in the next lemma, which shall become clear in the proof of Proposition 9.4.0.11 below.

Lemma 9.4.0.9. *Let Δ be a symplectic polar space of rank at least 3 and order (s, s). Suppose that every set of $s + 1$ singular planes admits a common opposite plane, except if they all intersect in a line and are contained in a common 3-space of the underlying projective space. Then a set T of $s + 1$ lines of Δ admits a common opposite in Δ if and only if T is not a line pencil in some plane of the underlying projective space.*

PROOF. Let T be a set of $s + 1$ lines of the symplectic polar space Δ of rank r naturally embedded in $\mathsf{PG}(2r - 1, s)$, $r \geq 3$. Suppose T is not a line pencil in some plane of $\mathsf{PG}(2r - 1, s)$. We show that there exists a line of Δ opposite all members of T. As usual, we set $T = \{L_0, L_1, \ldots, L_s\}$.

We include L_i in a plane α_i in such a way that the α_i do not contain a common line (which can be easily accomplished). Let β be a plane opposite all α_i, $i \in \{0, 1, \ldots, s\}$. Set $m_i := L_i^\perp \cap \beta$. If $\{m_i \mid i \in \{0, 1, \ldots, s\}\}$ is not a line, then we can find a line L in β not containing any of the m_i and hence opposite all of the L_i. So the m_i constitute a line M. Let $b \in \beta \setminus M$ be arbitrary. Then $b_i := b^\perp \cap L_i$ is a unique point. Suppose the lines bb_i do not form a line pencil in a plane of $\mathsf{PG}(2r - 1, s)$. Then we can find a line L through b locally opposite all of the bb_i. Then L is opposite all of the L_i by Lemma 3.6.2.21. Hence we may assume that the lines bb_i form a line pencil in a plane π_b of $\mathsf{PG}(2r-1, s)$. Suppose now that for two choices of $b \in \beta \setminus M$, the points b_i are contained in a common line K_b of $\mathsf{PG}(2r - 1, s)$. Let b, c be those two points and adapt the same notation for c as we introduced for b. Without loss of generality, we may assume that the line bc contains the point m_0. Then the lines L_i, $i \in \{1, 2, \ldots, s\}$ are contained in the plane γ of $\mathsf{PG}(2r-1, s)$ spanned by K_b and K_c. The point $b_0 = c_0$ is also contained in γ.

Suppose first that also L_0 is contained in γ. Then, since we assumed that T is not a line pencil, one can see that γ is a singular plane, and it contains a point x not on any of the L_i. Then the line $x^\perp \cap \gamma'$, with γ' a plane opposite γ in Δ, is opposite each member of T.

Suppose now that L_0 is not contained in γ. Let $z := L_1 \cap L_2$. Then $z \perp b_0$ as $z \perp \{b_1, b_2\} \subseteq K_b \ni b_0$. We can select a singular plane β_z containing zb_0 such that L_0 is not collinear to β_z, and β_z is not in a common singular 3-space with γ if singular.

In $\beta_z \setminus zb_0$ we can find a point y not collinear to b_1. It follows that y is not collinear to any of the b_i, $i \in \{1, 2, \ldots, s\}$. The lines joining y with the unique projection point of y on the L_i do not form a line pencil in any plane, as two of these lines coincide (namely, the line yz joins y with the point z of both L_1 and L_2). Hence we can find a line L through y locally opposite all these lines. Again, by Lemma 3.6.2.21, L is opposite each member of T.

Hence we may assume that for at most one point $b \in \beta \setminus M$, the points $b_0, b_1, \ldots, b_s$ are on one line (and we denote that point, if it exists, from now on with b^*). Note that we may also assume that L_0 and L_1 do not intersect. Indeed, if all members of T pairwise intersect, then either they are contained in a plane, and we treated that case above, or they all contain a common point p. In the latter case, the result follows from considering the residue at p (indeed, we can then select a line K locally

opposite each member of T; then select a point $q \in K \setminus \{p\}$ and a line L locally opposite K at q. The line L is opposite each member of T by Lemma 3.6.2.21).

First let $s > 2$. Choose points $b, c \in \beta \setminus M$ with $m_2 \in bc$ and $b^* \notin \{b, c\}$. As before, the lines L_0 and L_1 are contained in $\langle \pi_b, \pi_c \rangle$, which is a 4-space U_2 of $\mathsf{PG}(2r-1, s)$. If we choose $d \in \beta \setminus M$ with $m_3 \in bd$ and $d \neq b^*$, then we obtain likewise that L_0 and L_1 are contained in a 4-space U_3 of $\mathsf{PG}(2r-1, s)$ spanned by π_b and π_d. Suppose first these two 4-spaces coincide. Then $U_2 = U_3$ contains β. The polar space induced in U_2 is degenerate, but, as we have opposite lines (L_0 and some line in β), the radical is a point, which must coincide with the intersection of any pair of singular planes, contradicting $m_0 \neq m_1$. Hence $U_2 \neq U_3$ and these intersect in a 3-space containing L_0, L_1 and b. However, L_0 and L_1 already span $U_2 \cap U_3$, and by the arbitrariness of b, the 3-space $\langle L_0, L_1 \rangle$ contains β, which is ridiculous as β is disjoint from $L_0 \cup L_1$.

Now let $s = 2$. Similar arguments as in the previous paragraph show that

$$W := \langle L_0, L_1, L_2 \rangle$$

contains β. Then again $\dim W = 3$ leads to the contradiction that $L_0 \cap \beta$ is non-empty, and if $\dim W = 4$, then we have a degenerate symplectic polar space induced in W, leading to the same contradiction as before. Hence $\dim W = 5$. Also as before, W is a non-degenerate symplectic polar space. We coordinatize W as follows (using obvious shorthand notation). The two points on L_0 not collinear to all points of M are labelled 100000 and 010000. Likewise those on L_1 and L_2 by 001000, 000100 and 000010, 000001, respectively. We may assume that the points 100000, 001000 and 000010 are together in a plane π_b, $b \in \beta$, and then b has labels 101010. Similarly we have the points 100101, 011001 and 010110 in $\beta \setminus M$. It follows that $m_0 = 001111$, $m_1 = 110011$ and $m_2 = 111100$. Then the point 111111 is contained in each of the planes $\langle m_i, L_i \rangle$, $i = 0, 1, 2$, and hence collinear in Δ with all points of $L_0 \cup L_1 \cup L_2$, which spans W, contradicting non-degeneracy.

The proof is complete. □

Lemma 9.4.0.10. *Let Δ be a symplectic polar space of rank $r \geq 4$ and order (s, s), and let $i \in \mathbb{N}$ be such that $2 \leq i \leq r-2$. Suppose that every set of $s+1$ singular subspaces of dimension $i+1$ admits a common opposite, except if they all intersect in an i-dimensional singular subspace and are contained in a common $(i+2)$-space of the underlying projective space. Suppose also that* Theorem B *is true for symplectic polar spaces of rank at most $r-1$. Then a set T of $s+1$ i-dimensional singular subspaces of Δ admits a common opposite in Δ if and only if all members of T are not contained in a common $(i+1)$-dimensional subspace of the underlying projective space if they contain a common $(i-1)$-dimensional singular subspace of Δ.*

Proof. Let T be a set of $s+1$ singular subspaces of dimension i of the symplectic polar space Δ of rank r naturally embedded in $\mathsf{PG}(2r-1, s)$, $r \geq 4$. Suppose all members of T are not contained in an $(i+1)$-dimensional subspace of $\mathsf{PG}(2r-1, s)$ if they share a common $(i-1)$-dimensional singular subspace of Δ. We show that there exists a singular i-dimensional subspace of Δ opposite all members of T. We set $T = \{U_0, U_1, \ldots, U_s\}$.

First we assume that all members of T are contained in a common $(i+1)$-dimensional subspace W of $\mathsf{PG}(2r-1, s)$. By our assumption, not all members of T share the same

$(i-1)$-dimensional subspace. This implies that the radical of the polar space induced in W has dimension strictly larger than $i-1$; hence W is a singular subspace. Our assumption and Lemma 9.2.0.1 imply that we can find a point $x \in W$ not contained in any member of T. Let W' be a singular subspace of dimension $i+1$ opposite W. Then $x^\perp \cap W'$ is a singular subspace of dimension i opposite each member of T.

Henceforth we may assume that not all members of T are contained in the same subspace of $\mathsf{PG}(2r-1,s)$ of dimension $i+1$.

Include every member of T in a singular subspace of dimension $i+1$ in such a way that not all of them are contained in a common subspace of dimension $i+2$ of $\mathsf{PG}(2r-1,s)$. By assumption, we can find a subspace W opposite all of these $(i+1)$-dimensional subspaces. Set $\{m_i\} = W \cap U_i^\perp$. If the m_i do not form a line, then Lemma 9.2.0.1 yields a singular i-space $U \subseteq W$ not containing any of the m_i and hence opposite all members of T. Hence we may assume that the m_i form a line M. Set $H_i = U_i \cap M^\perp$. Then H_i is a hyperplane of U_i.

Assume now that two members of T, say U_0 and U_1, do not share a common $(i-1)$-dimensional singular subspace. Set $D = U_0 \cap U_1$. Let H be a hyperplane of U_0 not containing D, and hence distinct from H_0. Then $K := H^\perp \cap W$ is a line through m_0. Pick $x \in K \setminus \{m_0\}$ arbitrarily. Define $U_i^x := \langle x, x^\perp \cap U_i\rangle$. Suppose there exists an i-dimensional singular subspace U through x locally opposite all U_i^x. Then, by Lemma 3.6.2.21, U is opposite each member of T. Our hypotheses imply that the U_i^x are contained in an $(i+1)$-dimensional subspace A_x of $\mathsf{PG}(2r-1,s)$ and all U_i^x share a common $(i-1)$-space B_x. The singular subspace B_x contains a hyperplane of H and a hyperplane of $x^\perp \cap U_1$. As D is not contained in H, these two hyperplanes do not coincide, and hence they generate B_x. Now we do the same construction with $y \in K \setminus \{x, m_0\}$ and obtain the similarly defined subspaces A_y and B_y. It is elementary to check that $x^\perp \cap U_i \neq y^\perp \cap U_i$, for $i \in \{1,2,\ldots,s\}$. Hence $A := \langle A_x, A_y\rangle$ contains U_i for all $i \in \{1,2,\ldots,s\}$. The intersection $A_x \cap A_y$ contains H. Also, since H does not contain D, it does not contain the intersection $x^\perp \cap y^\perp \cap U_1$ (which is a singular subspace of dimension $(i-2)$). It follows that A_x and A_y share at least a subspace of dimension i (generated by H and $x^\perp \cap y^\perp \cap U_1$). Hence $\dim A \in \{n+1, n+2\}$. Suppose $\dim A = n+1$. Then $A_x = A_y$ and contains the line $\langle x, y\rangle$, which necessarily intersects U_1 for dimension reasons. But this contradicts the fact that W is opposite some singular $(i+1)$-space containing U_1. Consequently $\dim A = n+2$. We can now do the same thing with another hyperplane H' of U_0 not containing D and distinct from H_0, and obtain the similarly defined subspace A' of dimension $n+2$ and the line $\langle x', y'\rangle$ of W, with $m_0 \in \langle x', y'\rangle$. Since both A and A' share the subspace $\langle H, U_1\rangle$ of dimension at least $i+1$, we have $\dim\langle A, A'\rangle \in \{n+2, n+3\}$. If $A = A'$, then A contains the plane $\langle x, x', m_0\rangle$, which necessarily intersects U_1 in at least a point since $\dim A = n+2$ and $\dim U_1 = i$. This is again a contradiction. It follows that $A_0 := A \cap A'$ has dimension $i+1$ and contains all of $U_1, U_2, \ldots, U_s$, plus H and m_0. If A_0 were singular, then $m_0 = m_1$, a contradiction. Hence A_0 is not singular and all U_i, $i \in \{1,2,\ldots,s\}$, share a common $(i-1)$-space V_0. There is a unique i-space U_0^* through V_0 in A_0 distinct from U_i for any $i \in \{1,2,\ldots,s\}$. The space U_0^* contains H and m_0.

Now we can find a singular $(i+1)$-space W^* containing U_0^* with the property that not all of its points are collinear to all points of U_0. In W^*, we can then find a

point z not collinear to all points of U_1 (because $U_1^\perp$ cuts out a hyperplane of W^*). Since a point of Δ is collinear to either all points, or a hyperplane of points of W^*, we see that $z^\perp$ intersects each U_i in a hyperplane of U_i, and for $i = 1, 2, \ldots, s$, that hyperplane is necessarily V_0. Hence there exists an i-space U through z locally opposite the two spaces $\langle z, H\rangle$ and $\langle z, V_0\rangle$ at z, and U is opposite each member of T by Lemma 3.6.2.21.

Hence we may assume that each pair in T intersects in an $(i-1)$-space. Then either that $(i-1)$-space is unique, say V^*, or all members of T are contained in some $(i+1)$-space, say W^*. In the former case, our hypotheses permit finding an i-space U through V^* locally opposite each member of T, and then the projection U' of U onto a singular subspace opposite V^* is opposite each member of T at V^*. In the latter case, we are back in the situation of the first paragraph of this proof, which we already handled.

This completes the proof of the lemma. □

We can now prove Theorem B for symplectic polar spaces.

Proposition 9.4.0.11. *Let Γ be a symplectic polar space of rank r at least 2 and order (s, s). Let T be a set of $s+1$ singular subspaces of Γ of dimension $k \leq r-2$. Then there exists a singular subspace of dimension k opposite each member of T, except if either T is a line of the corresponding k-Grassmannian geometry, or all members of T contain a given $(k-1)$-dimensional subspace in the residue of which they form a hyperbolic line.*

PROOF. We prove this by induction on r, and for a given r we use induction on $r - i$. Let first $r = 3$. Then $i = 1$, and the assertion holds by Lemma 9.4.0.9, Lemma 9.4.0.7, and Lemma 9.4.0.5 (we need the latter in characteristic 2).

Now assume $r \geq 4$. Then the assertion holds for $i \geq 2$ by Lemma 9.4.0.10, and it holds for $i = 1$ by Lemma 9.4.0.9. □

Now Remark 9.4.0.4 yields Theorem C.

9.5. Points & Lines of $\mathsf{E}_{6,1}$

9.5.1. Points of Lie Incidence Geometries of Type $\mathsf{E}_{6,1}$

Proposition 9.5.1.1. *If every line of a parapolar space Γ of type $\mathsf{E}_{6,1}$ contains exactly $s+1$ points, then there exists a symp opposite each point of an arbitrary set S of $s+1$ (distinct) points, except when these points are contained in a single line.*

PROOF. Let S be a set of $s+1$ distinct points $p_0, \ldots, p_s$ in Γ that are not all on a common line. Assume first that all points of S are mutually collinear. Then $p_0, \ldots, p_s$ are either contained in a common 4-space or in a common 5-space. Since

both are residues in the corresponding building, Corollary 8.2.1.6 and Theorem B imply that S is a line, proving the assertion.

So, we may assume that at least two points of S have distance 2. Let p_0 and p_1 be two non-collinear points of S. Consider the symp $\xi_0 = \xi(p_0, p_1)$. For each point of S that is close to ξ_0 we choose an arbitrary point in ξ_0 collinear with it. Let S' be the set of points thus obtained, complemented with the points of S contained in ξ_0. Lemma 9.3.0.1 yields a point b in ξ_0 non-collinear to each member of S', and hence non-collinear to each member of S. Let Ξ be the set of symps $\xi(b, p_i)$, for $i \in \{0, 1, \ldots, s\}$. There are at most s such different symps as $\xi(b, p_0) = \xi(b, p_1)$. Applying Proposition 7.1.0.1 in $\mathsf{Res}(b)$, we find a symp ξ through b intersecting each member of Ξ in exactly b. Then, by Lemma 5.4.1.4, the symp ξ is far from all points of S. □

9.5.2. Lines of Lie Incidence Geometries of Type $\mathsf{E}_{6,1}$

Lemma 9.5.2.2. *Let Δ be a Lie incidence geometry of type $\mathsf{E}_{6,1}$ such that every line has $s+1$ points. Let p be a point and M a 3-space such that no point of M is collinear to p. Then the unique maximal 4-space containing M does not contain any point collinear to p.*

Proof. Suppose a point p is collinear to a point q of a 4-space C containing a 3-space M which does not contain any point collinear to p. Put C in a symp ξ. If p is in ξ, then p is collinear to a 3-space of C, which intersects M, a contradiction. If p is not in ξ, then p is collinear to a $4'$-space W, which already intersects C in q. But the intersection must have odd codimension; hence the intersection is either a line or a 3-space. Both would intersect M, a contradiction. □

Lemma 9.5.2.3. *Let Δ be a Lie incidence geometry of type $\mathsf{E}_{6,1}$ such that every line has $s+1$ points. Let L be a line, and let b be a point not collinear to any point of L. Then $\langle b, b^\perp \cap L^\perp \rangle$ is a maximal 4-space.*

Proof. According to Proposition 4.5.0.14, $b^\perp \cap L^\perp$ is a 3-space. Furthermore, $\langle L, b^\perp \cap L^\perp \rangle$ is a 5-space. That means, for every point ℓ_i on L ($i \in \{0, 1, \ldots, s\}$), $\langle \ell_i, b^\perp \cap L^\perp \rangle$ is a $4'$-space contained in $\langle L, b^\perp \cap L^\perp \rangle$. But then $b^\perp \cap L^\perp$ is already contained in $(s+1)$ $4'$-spaces, and thus $\langle b, b^\perp \cap L^\perp \rangle$ has to be a maximal 4-space. □

Proposition 9.5.2.4. *Let Δ be Lie incidence geometry of type $\mathsf{E}_{6,1}$ such that every line has $s+1$ points. Let $L_0, \ldots, L_s$ be $s+1$ different lines in Δ such that they do not form a line pencil in a plane. Then there exists a 4-space opposite all lines $L_0, \ldots, L_s$ in Δ.*

Proof.

(i) Suppose first that all $L_0, \ldots, L_s$ were contained in a common symp ξ. Then in ξ we can find a line opposite all $L_0, \ldots, L_s$ and we will denote it by M. In Δ we can find a point b opposite ξ. Take two arbitrary points m and m' on M. Then $m^\perp \cap m'^\perp \cap b^\perp$ forms a 3-space and b together with that 3-space forms a 4-space. This 4-space is opposite all $L_0, \ldots, L_s$.

(ii) Suppose that $L_0, \ldots, L_s$ are not in a common symp. Then we can put them in different symps and find a point b opposite to all of those symps. The intersection of the perps of any L_i and b is a 3-space that we will denote by U_i.

(ii.a) If at least two U_i are the same, we can find a 3-space U that is locally opposite each U_i in $\mathsf{Res}(b)$. Then the 4-space $\langle b, U\rangle$ is maximal according to Lemma 9.5.2.3, and it is opposite each $L_0, \ldots, L_s$.

(ii.b) Suppose all U_i are distinct. The residue of a point of Δ is a geometry of type $D_{5,5}$. In $\mathsf{Res}(b)$ we can see all 3-spaces U_i as maximal 3-spaces, since the $\langle b, U_i\rangle$ are maximal 4-spaces. Translating into a geometry of type $D_{5,1}$, we can see the U_i as lines and find an opposite to all of them except if they form a planar line pencil.

(ii.b.1) Suppose it is not the case that in $\mathsf{Res}(b)$, viewed as a $D_{5,1}$ geometry, the U_i form a planar line pencil. Then we can find a line U opposite all of them and can see it as a 3-space in $\mathsf{Res}(b)$ viewed as $D_{5,5}$. In that case the 4-space $\langle b, U\rangle$ in Δ is a 4-space opposite all L_i.

(ii.b.2) Suppose it is the case that in $\mathsf{Res}(b)$, viewed as a $D_{5,1}$ geometry, the U_i form a planar line pencil. Then, translating back to $D_{5,5}$, that means that the 3-spaces U_i all intersect in a line and are contained in a common symp (since points of $D_{5,1}$ correspond to symps in $D_{5,5}$ and planes in $D_{5,1}$ to lines in $D_{5,5}$). In Δ that means that all $\langle b, U_i\rangle$ intersect in a plane that we will denote by β. All U_i intersect β in lines N_i not containing b. We can find a 5-space opposite all $\langle U_i, L_i\rangle$ unless all $\langle U_i, L_i\rangle$ intersect in a common plane α.

We now have two cases to consider for (ii.b.2).

(ii.b.2) Case 1. *(cf. Fig. 1)*

Suppose first that all $\langle U_i, L_i\rangle$ do not intersect in a common plane and denote by B the 5-space opposite to all of them.

The projection of a point $l_{i,j}$ of some L_i onto B is a unique point.

We get lines L_i' such that each point of L_i is collinear to a unique point of L_i', but no point of $B - \bigcup_i L_i'$ is collinear to any point of any L_i.

If the L_i' do not form a planar line pencil in B, then we can find a 3-space M in B opposite each L_i'.

That 3-space is contained in a unique maximal 4-space that we denote by C, and C is opposite all L_i with 9.5.2.2.

So suppose the L_i' form a planar line pencil in B. We denote the corresponding plane by γ.

In an apartment, the set of points collinear to a unique point of a given 5-space forms exactly the opposite 5-space, and since 5-spaces do not contain 4-spaces, each 4-space must contain some point that is collinear to at least a 3-space.

That means we can find a point q_i in $\langle b, U_i\rangle$ (for $i \in \{0, \ldots, s\}$) that is collinear to a 3-space of B.

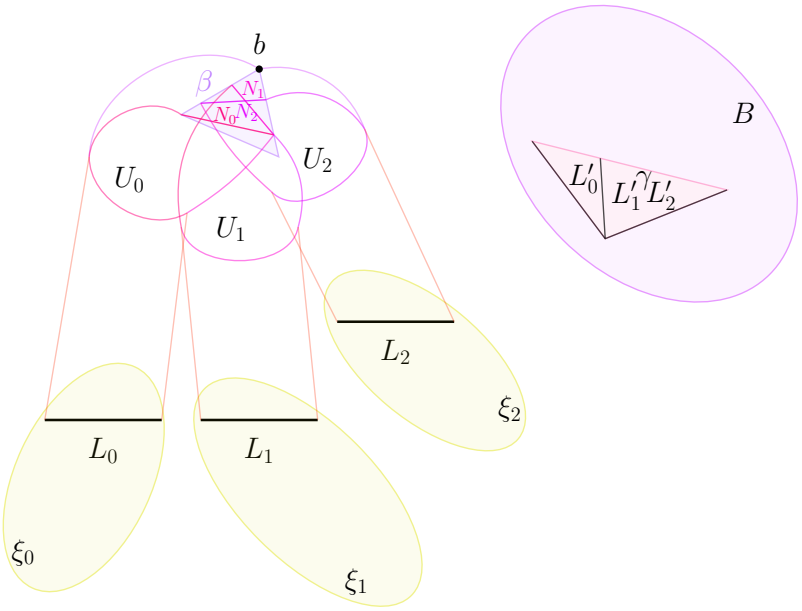


FIG. 1. An inaccurate visualization for (ii.b.2) Case 1. This visualization is inaccurate since symps in Δ always intersect, all lines contained in a plane pairwise intersect, etc. We still provide it in the hopes that it will help the reader to follow the proof.

The point q_i cannot be in U_i, because $\langle L_i, U_i\rangle$ and B are opposite. Since every 3-space intersects γ, q_i has to be collinear to a point q_i' in γ that is collinear to some point $l_{k,j}$ of some L_k.

In the following, we will consider q_0 and q_0'.

The point q_0' is collinear to some point $l_{j,k}$ of some L_j. The symp $\xi(q_0, l_{j,k})$ contains q_0' and the line $N_j = \langle U_j, L_j\rangle \cap \beta$. That means q_0' has to be collinear to a point of N_j.

Since q_0' is only collinear to a unique point of $\langle U_j, L_j\rangle$, it has to be the case that $(q_0')^\perp \cap N_j = l_{j,k} = (q_0')^\perp \cap L_j$. This is impossible since b is collinear to each point of N_j but to no point of L_j.

(ii.b.2) Case 2. *(cf. Fig. 2)*

Now suppose that all $\langle U_i, L_i\rangle$ intersect in a common plane α.

Suppose one of the lines $L_0, \ldots, L_s$ were contained in α. We may assume $L_0 \subseteq \alpha$. Since $\alpha \subseteq \langle U_i, L_i\rangle$ for all $i \in \{0, \ldots, s\}$, $L_0 \subseteq \langle U_1, L_1\rangle$. But then every point of L_0 is collinear to every point of U_1 and to every point of U_0, and since $L_0^\perp \cap b^\perp$ is a 3-space, it would follow that $U_0 = U_1$, which contradicts the assumptions.

The N_i cannot form a line pencil in β since they do not contain b.

Suppose U_0 and U_1 only intersect in a point z in β.

Then α is a plane through z. More generally, if $z_{i,j} := N_i \cap N_j$, then α contains all $z_{i,j}$. If the $z_{i,j}$ were not all on a line, then $\alpha = \beta$, which is impossible.

But if the $z_{i,j}$ are all on a line, then all the U_i have to intersect in that line. We will denote that line by L.

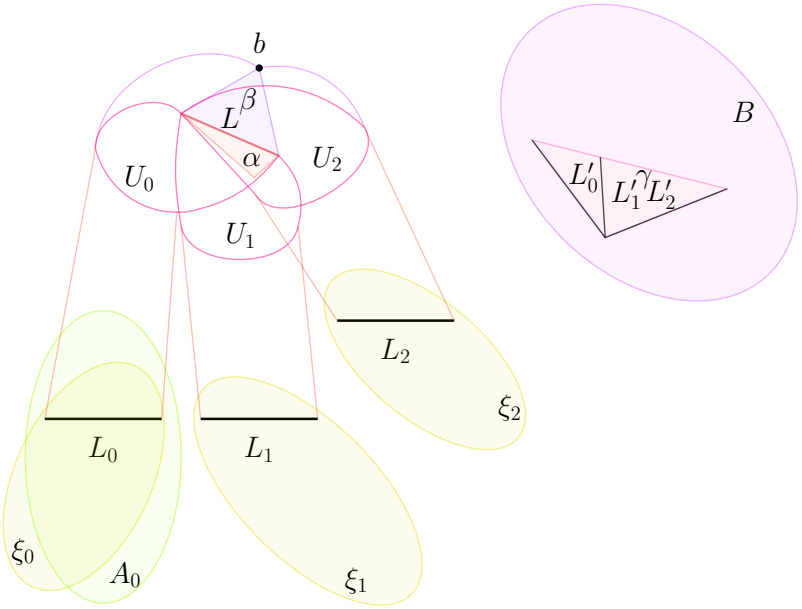


FIG. 2. An inaccurate visualization for the beginning of (ii.b.2) Case 2. This visualization is inaccurate since, for example, symps in Δ always intersect.

Let A_0 be a 5-space containing L_0 that does not contain the plane α. Then we can find a 5-space B opposite all 5-spaces A_0, $\langle U_1, L_1\rangle, \ldots, \langle U_s, L_s\rangle$.

Again, since $\langle b, U_i\rangle$ is a 4-space and B a 5-space, we can find a point q_i in $\langle b, U_i\rangle$ that is collinear to a 3-space of B. For $j \in \{1, \ldots, s\}$, the point q_j cannot be in U_j, since $\langle L_j, U_j\rangle$ and B are opposite. Since every 3-space intersects γ, q_j has to be collinear to a point q'_j in γ that is collinear to some $l_k \in L_k$. In the following, we will consider q_1 and set $q_1 = q$ and $q'_1 = q'$.

If l_k were not on L_0, we would get a contradiction as before with

$$(q')^\perp \cap L = l_k = (q')^\perp \cap L_k,$$

$b \perp L$, and b not collinear to any point of L_k. That means l_k has to be on L_0 (so $k = 0$). Then q' is collinear to the line $l_0 l$. We now make some observations and define planes δ, ϵ, and ϵ'.

- Since $\langle L_0, U_0\rangle$ intersects the space A_0 in L_0, $A_0 \cap \langle L_0, U_0\rangle$ is at least a plane. Let δ be a plane contained in $A_0 \cap \langle L_0, U_0\rangle$. Since δ is in A_0 and A_0 is opposite B, every point of δ far from B.
- Since $\langle L_0, U_0\rangle$ and B cannot be opposite, there exists at least a 5-space W intersecting both B and $\langle L_0, U_0\rangle$ in planes. We will denote the plane $W \cap \langle L_0, U_0\rangle$ by ϵ and the plane $W \cap B$ by ϵ'.
- Every point of ϵ is close to B, and every point of α is far from B. Therefore, the planes α and ϵ — and similarly the planes δ and ϵ — cannot intersect.

Since q' is collinear to the line $l_0 l$ in $\langle L_0, U_0\rangle$, it already follows that q' has to be collinear to a 3-space of $\langle L_0, U_0\rangle$. That means q' is in ϵ'. Then q' is collinear to ϵ, and to the line ll_0 (with $l \in \alpha$ and $l_0 \in \delta$), and only collinear to a 3-space of $\langle U_0, L_0\rangle$. That means it has to be the case that

$$(q')^\perp \cap \langle U_0, L_0\rangle = \langle l, \epsilon\rangle = \langle l_0, \epsilon\rangle.$$

That means ϵ has to intersect the line ll_0 in a point that is neither in α nor in δ. We denote that point by x.

No point of L_0 is in ϵ. But since each point of L_0 has to be collinear to a point of ϵ', and every point of L_0 is collinear to a unique point of B on L_0', it follows that L_0' has to be contained in ϵ'.

Let α' be the plane in B that is in bijection to α. Suppose $\alpha' \cap \gamma = \varnothing$.

The points $l'_{0,i}$ of L_0' are collinear to 3-spaces of $\langle U_0, L_0\rangle$ since they are contained in ϵ'. These 3-spaces intersect α. But points of α are collinear to unique points of B. This is a contradiction.

Hence $\alpha' \cap \gamma \neq \varnothing$. In fact, α' has to intersect γ in L_0', since it is true for every point $l'_{0,i}$ of L_0' that it is collinear to a point, say $l''_{0,i}$, of α. The $l''_{0,i}$ have to be distinct, since otherwise points of α would not be collinear to unique points of B.

With that, α' has to intersect γ in L_0'. Since all L_i' form a planar line pencil, there is a point d in the intersection $\bigcap_{i\in\{0,\dots,s\}} L_i'$. With $d \in L_0'$, we have $d \in \alpha'$, and therefore d is collinear to a unique point d_α of α.

Since d is in the intersection of all L_i', d is also in L_1' and therefore collinear to a unique point d_1 on L_1. Since d is only collinear to a unique point of $\langle U_1, L_1\rangle$, d_α and d_1 have to be equal.

With this argument applied to all $L_1', L_2', \dots, L_s'$, we can see that

$$d_\alpha = d_1 = d_2 = \cdots = d_s,$$

which means that all $L_1, L_2, \dots, L_s$ have to intersect in the point d_α.

Now let A_1 be a 5-space containing L_1 and not containing α. Then we can find a 5-space $\tilde{B}$ opposite to all

$$\langle U_0, L_0\rangle, A_1, \langle U_2, L_2\rangle, \langle U_3, L_3\rangle, \dots, \langle U_s, L_s\rangle.$$

We repeat all the exact same steps as before while interchanging the roles of L_0 and L_1. We can conclude that all $L_0, L_2, \dots, L_s$ have to intersect in a common point $\tilde{d}_\alpha$.

Now either all L_i intersect in the same point, or $s = 2$ and the L_i form a triangle in a plane.

In the latter case, the lines L_i are contained in a common 5-space and do not form a line pencil. In an opposite 5-space we find an opposite 3-space to all L_i and therefore an opposite 4-space with 9.5.2.2.

In the former case consider the point residue at d_α. We can find a 5-space C through d_α that is locally opposite (since the L_i in that residue are not collinear points). We can take a symp ξ through some 4′-space U' of C, and then any 4-space U of ξ disjoint from U' is opposite all L_i.

□

9.6. Points & Lines of $\mathsf{E}_{7,7}$

9.6.1. Points of Lie Incidence Geometries of Type $\mathsf{E}_{7,7}$

Proposition 9.6.1.1. *If every line of a parapolar space Γ of type $\mathsf{E}_{7,7}$ contains exactly $s+1$ points, then there exists a point at distance 3 from each point of an arbitrary set S of $s+1$ (distinct) points, except if these points are contained in a single line.*

Proof. Suppose S is not a line. Set $S = \{p_0, \dots, p_s\}$. We distinguish several cases.

(*i*) *Suppose all points in S are pairwise collinear.* Then S is contained in a (maximal) singular subspace. As this corresponds to a residue in the corresponding building, Corollary 8.2.1.6 and Theorem B lead to a contradiction.

(*ii*) *Suppose some pair of points from S is symplectic.* Suppose p_0 and p_1 are symplectic and let ξ be the symp containing them. Let the set T of points of Δ consist of the points of S in ξ, the set of points of ξ collinear to a point of T far from ξ, and, for each point p_i of S close to ξ, an arbitrary point of ξ collinear to p_i. Then T is a set of at most $s+1$ points not forming a line (as p_0 and p_1 belong to T and are not collinear). Hence by Lemma 9.3.0.1, we find a point $b \in \xi$ not collinear to any member of T. Hence for each point $p_i \in S$ not far from ξ, there is a unique symp ξ_i containing b and p_i. Since $\xi = \xi_0 = \xi_1$, there are at most s such symps. Proposition 7.1.0.1 yields a line L through b locally opposite each such symp, which means that a point of $L \setminus \{b\}$ is far from each such symp. This implies that each point of $L \setminus \{b\}$ is opposite each point of S that is not far from ξ. But since b is opposite every point of S that is far from ξ by construction, the line L contains points opposite each member of S. Hence each point p_i, $i = 0, 1, \dots, s$, has a unique projection p_i' onto L. Since $b = p_0' = p_1'$, there is at least one point of L opposite each point of S.

(*iii*) *Some pair of points from S is opposite and there are no symplectic pairs in S.* It follows that collinearity is an equivalence relation in S. Let $C \subseteq S$ be a corresponding equivalence class of minimal size (then certainly $|C| < 1 + s/2$ and $|S \setminus C| \geq 2$). Let ξ be a symp either containing C (if the latter is contained in a singular 5-space or a singular 5′-space; we assume $p_0 \in C$), or containing at least one point, say again p_0, of C and intersecting the 6-space spanned by C in a 5′-space (if C generates a 6-space; Facts 4.6.2.11(v) allows for this). Either way, let W be the singular subspace of ξ obtained by intersecting ξ with the singular subspace generated by C. Let $S' \neq \varnothing$ be the set of points of ξ collinear to a point of $S \setminus C$.

Select $q \in S'$ arbitrarily and suppose $q \perp p_1 \in S \setminus C$. As $|S \setminus C| \geq 2$, we can select a line M in ξ through q disjoint from W and not containing S'. With that, M contains at least one point $b \perp q$ such that $b \notin S'$ and b is not collinear to a member of C. It follows that b is not collinear to any point of S. So we can consider the set Ξ of symps containing b and some point of S.

Since p_0 and p_1 are opposite, the symps $\xi = \xi(b, p_0)$ and $\xi(b, p_1)$ only intersect in the line bq. Hence in the residue of b, the set Ξ does not correspond to the set of symps containing a given maximal singular subspace. Hence the dual of Proposition 9.5.1.1 yields a line L through b locally opposite each member of Ξ. As in (i) above, the line L is far from each member of S but p_0 and p_1 project to the same point of L, implying that there is at least one point on L opposite every member of S. □

9.6.2. Lines of Lie Incidence Geometries of Type $\mathsf{E}_{7,7}$

Proposition 9.6.2.2. *If every line of a parapolar space Γ of type $\mathsf{E}_{7,7}$ contains exactly $s+1$ points, then there exists a line opposite each member of an arbitrary set T of $s+1$ (distinct) lines, except if these lines form a planar line pencil.*

Proof. Since symps are vertices of type 1 in the corresponding building, Proposition 9.8.2.5 and Proposition 8.3.0.7, proved independently, allow us to consider symps $\widehat{\xi_0}, \widehat{\xi_1}, \ldots, \widehat{\xi_s}$ such that $L_i \in \widehat{\xi_i}$ and such that there exists a symp ξ opposite every $\widehat{\xi_i}$, for $i \in \{0, 1, \ldots, s\}$. Then, in view of Fact 4.6.2.13, every point of any $\widehat{\xi_i}$ is collinear to a unique point of ξ. The points of ξ, which are in bijection with the points of $L_i \in \widehat{\xi_i}$, form a line again that we will denote by L_i'. Suppose the lines L_i' do not form a planar line pencil. Then, by Lemma 9.4.0.8, we can find a line M in ξ that is locally opposite all L_i' at ξ (viewed as a residue). With Proposition 2.5.0.4, it follows that M is opposite all L_i.

Now suppose the L_i' do form a planar line pencil in ξ and denote the plane by π. Let p be the point in which all L_i' intersect, and let q be a point in $\xi \setminus \pi$ not collinear to p. Then $\mathsf{proj}_\pi(q)$ (in ξ) is a line K, which intersects each L_i' in some point p_i'. Set $p_i = \mathsf{proj}_{L_i} p_i'$. Since p_i and q are both collinear to p_i', they are symplectic and we denote the symp spanned by p_i and q by ξ_i. If the ξ_i do not all intersect in a common 5-space, then we can find a line M through q locally opposite all ξ_i, and by Proposition 2.5.0.4, M is opposite all L_i.

So we may assume from now on that all ξ_i intersect in a common 5-space Q. Note that they are pairwise distinct, as every ξ_i intersects ξ in exactly the line qp_i'. Let q_0 be a point on qp_0' distinct from q and p_0'. The point q_0 is in the plane $\alpha := \langle q, K \rangle$, and with that $\mathsf{proj}_\pi(q_0) = \mathsf{proj}_\pi(q) = K$. Hence, q_0 is symplectic to each p_i as well. We will denote the symps spanned by p_i and q_0 by ξ_i^0. Since q_0 is on qp_0', ξ_0^0 coincides with ξ_0. Similarly to before, we may assume that all ξ_i^0 intersect in a common 5-space Q^0. For each $j \in \{1, 2, \ldots, s\}$, we have that $\xi_j \neq \xi_j^0$ and ξ_j^0 intersects ξ in exactly the line $q_0 p_j$.

The 5-spaces Q and Q^0 are distinct but both contained in ξ_0. Since symps in Δ are polar spaces of type $\mathsf{D}_{6,1}$, two 5-spaces in ξ_0 intersect in even codimension and hence either in nothing, a line, a 3-space, or they coincide.

Note that p_j, for $j \in \{1, 2, \ldots, s\}$, cannot be contained in ξ_0, since otherwise

$$\xi_0 = \xi(q, p_j) = \xi_j,$$

a contradiction. We have that $\mathsf{proj}_{\xi_0}(p_j)$ (in Δ) is a 5′-space A_j. We know that $\mathsf{proj}_Q(p_j)$ is a 4-space $U_j \subseteq A_j$ and $\mathsf{proj}_{Q^0}(p_j)$ is a 4-space $U_j^0 \subseteq A_j$. Since

$$U_j \cap U_j^0 \subseteq Q \cap Q^0,$$

it follows that $Q \cap Q^0 = U_j \cap U_j^0 =: V$ is a 3-space independent of $j \in \{1, 2, \ldots, s\}$.

Let q_1 be a point of $qp_1' \setminus \{q, p_1'\}$. Without loss of generality, assume $p_2' \in q_0q_1$. Then the symps $\xi(q_1, p_1)$ and $\xi(q_1, p_2)$ contain both V, and it follows again that all symps $\xi(q_1, p_i)$ contain V. Since they also contain q_1, we find $q_1 \perp V$. Since q_1 was essentially arbitrary, we conclude that V is the intersection of all symps defined by p_i and some point of $\alpha \setminus K$. Moreover, $\alpha \perp V$ and $W := \langle \alpha, V \rangle$ is a (maximal) 6-space. Since both p_0 and V are in the intersection of $\xi(q, p_0)$ and $\xi(q_1, p_0)$, we also deduce $p_0 \perp V$.

We observe that $A_j = \mathsf{proj}_{\xi_0}(p_j) = \langle \mathsf{proj}_Q(p_j), \mathsf{proj}_{Q^0}(p_j) \rangle = \langle U_j, U_j^0 \rangle$ and that all A_j contain V. We define U_0 and U_0^0 as the 4-spaces $\mathsf{proj}_Q(p_0)$ and $\mathsf{proj}_{Q^0}(p_0)$, respectively. By the foregoing, $V = U_0^0 \cap U_0$.

The point q is contained in Q and the point q_0 is contained in Q^0. In a 5-space, there can be only $s + 1$ different 4-spaces that contain a given 3-space. Since none of the U_i (U_i^0 respectively) can contain q (q_0 respectively), at least two of the U_i (U_i^0 respectively) have to coincide. If $U_m = U_n$ for some $m, n \in \{0, 1, \ldots, s\}$, then it follows that $U_m^0 = U_n^0$, since otherwise A_m and A_n would intersect in a 4-space. We conclude that two A_i, for $i \in \{0, 1, \ldots, s\}$, have to coincide. Let $m, n \in \{0, 1, \ldots, s\}$ be such that $A_m = A_n$. Since, by Facts 4.6.2.11(v), 6-spaces cannot intersect in 5′-spaces, $\langle p_m, A_m \rangle$ and $\langle p_n, A_m \rangle$ have to be equal. Hence p_m and p_n are collinear.

We now claim that all p_i are contained in a common symp. Indeed, by Facts 4.6.2.11(v), there is a unique symp ζ containing the 5′-space $\langle V, K \rangle$. Now ξ_0 and ζ have the 4-space $\langle p_0', V \rangle$ in common, and hence they intersect in the 5-space $\langle p_0, p_0', V \rangle$, since this is the unique 5-space of ξ_0 containing $\langle p_0', V \rangle$. We get $p_0 \in \zeta$. Similarly, $p_i \in \zeta$ for all $i \in \{1, 2, \ldots, s\}$ and the claim is proved.

We now vary K over all lines of π not containing p. We obtain s^2 distinct pairs of collinear points, where each pair of points is contained in two different lines L_i, L_j, $i \neq j$, and no point of such a pair is collinear to p. Since we only have $\binom{s+1}{2} = \frac{1}{2}s(s+1) < s^2$ pairs of lines, there must exist $n, m \in \{0, 1, \ldots, s\}$, $n \neq m$, and distinct point pairs $\{a_n, a_m\}$ and $\{b_n, b_m\}$, with $a_n, b_n \in L_n$ and $a_m, b_m \in L_m$ such that $a_n \perp a_m$ and $b_n \perp b_m$. There are two cases.

Case 1: *$a_n \neq b_n$ and $a_m \neq b_m$.*
In this case one can see that L_n and L_m are contained in a common symp ξ^*. We claim that all points of $L_0 \cup L_1 \cup \cdots \cup L_s$ are mutually collinear. Indeed, suppose not, then there exists some symplectic pair of points on that union, contained in a unique symp ξ^{**}. Then we re-choose $\widehat{\xi_n}$ as ξ^*, and we re-choose $\widehat{\xi_m}$ as ξ^{**}. Since we have some freedom to choose the other $s - 1$ symps, we have a new symp ξ' opposite each of these $s + 1$ symps. But the projection of the L_i onto ξ' contains a symplectic pair (due to the presence of ξ^{**}), hence cannot be a planar line pencil. As before, this leads to a line opposite all of the L_i. The claim is proved. Hence S is contained in a singular subspace and Corollary 8.2.1.6 and Lemma 9.2.0.1 lead to the assertion.

Case 2: *without loss of generality* $a_n = b_n$ *and* $a_m \neq b_m$.
In this case a_n is collinear to the line L_m. Let $c_n \in L_n$ be different from a_n and not collinear to p. If $c_n \perp a_m$, then we are back in Case 1. So we may assume that c_n is not collinear to a_m. Then, interchanging the role of K with that of the line of π containing the respective points collinear to c_n and a_m, we see that there exists a unique symp ζ_n containing c_n, a_m, and a point c_i of each L_i, $i \in \{0, 1, \ldots, s\} \setminus \{n, m\}$. It also contains L_n. We may now re-choose $\widehat{\xi_n}$ as ζ_n. As c_n and a_m are not collinear, this again leads, as in Case 1 above, to a line opposite each L_i and the proposition is proved. □

9.7. Points of Exceptional Hexagonic Geometries of Rank ≥ 3

9.7.1. Points of Hexagonic Lie Incidence Geometries

Here we prove Theorem D and Theorem E for the vertices of the polar type in a building Δ of type E_i, $i = 6, 7, 8$, and for types 1 and 4 for a building Δ of type F_4; that is, for the exceptional hexagonic geometries distinct from generalized hexagons.

9.7.1.1. Reduction to Geometric Lines

Proposition 9.7.1.1. *Let* Γ *be an exceptional hexagonic geometry with* $s+1$ *points per line. Then a given set* T *of* $s+1$ *points of* Γ *admits an opposite point if and only if* T *is not a geometric line of* Γ.

Proof. If T is a geometric line, then T does not admit any point opposite all its points. Now suppose T does not admit any point opposite all of its points. We show that T is a geometric line. Suppose, for a contradiction, that T is not a geometric line. Then there exists a point x not opposite at least two points of T but opposite at least one point of T, and we shall call each such point a *spoilsport*. Suppose x is not opposite $r \geq 2$ points of T, with $r \leq s$, and let S be that set of points. We adopt the following notation. For each point $p \in S$ not equal or symplectic to x, we denote the line through x closest to p by $L_{x,p}$. If $p \in S$ is symplectic to x, then we denote by $\mathscr{L}_{x,p}$ the set of lines of $\xi(x, p)$ through x. Note that each point $z \neq x$ on any line K through x locally opposite some member of $\mathscr{L}_{x,p}$ is special to p.

(*i*) *Suppose* $x \in T$. For each point $p \in S$ symplectic to x, we choose an arbitrary line $L_{x,p} \in \mathscr{L}_{x,p}$. Then, by Corollary 9.1.1.5 for F_4, and by Proposition 7.1.0.1 for the other cases, we find a line $L \ni x$ locally opposite all of $L_{x,p}$, for p ranging through $S \setminus \{x\}$. Since $T \setminus S$ contains at most $s - 1$ elements, there is a point x' on L opposite all members of $T \setminus S$. If S contains at least one point collinear or symplectic to x, then $x' \notin T$ is a spoilsport. If $S \setminus \{x\}$ only contains points special to x, then some point of L at distance 2 from at least

one member of $T \setminus S$ is a spoilsport not contained in T. Hence we may assume from now on that $x \notin T$.

(*ii*) *Suppose $x \notin T$ and S contains at least one point collinear or symplectic to x.* Again, we choose an arbitrary line $L_{x,p} \in \mathscr{L}_{x,p}$ for each $p \in S$ symplectic to x, and we find a line L through x locally opposite all $L_{x,p}$, $p \in S$. There are two possibilities. First assume that S contains at least one point special to x. Then we select a point $x' \in L$ at distance 2 from at least one member of $T \setminus S$, and we see that x' is a spoilsport not collinear and not symplectic to any point of T. Secondly, assume that S does not contain any point special to x. Then we select a point $x'' \in L \setminus \{x\}$ distinct from the at most $(s+1) - r \leq s - 1$ points at distance 2 from some member of $T \setminus S$. Then x'' is opposite every member of $T \setminus S$ and special to each member of S, and hence x'' is a spoilsport. So, in both cases we constructed a spoilsport not collinear and not symplectic to any point of T. So from now on we may assume that x is special to each point of S.

(*iii*) *Suppose x is special to each point of S.* Then we can find a line L locally opposite each $L_{x,p}$, with $p \in S$, and a point $y \in L \setminus \{x\}$ opposite each member of $T \setminus S$. The point y is opposite each member of T, a contradiction.

We conclude that T is a geometric line. □

Proposition 9.7.1.1 reduces the classification of point sets T in a finite exceptional hexagonic geometry, where T has the size of a line and does not admit a point opposite each of its members, to the classification of geometric lines in such geometries. In view of Theorem 6.5 in [**KV13**], see Proposition 8.3.0.1, and Theorem 8.3.0.3, we can state the following consequence.

Corollary A. *Let Γ be an exceptional hexagonic geometry with $s+1$ points per line. Then a given set T of $s+1$ points of Γ admits an opposite point if and only if none of the following occurs.*

(1) *T is an ordinary line;*
(2) *T is a hyperbolic line in a symplecton of Γ isomorphic to a symplectic polar space (and this only occurs in Lie incidence geometries of type $\mathsf{F}_{4,4}$ arising from a split building of type F_4);*

9.8. Lines of Exceptional Hexagonic Geometries of Rank ≥ 3

We now treat the case of the lines. The idea here is to use the combing of the distinct types of mutual distances between lines as introduced in the previous chapter. We are going to define two algorithms that work under some conditions. In order to check these conditions, we need some properties of the point residues. We start with these.

9.8.1. Two Lemmata in the Residues

We begin with two results in the point residuals of hexagonic geometries. The first one summarizes earlier findings.

Lemma 9.8.1.1. *Let Δ be either a Lie incidence geometry of type $\mathsf{A}_{5,3}$, $\mathsf{D}_{6,6}$, or $\mathsf{E}_{7,7}$, or a dual polar space of rank 3. Suppose each line has exactly $s+1$ points. Suppose also that Δ is not isomorphic to $\mathsf{B}_{3,3}(\sqrt{s}, s)$. Then a set of at most $s+1$ points of Δ admits no common opposite point if and only if the points form a geometric line. In particular, if there are at most s points, or if there exists a point opposite at least one point of the set, and not opposite at least two points of the set, then the set admits an opposite point.*

Proof. This follows from Lemma 9.2.0.1 for type $\mathsf{A}_{5,3}$, from Lemma 9.3.0.2 for type $\mathsf{D}_{6,6}$, from Lemma 9.4.0.5 and Lemma 9.4.0.7 for dual polar spaces, and from Proposition 9.6.1.1 and [**KV13**, Corollary 5.6] for type $\mathsf{E}_{7,7}$. □

Lemma 9.8.1.2. *Let Δ be either a Lie incidence geometry of type $\mathsf{A}_{5,3}$, $\mathsf{D}_{6,6}$, or $\mathsf{E}_{7,7}$, or a dual polar space of rank 3. Suppose each line contains precisely $s+1$ points, and let p be a point. Let $Q := \{q_1, \ldots, q_\ell\}$ be a set of points not containing p, $\ell \leq s$. Then there exists a point p' opposite each member of Q, and not opposite p.*

Proof. We first construct a symp ξ through p far from each member of Q. If $q \in Q$ is opposite p, each symp through p qualifies. If $q \in Q$ is symplectic to p, let K_q be an arbitrary line through p in the symp containing p and q; if $q \perp p$, then let K_q be the line containing p and q. Since $\ell \leq s$, we infer from Proposition 7.1.0.1 and Corollary 9.1.1.5 that there exists a symp ξ through p locally opposite all K_q, $q \in Q$. Then ξ is far from each member of Q. Now the same references yield a point p' in ξ (locally) opposite in ξ each intersection $\xi \cap q^\perp$, for $q \in Q$. The point p' is opposite each member of Q and not opposite p. □

9.8.2. Combing Algorithms

Recall from Chapter 8 that the level of a pair of lines is the number of applications of Lemma 8.1.0.9 it takes to arrive at an opposite configuration. Now we introduce two algorithms, which we will call *combing algorithms* that reduces these levels. They require that certain conditions are met, and we will also introduce these. Naturally, we will only run them when all conditions are satisfied. They are defined as follows.

Definition 9.8.2.3 (**The combing algorithms**)**.** Let $L_0, L_1, \ldots, L_s$ be $s+1$ lines of Γ and let L be another arbitrary line. Suppose

(ALG1) There exists a point $x \in L$ which is free for every pair (L, L_i), $i = 0, 1, \ldots, s$.

Condition (ALG1) just means that the set of projection points on L with respect to the lines L_i, $i = 0, 1, \ldots, s$, does not cover L.

For each L_i, $i \in \{0, 1, \ldots, s\}$, and each point x on L that is free with respect to each line L_j, $j \in \{0, 1, \ldots, s\}$, we define a line M_i as follows. If L_i is opposite L, then M_i is the unique line through x containing a point collinear to L_i. The pair $\{L, M_i\}$ has distance (0112), or is, in other words, locally opposite at x. If L_i is not opposite L, then we set M_i equal to the line K, as (perhaps not uniquely) defined in Lemma 8.1.0.9. If the line K is not uniquely defined there, then we arbitrarily choose one (and one may think of taking the closest to L, if it exists). We set $\mathscr{M} = \{M_i \mid i \in \{0, 1, \ldots, s\}\}$.

(ALG2) There exists a line L' through x locally opposite each member of $\mathscr{M}$ at x.

The line L' is the outcome of the *first combing algorithm*.

We then replace L with L'.

(ALG3) There exists a line $M \in \mathscr{M}$ locally opposite L at x and there exists a line L'' through x locally opposite each member of $\mathscr{M} \setminus \{M\}$ (where we view $\mathscr{M}$ as a set and not as a multiset) and not opposite M.

The line L'' is the outcome of the *second combing algorithm* ("combing back at M"). We then replace L with L''.

We observe:

Lemma 9.8.2.4. *Under application of the second combing algorithm, a line opposite L (level 0 with respect to L) always goes to a line at level at most 1 with respect to L'.*

We will always use Lemma 9.8.1.1 to be able to perform the first combing algorithm, whereas Lemma 9.8.1.2 will allow us to perform the second combing algorithm. This is roughly the content of the proof of the next result.

Proposition 9.8.2.5. *Every set $T = \{L_0, L_1, \ldots, L_s\}$ of $s+1$ lines in a metasymplectic space, not isomorphic to $\mathsf{F}_{4,4}(\sqrt{s}, s)$, or in an exceptional long root subgroup geometry of type E_6, E_7, or E_8, where every line has exactly $s+1$ points such that every other line is not opposite at least one member of T, is a geometric line in the line-Grassmannian geometry, that is, has the property that every other line is either not opposite a unique member of T, or opposite no member of T.*

PROOF. A geometric line has the stated property. So assume now that T is not a geometric line, but every other line is not opposite at least one member of T. The only way in which we can violate the defining property of a geometric line is to assume the existence of a line L not opposite at least 2 members of T and opposite at least one member of T. We prove that this leads to a contradiction. The rough idea is to apply the combing algorithms to L and T until we find a line opposite all members of T. Since our proof will be inductive in some sense, it is important that after each application of the combing algorithm, the new line L satisfies the same assumption, that is, the new line L is not opposite at least two members of T and opposite at least one member of T, or the proof (locally) ends and L is opposite each member of T.

We say in this case that the new line L is *legal*.

This requirement indicates that we must apply the combing algorithms carefully, selecting the appropriate one at each step. The way we do this goes as follows.

We start by noting that a line $M_i \in \mathscr{M}$ is locally opposite L if and only if $L_i \equiv L$. Hence, since at least two lines of T are not opposite L, and at least one line of T is opposite L, Lemma 9.8.1.1 implies that (ALG2) is satisfied. Also, since opposite lines do not define projection points, and there is at least one line in T opposite L, also (ALG1) is satisfied. Moreover, Lemma 9.8.1.2 allows us to run the second combing algorithm since T contains at least one line opposite L.

Now we combine the two combing algorithms in one overarching algorithm that proves the theorem. That algorithm goes as follows.

- If T contains at least two members at level at least 2, then we apply the first combing algorithm. Note that elements of T opposite L remain opposite L', and elements of T at level $k \geq 1$ with respect to L are at level $k-1$ with respect to L'. Hence L' is legal, and the maximum level decreases.
- If T contains exactly one member L_0 at level at least 2 (and hence at least one member L_1 of level 1), then we apply the second combing algorithm, combing back at an arbitrary $M_2 \in \mathscr{M}$ locally opposite L. Then L_0 comes at level at least 1, L_1 at level 0, and L_2 at level 1. Hence L'' is legal. Moreover, Lemma 9.8.2.4 guarantees that the maximum level again decreases.
- We apply the previous two steps as long as the maximal level is at least 2. If the maximal level is or becomes 1, then we apply the first combing algorithm and obtain a line L' opposite each member of T, a contradiction that proves the assertion. □

9.8.3. The F_4 Case

The above approach does not work for lines of metasymplectic spaces Δ isomorphic to $\mathsf{F}_{4,4}(q, q^2)$, because we cannot apply Lemma 9.8.1.1. In the polar space $\mathsf{B}_{3,1}(q, q^2)$ corresponding to the residue of a point p, there exist sets of q^2+1 planes that admit no common opposite plane, yet are not isomorphic to a geometric line (a pencil of planes). Such examples consist of planes through a common point b forming a spread in a subquadrangle of order (q, q) of the residue at b. We will call such a set an *ovoidal blocking set (obs)*. Hence, a different proof is required.

Viewed in Δ, the point b is a symp, and the elements of an obs are lines through a common point forming an ovoid in a subquadrangle of order (q, q) of the residue at that point. We will also call such a set an obs.

Viewed in $\mathsf{F}_{4,1}(q, q^2)$, the point p is a symp, and the elements of an obs are planes through a common point b of the symp p, forming a spread in a subquadrangle of order (q, q) of the residue at b.

We first observe that this is indeed a blocking set.

Lemma 9.8.3.6. *Let T be a set of q^2+1 lines of $\Delta = \mathsf{F}_{4,4}(q, q^2)$ incident with a common point b and forming an ovoid in a subquadrangle of the residue $\mathsf{Res}_\xi(b)$ at b of some symp ξ through b. Then no line of Δ is opposite each member of T.*

PROOF. This follows directly from Corollary 8.2.1.6. □

We now show a converse to Lemma 9.8.3.6, that is, any set T of q^2+1 lines of $\Delta = \mathsf{F}_{4,4}(q,q^2)$ with the property that no line of Δ is opposite each member of T is either a planar line pencil or an obs.

Let $T = \{L_0, \ldots, L_t\}$, $t = q^2$, be a set of lines of $\Delta \cong \mathsf{F}_{4,4}(q,q^2)$ admitting no common opposite line.

Note that each point of a singular subspace S is opposite some point of a given symp ξ if and only if S and ξ are far. Indeed, if there is a symp through S opposite ξ, then each point of S is opposite some point of ξ. Now suppose each point of S is opposite some point of ξ. Pick $x \in S$ and let ζ be the unique symp through x intersecting ξ. Our assumption implies that S and ζ intersect just in x. Hence we can find a symp ζ' through S locally opposite ζ in x. Then ζ' is opposite ξ by Proposition 2.5.0.4.

Lemma 9.8.3.7. *There exist a point b and a symp ξ in Δ, with $b \in \xi$ such that both b and ξ are far from each member of T. For each such b and ξ we have that the projections of the members of T onto b and ξ, respectively, form either both a planar line pencil, or both an obs.*

PROOF. We can choose points b_i contained in L_i such that the b_i do not form a geometric line in Δ. Then Proposition 9.7.1.1, Proposition 8.3.0.1 and Theorem 8.3.0.3 yield a point b opposite all b_i, $i \in \{0,1,\ldots,s\}$. So, b is far from each member of T. Now set

$$T' = \{L_i' = \mathsf{proj}_b^{b_i}(L_i) \mid i \in \{0,1,\ldots,s\}\}.$$

If T' is not an obs and not a planar line pencil, then we can find a line L through b locally opposite each member of T', and so, by Proposition 2.5.0.4, L is opposite each member of T, a contradiction. We conclude that T' is contained in a symp ζ through b. Now let ξ be a symp locally opposite ζ at b. Then, again by Proposition 2.5.0.4, the projection ξ_i of ζ onto b_i is opposite ξ. However, ξ_i contains L_i as ζ contains L_i', $i = 0,1,\ldots,s$. Hence ξ is far from each member of T.

Now let L_i'' be the projection of L_i onto ξ and let L_i''' be the unique line of ξ collinear to L_i'. We claim that L_i'' intersects L_i''', which then shows that the projection of T' onto ξ coincides with the projection onto b of the projection of T onto ξ, and hence T' is isomorphic to the projection of T onto ξ and the lemma follows.

Let M_i''' be the unique line through b intersecting L_i'', say in the point x_i''. Since $x_i'' \in L_i''$, there is a unique point $x_i \in L_i$ symplectic to x_i''. Then b is collinear to a unique line K_i of $\xi(x_i, x_i'')$ through x_i'', and x_i is collinear to a unique point x_i' of K_i. Now $x_i \perp x_i' \perp b$ defines a path of length 2 from $x_i \in L_i$ to b, hence $bx_i' = L_i'$ and $M_i'' = L_i'''$ and the claim follows. □

Lemma 9.8.3.8. *Each pair of members of T is either contained in a symp or has a point in common.*

PROOF. It is convenient to consider the dual situation, that is, T corresponds to a set T^* of planes $\{\alpha_0, \ldots, \alpha_s\}$, $s = q^2$, of $\mathsf{F}_{4,1}(q,q^2)$. By Lemma 9.8.3.7 we can find a symp ξ far from each member of T^*. Hence we can project all planes α_i onto ξ and obtain planes α_i'.

By Corollary 8.2.1.6 and Corollary 9.1.1.2, the α_i' form a full plane pencil or an obs. In particular, all planes α_i' contain a common point q, and for each line L_0 of α_0' through q, except for the possible intersection line with α_1', there exist q^2 lines L_1 of α_1' through q not coplanar with L_0. Let z_0 and z_1 be two arbitrary points on L_0 and L_1, respectively. Select a point p in ξ not collinear to q, but collinear to both z_0 and z_1. Let p_i and x_i be the unique points in α_i symplectic to z_i and q, respectively, $i = 0, 1$. Since p is collinear to a unique line of $\xi(p_i, z_i)$, there is a unique point y_i in $\xi(p_i, z_i)$ collinear to p_i, z_i and p, $i = 0, 1$. The line $L_i'' = py_i$ is the projection of L_i from x_i onto p. By the "dual" of Lemma 9.8.3.7, the points y_0 and y_1 are either collinear or symplectic. But since $\xi(p_i, z_i)$ is symplectic to ξ, and z_0 is symplectic to z_1, the symps $\xi(p_0, z_0)$ and $\xi(p_1, z_1)$ are opposite (use Proposition 2.5.0.4 together with the observation that symplectic symps are locally opposite at their intersection point). Hence y_0 and y_1 are symplectic. Let q_i be the unique point of $\xi(p_i, z_i)$ collinear to q, p_i and z_i, $i = 0, 1$. Then, varying p over all points of ξ not collinear to q, but collinear to both z_0 and z_1, we deduce that $p_0^\perp \cap z_0^\perp \setminus q_0^\perp$ corresponds to $p_1^\perp \cap z_1^\perp \setminus q_1^\perp$ under the projection map from $\xi(p_0, z_0)$ to $\xi(p_1, z_1)$ given on the points by "being symplectic". It easily follows that $p_0^\perp \cap z_0^\perp$ corresponds to $p_1^\perp \cap z_1^\perp$. Hence $(p_0^\perp \cap z_0^\perp)^\perp$ corresponds to $(p_1^\perp \cap z_1^\perp)^\perp$. Since symps are isomorphic to quadrics $Q^-(7, q)$, which are embedded in non-degenerate (symplectic) polarities, we have $(p_i^\perp \cap z_i^\perp)^\perp = \{p_i, z_i\}$, $i = 0, 1$. Since z_0 corresponds to z_1, we conclude that p_0 and p_1 are symplectic.

We have shown that p_0 is symplectic to all points of α_1, except possibly the points of a unique line. It then easily follows that p_0 is collinear or symplectic to any given point of α_1. By the arbitrariness of p_0 in $\alpha_0 \setminus \{x_0\}$, we deduce that any pair of points in $\alpha_0 \cup \alpha_1$ is symplectic, collinear, or identical. Consider any symp ξ_1 through α_1. If $p_0 \in \xi_1$, then p_0 is collinear to at least a line of α_1. If $p_0 \notin \xi_1$, then it must be close to it and the line $p_0^\perp \cap \xi_1$ must be contained in α_1. Hence in any case, there is a line of α_1 collinear to p_0 and so we can assume that ξ_1 contains p_0. Suppose some point $r_0 \in \alpha_0$ does not belong to ξ_1. Then $r_0^\perp \cap \xi_1 \subseteq \alpha_1$, as before, showing $p_0 \in \alpha_1$.

Hence we have shown that either α_0 and α_1 are contained in a symp, or they have a point in common. This means that, if α_i corresponds to L_i, then L_0 and L_1 either intersect in a point or are contained in a common symp.

The assertion follows by the arbitrariness of L_0 and L_1 in T. □

We can now classify the blocking sets of lines of size $q^2 + 1$ in $\mathsf{F}_{4,4}(q, q^2)$.

Theorem 9.8.3.9. *Let T be a set of q^2+1 lines of $\Delta = \mathsf{F}_{4,4}(q, q^2)$. Then all members of T are incident with a common point b and form an ovoid in a subquadrangle of the residue $\mathsf{Res}_\xi(b)$ at b of some symp ξ through b if and only if no line of Δ is opposite each member of T.*

Proof. The "only if" part is Lemma 9.8.3.6. We now show the "if" part. We first claim that each pair of members of T intersect non-trivially. Indeed, we may assume for a contradiction that L_0 and L_1 do not intersect. Then by Lemma 9.8.3.8 they are contained in a common symp ζ. Lemma 9.8.3.7 yields a symp ξ far from each member of T. Also, the same Lemma 9.8.3.7 implies that the projection of L_0, L_1 from ζ onto ξ is a pair of intersecting lines. Since the projection from ζ to ξ

is an isomorphism of polar spaces, this implies that L_0 and L_1 also intersect. The claim follows.

We next claim that all members of T are either contained in a plane or contain a common point. Suppose the latter does not hold. Then there are three lines L_0, L_1, L_2 forming a triangle in a plane. All other members of T have to be contained in that plane. The claim is proved.

Since we now have that T belongs to a point residual, or the residue of a plane, the theorem follows from Corollary 8.2.1.6 and Theorem B. □

9.9. Generalized Hexagons

We start with a nonexistence result of a class of hexagons with certain parameters.

Lemma 9.9.0.1. *Let t be a natural number at least 2. Then there does not exist a generalized hexagon of order (s,t), with $s = t + t^2$.*

Proof. Since, by [**FH64**], the number st is a perfect square, we have t^2+t^3 is a perfect square. Hence $t+1$ is a perfect square, say $t = a^2 - 1$. Then $s = a^2(a^2-1)$. Now, by [**FH64**], we know that the rational number

$$\frac{st(1+s+t+st)(1+\sqrt{st}+st)}{2(s+t+\sqrt{st})}$$

is an integer. The denominator of that expression is equal to

$$2(a-1)(a+1)(a^2+a+1) = 2t(a^2+a+1).$$

Hence a^2+a+1 divides the numerator divided by t. We now observe, taking into account that a^2+a+1 is odd, the following facts.

- $\gcd(a^2, a^2+a+1) = 1$.
- Since $a^2-1 = (a^2+a+1)-(a+2)$ and $a^2+a+1 = (a+2)^2-3(a+2)+3$, we find $\gcd(a^2-1, a^2+a+1) \in \{1,3\}$.
- We have $1+s+t+st = a^2(a^4-a^2+1)$. Since

$$a^4-a^2+1 = (a^2(a^2+a+1) - a(a^2+a+1) - (a^2+a+1) + 2(a+1)),$$

  we find $\gcd(a^4-a^2+1, a^2+a+1) = \gcd(a+1, a^2+a+1) = 1$.
- We have $1+\sqrt{st}+st = a^6-2a^4+a^3+a^2-a+1$. Since

$$a^6-2a^4+a^3+a^2-a+1 = (a^4-a^3-2a^2+4a-1)(a^2+a+1) - 4a + 2,$$

  we find

$$\gcd(1+\sqrt{st}+st, a^2+a+1) = \gcd(2a-1, a^2+a+1),$$

  which, in view of $4a^2+4a+4 = (2a-1)^2+4(2a-1)+7$, is either 1 or 7. In the latter case, $2a-1$ is divisible by 7, implying in particular $a \geq 4$.

We conclude that the greatest common divisor of $s(1+s+t+st)(1+\sqrt{st}+st)$ and a^2+a+1 is one of $1, 3, 7, 21$. It follows that $a^2+a+1 \in \{3,7,21\}$, hence $a \in \{2,4\}$ (remember $a > 1$). But then, by the last bullet point above, $a = 4$.

But in this case, one calculates easily that the number

$$\frac{st(1+s+t+st)(1-\sqrt{st}+st)}{2(s+t-\sqrt{st})}$$

is not an integer (the denominator is divisible by 13, whereas this is not the case for the numerator), as is required by [**FH64**]. □

Every point x of a generalized hexagon has a projection onto a given line L, which is x itself if $x \in L$, which is the unique point of L collinear to x if x is close to L, and which is special to x if x is far from L. We call this projection occasionally the *nearest point to x on L*. Also, recall that, if the nearest point to x on L is collinear to but distinct from x, then we call x and L close.

Proposition 9.9.0.2. *If a set of $s+1$ points $S = \{p_0, p_1, \ldots, p_s\}$ of a generalized hexagon Δ of finite order (s,t), $s,t > 1$, admits no opposite point, then it is either*

(*i*) *a line, or*
(*ii*) *a hyperbolic line (and then $s = t$), or*
(*iii*) *a regular distance-3 trace (and then $s \geq t$).*

PROOF. Suppose $S = \{p_0, p_1, \ldots, p_s\}$ is a set of $s+1$ points in Δ such that no point of Δ is opposite every point of S. We proceed with proving some claims.

Claim 1. *If $p_0 \perp x \perp p_1$, with p_0 not collinear to p_1, and $x \notin S$, then every line through x contains at least one member of S.*
Indeed, suppose the line L through x is disjoint from S. Since p_0 and p_1 project onto the same point x of L, there exists some point $y \in L$ not collinear to any member of S. Consider a line $M \neq L$ through y. No point of S is contained in M or is close to M (since this would lead to a 4-gon or 5-gon containing y). Hence they are all far from M. Since p_0 and p_1 project onto the same point y of M, there is a point $z \in M$ opposite each member of S, a contradiction.

Claim 1 is proved.

Claim 2. *If p_0 and p_1 are collinear, then S is a line of Δ.*
Indeed, suppose first that there exists a point $x \in L := p_0p_1$ that does not coincide with a projection of some member of S onto L. Consider a line $M \neq L$ containing x. Then the only points of S close to M are on L. Since p_0 and p_1 project onto the same point x of M, there exists some point $y \in M$ which is not the projection of any member of S onto M. We deduce that every line $K \neq M$ through y is far from every member of S. Since p_0 and p_1 project onto the same point y of such a line K, there exists a point opposite every member of S on each such line K, a contradiction.

Hence every point q_i on L is the projection of a unique point p_i of S. Suppose $S \neq L$. Then we may assume that $p_2 \notin L$ and so p_2 is either special to or opposite p_0. If p_2 is special to p_0, then by Claim 1 each line through q_2 contains a point, say p_3, of S, implying that $q_2 = q_3$, contradicting the uniqueness of p_2. So p_2 is far from L. Select a line M' through q_2 distinct from L and far from p_2 (M' exists since $t \geq 2$). Then, since $q_2 \neq q_i$, the point p_i is either on L or opposite q_2, for each $i \in \{3, 4, \ldots, s\}$. It follows that, with M' in the role of M in the previous paragraph, we again reach the same contradiction.

Claim 2 is proved. From now on, we may assume that S does not contain two collinear points.

Claim 3. *If p_0 and p_1 are opposite, and some line L close to both contains no point of S, then L is close to each point of S and each point of L is collinear to a unique point of S.*
Let x_i be the nearest point to p_i on L, $i \in \{0, 1, \ldots, s\}$, and note that $x_i \neq p_i$ by assumption. Suppose there exists a point $x \in L \setminus \{x_0, x_1, \ldots, x_s\}$. Let $M \neq L$ be any line through x. Then M is far from each point of S. Since x is special to at least two points p_0, p_1 of S, there is some point of M opposite each point of S, a contradiction. Hence $L = \{x_0, x_1, \ldots, x_s\}$. Suppose some point, say p_2, of S is not collinear to its projection x_2 onto L. Let M_2 be a line through x_2 not close to x_2 and distinct from L (which exists as $t > 1$). Then M_2 is far from each point of S, but p_0, p_1 and p_2 have the same projection x_2, yielding a point $y_2 \in M_2$ opposite each point of S, a contradiction. Hence $p_i \perp x_i$, for all $i \in \{0, 1, \ldots, s\}$.

Claim 4. *If S only contains pairwise opposite points, then $s \geq t$ and S is a regular distance-3 trace.*
The assumptions of Step 3 are satisfied for each line close to both p_0 and p_1. Hence every point of S is collinear to some point of each line that is close to both p_0 and p_1. We conclude that S is a regular distance-3 trace. We now show that $s \geq t$.

Let z be any point special to both p_0 and p_1, and not on a line close to p_0 and p_1. Note that z is not collinear to any point of S, as $z \perp p_2$ would imply, by interchanging the roles of p_1 and p_2, that the line through z close to p_0 is also close to p_1, which is not the case by the assumptions on z. Then, similarly as before, every line through z is close to some point of S, but not to two such points as this would mean that z is already collinear to some point of S (by the definition of distance-3 trace), contradicting our note above. We conclude $t \leq s$.

Claim 5. *If S only contains pairwise special points, then $s = t$ and S is a hyperbolic line.*
Indeed, set $x = [p_0, p_1]$. By Claim 1, every line through x contains a point of S. Hence $t \leq s$. If $t = s$, then let y be a point opposite x, but not opposite either p_0 or p_1. We claim that p_i is not opposite y, for every $i \in \{2, 3, \ldots, s\}$. Indeed, suppose p_2 is opposite y and let L_y be the unique line through y not opposite xp_2. No point p_i, $i \in \{0, 1, \ldots, s\}$ is collinear to some point q_i of L_y as this would induce a 5-gon containing x, p_i, q_i and the lines L_y and xp_i. Hence all points p_i have a unique point on L_y to which it is not opposite. But p_0 and p_1 are not opposite the same point, yielding a point on L_y opposite all members of S, a contradiction. The claim is proved. This now implies that S is a hyperbolic line.

Suppose now that $t < s$. Claim 1 implies that S is a t-cloud, in the terminology of [**BTV02**]. By [**BTV02**, Lemma 1] and the remark following Lemma 1 of [**BTV02**], it follows that $S \cup S^*$, where S^* is the set of points collinear to at least two points of S, is the point set of a subhexagon of order $(1, t)$, and as such S and S^* are the point and line set, respectively, of a projective plane of order t. Hence $s = t^2 + t$. This contradicts Lemma 9.9.0.1.

There remains one case to take care of.

Claim 6. *If S contains opposite pairs, then it does not contain special pairs.*
Indeed, let $p_0 \equiv p_1$. Suppose for a contradiction that S contains a special pair, too. We first show that every line L close to p_0 and p_1, respectively, contains a (unique) point of S. Indeed, suppose not. Then Claim 3 implies that each point of L is collinear to a unique point of S, implying that each pair of points of S is opposite, contradicting our assumption. Hence L contains some point p_2 of S, unique by Claim 2. It also follows from Step 1 that each line through $[p_i, p_2]$, $j = 0, 1$, contains a unique point of S.

Now let T be the set of points of Δ with the property that each line through them contains a point of S. Let $\mathscr{L}$ be the set of lines of Δ through such points and note that each member of $\mathscr{L}$ contains at least one point of T and exactly one point of S. Then we prove that $\Gamma = (S \cup T, \mathscr{L})$ is a subhexagon. Indeed, if L, L' are two distinct lines containing points collinear to p_0 and p_1, respectively, then p_0, p_1, L, L' are contained in an ordinary hexagon H, implying that the girth of the incidence graph of Γ is equal to 12.

In order to show that the diameter of said graph is 6, it suffices to prove that for every point $x \in S \cup T$ and every line $L \in \mathscr{L}$, the unique minimal path joining x and L in Δ belongs to Γ. If $x \in L$, then this is trivial. Suppose now $x \in M \ni y \in L$, with $x \notin L$. If $x \in S$, then $y \notin S$ and so L contains a point of S distinct from y. It follows that $y \in T$ and consequently $M \in \mathscr{L}$. Suppose now $x \in T$. Then there exists some point $x' \in S \cap M$. It again follows $y \in T$ and $M \in \mathscr{L}$. At last, suppose $x \in M \ni y \in K \ni z \in L$, with $x \neq y \neq z$ and $M \neq K \neq L$. If $x \in T$, then there exists $x' \in M \cap S$. If $x' = y$, then the previous case proves the assertion; if $x' \neq y$, then we replace x with x' and hence we may assume $x \in S$. If $z \in S$, then $y \in T$ and the assertion follows easily. So suppose $z \notin S$. Then some point $q \in L$ different from z belongs to S. By the first paragraph, the line K contains a point of S. It follows that $y, z \in T$ and the assertion follows. Since one can see that every point of Δ (and hence of Γ) is opposite at least one point of H, we see that all lines through any point of $S \cup T$ belong to $\mathscr{L}$. Hence, by [**Van98**, Lemma 1.3.6] in combination with [**Van98**, Theorem 1.6.2], Γ is a subhexagon of order (s', t), $1 \leq s' \leq s$. Since, with above notation, the line L contains at least three points of $S \cup T$, we have $s' > 1$. Now, by the definition of $\mathscr{L}$, every member of $\mathscr{L}$ contains a unique member of S. A standard count reveals that $|S| = 1 + s't + (s't)^2$ points. It follows that $s = s't + (s't)^2$. Now, both st and $s't$ are perfect squares by [**FH64**]. It follows that $s/t = s'(1 + s't)$ is a perfect square. Since s' and $1 + s't$ are relatively prime, both s' and $1 + s't$ are perfect squares, which contradicts the fact that $s't$ is a perfect square. This completes the proof of Claim 6.

This also completes the proof of the proposition. □

The following result classifies very explicitly all sets of size $s+1$ admitting no global opposite point in finite Moufang hexagons of order (s, t).

Corollary 9.9.0.3. *A set S of $s + 1$ points of a Moufang generalized hexagon of finite order (s, t), $s, t > 1$, admits no opposite point if and only if it is either*

(i) a line, or
(ii) a hyperbolic line in the split Cayley hexagon $\mathsf{G}_{2,2}(s, s)$, or

(iii) a distance-3 trace in the split Cayley hexagon $\mathsf{G}_{2,2}(s,s)$ *with* s *even, or in the twisted triality hexagon* $\mathsf{G}_{2,2}(t,s)$*, with* $t^3 = s$ *even.*

PROOF. In view of Proposition 9.9.0.2, (*ii*) follows from [**Van98**, Remark 6.3.5] and (*iii*) follows from [**Gov97**, Theorem 1]. □

Note that the previous corollary implies that not every regular distance-3 trace is a set of $s+1$ points such that no point is opposite each point of that set. Indeed, the split Cayley hexagons and the twisted triality hexagons in odd characteristic are counterexamples.

Remark 9.9.0.4. In Claim 4 of the proof of Proposition 9.9.0.2, hexagons of order (t^2+t,t) appear as possible counterexamples, but, as we assume thickness, they are killed by Lemma 9.9.0.1. If we drop the thickness assumption, it is curious to note that the arguments of that step give rise to a rather exceptional example of a set F of $s+1$ point-line flags in a projective plane of order s such that no point-line flag is opposite all members of S. Indeed, putting $t=1$, we obtain a hexagon of order $(2,1)$, which arises from $\mathsf{PG}(2,2)$, and S consists of three flags $\{p_0, p_0p_1\}, \{p_1, p_1p_2\}, \{p_2, p_0p_2\}$ from a triangle $\{p_0,p_1,p_2\}$. The points of these flags do not form a line, and the lines of these flags do not form a line pencil. We conjecture that this is the only example of size $s+1$ in any projective plane of order s with that property and such that no flag of the plane is opposite all of its members.

Note 9.9.0.5. Let Δ be a finite generalized octagon of order (s,t), $s,t>1$. Then similar arguments as in the proof of Proposition 9.9.0.2 show that, if $s \leq t$ (and then $s<t$ since $s \neq t$), the only sets of $s+1$ points admitting no common opposite are the lines. If $s>t$, then the problem of determining all minimal blocking sets is still open.

10. Groups of Projectivities

In this chapter, we determine the special and general groups of projectivities of all irreducible residues of non-maximal simplices F, for all thick spherical buildings Δ of rank at least 3. We will list our main results shortly, for an overview, and then proceed to prove them consecutively. The results in this chapter are based on [**BSV25**] and [**BV25a**].

Recall from Definitions 2.5.0.6, for a thick spherical building Δ and a non-maximal simplex F of Δ, the general group of projectivities $\Pi(F)$ is given by all projectivities starting and ending in the residue of F, and the special group of projectivities $\Pi^+(F)$ is given by all the aforementioned ones that have even length. Furthermore, recall from Note & Notation 2.5.0.7 that $n(J)$ is defined to be the index $[\Pi(F) : \Pi^+(F)]$.

Definition 10.0.0.1 (**Mixed residues**)**.** In the following, given a polar space Δ, we call a residue that is a proper upper residue in a singular subspace of Δ a *mixed residue* of Δ.

The outline of this chapter is the following.

(i) The first results are about the number $n(J)$. If we can figure out $n(J)$ in each case, and especially when $n(J)$ equals 1, then we already know when $\Pi(F)$ and $\Pi^+(F)$ coincide. That will mean that, in some cases, we will be done as soon as we determine one of these groups.

(ii) After that, we will show that, under certain conditions, the general group of projectivities is generated by the projectivities of length 3 or 4. That will make it easier for us to determine these groups, since we only have to consider projectivities of length 3 or 4.

(iii) We will first determine the general and special projectivity groups for panels, and then for simplices with irreducible residues of rank ≥ 3, for thick, spherical buildings of simply laced type.

(iv) Thereafter, we move on to non-simply laced buildings (i.e., buildings of type B_n, C_n, and F_4), and first focus on the determination of the projectivity groups of the upper residues, and then of the lower and mixed residues of buildings of types B_n and C_n. For the upper residues, we first consider the projectivity groups of simplices corresponding to points, and afterwards the projectivity groups of simplices corresponding to singular subspaces of dimension at least 1. For the lower residues, we first determine the projectivity groups of simplices corresponding to non-maximal singular subspaces, and later the projectivity groups of simplices corresponding to maximal ones. Mixed residues only need special attention if the building corresponds to a non-embeddable polar space and we discuss them in that case after we determined the projectivity groups

of lower residues of maximal singular subspaces of these buildings. Lastly, we turn to buildings of type F_4, for which we first determine the projectivity groups of simplices corresponding to points, and then of simplices corresponding to non-maximal singular subspaces.

In the case that Δ has rank 2 — that is, Δ is the building of a generalized polygon — these groups have already been determined. In 1988, Norbert Knarr determined them in the finite case in [**Kna88**], and this was further generalized to large infinite classes by Hendrik Van Maldeghem in 1998 in [**Van98**].

Note that, in the rank 2 case, the simplex F of Δ, whose projectivity groups we want to determine, can be thought of as either a point (type 1) or a line (type 2) of the generalized polygon. For Moufang buildings of rank 2, it was shown in Chapter 8 of [**Van98**] that $n(\{1\}) = n(\{2\}) = 1$, as soon as Δ is a so-called *Pappian polygon*.[1] In fact, we always have $1 \in \{n(\{1\}), n(\{2\})\}$ due to Lemma 8.4.6 of [**Van98**]. One of the goals of this chapter is to generalize this to all thick, spherical buildings. This will be achieved by proving a general sufficient condition in J for $n(J)$ being equal to 1. To state this, we introduce *polar closed types*. Before doing so, we finally give a sketch of a proof for the existence of centers of roots.

Lemma 10.0.0.2. *Let Σ be an irreducible spherical Coxeter complex of rank at least three. Then every root α of Σ contains a unique simplex fixed by every (not necessarily type-preserving) automorphism of Σ that preserves α.*

Proof. Let α be a root of Σ. We prove the lemma using an opposite approach: we show that every simplex of polar type — where polar type is here defined as the set of types joined to the node turning the spherical Coxeter diagram to an affine one — is the unique simplex that is fixed under each automorphism of Σ preserving some root. The claim will then follow by noticing that the number of roots is equal to the number of such simplices. We will not explicitly do this elementary count.

From the discussion in Section 5.4, it follows that the simplices of polar type form the point set of a hexagonic geometry. Using the representations of apartments in Section 5.6, we see that an apartment Σ, where the simplices of polar type are represented as points, consists of two opposite points, say x and y, each with its neighbours $\Sigma(x)$ and $\Sigma(y)$, respectively, and the equator $E(x, y)$ of the two opposite vertices. It can be seen that collinearity defines an isomorphism between $\Sigma(x)$ and $\Sigma(y)$, and that the permutation that interchanges x and y fixes every point of $E(x, y)$, and maps a point $p \in \Sigma(x)$ (resp. $\Sigma(y)$) to the unique point of $\Sigma(y)$ (resp. $\Sigma(x)$) collinear to p, is an automorphism of the apartment arising from two opposite foldings. Thus $x \cup \Sigma(x) \cup E(x, y)$ defines a root α with boundary $E(x, y)$. The full automorphism group of $x \cup E(x, y)$ stabilizes α. Therefore, the stabilizer of α fixes only x and no other simplex. This proves the claim.
Notice that $\Sigma(x)$ can be empty — this is the case for type B_n and polar type 1. □

This confirms the claim in Definition & Facts 2.3.0.8 that, in the simply laced case, the unique type of the central simplex in a root coincides with the types joined to the highest root in the affine diagram; this type is thus called the polar type.

[1] For a definition of *Pappian polygon* see [**Van98**, §3.5].

Definition 10.0.0.3 (**Polar closed types**). Let J be the type of a simplex. If there exists a partition $J_1, \ldots, J_k$ of J into singletons and pairs, such that, for each $\ell \in \{1, \ldots, k\}$, we can order the elements of the partition such that the type J_ℓ is a polar type in the residue of $J_1 \cup \cdots \cup J_{\ell-1}$, then we call J *polar closed*.

10.1. Main results

Special Versus General Projectivity Groups

Theorem F. *Let Δ be a thick spherical building with type set I. If either $J \neq J^{\equiv}$ or $J \subseteq I$ is polar closed, then $n(J) = 1$.*

To see a partial converse of this statement, we first restrict to the simply laced case.

Theorem G. *Let Δ be a thick irreducible spherical building of simply laced type with type set I. If $J \subseteq I$, $J^{\equiv} = J$, and $I \setminus J$ has at least one connected component K of size at least 2, such that $I \setminus K$ is not polar closed, then $n(J) = 2$.*

Note that, if J is polar closed, then for each connected component K of $I \setminus J$ the type set $I \setminus K$ is polar closed. The above theorems imply the following combinatorial characterization of the polar type in connected simply laced spherical diagrams. For $K \subseteq I$, we denote by $\overline{K}$ the union of all connected components of K of size at least 2.

Corollary B. *The polar type of a connected simply laced spherical diagram D_I over the type set I is the unique smallest subset $J \subseteq I$ with the property that opposition in $D_{\overline{I \setminus J}}$ coincides with opposition in D_I.*

We had to restrict to the simply laced case here, as there exist polar spaces of rank n for which $\Pi^+(F)$ is not permutation equivalent to $\mathsf{PGL}_2(\mathbb{K})$ for F of cotype n. This happens even if the set of maximal singular subspaces through a submaximal singular subspace carries the structure of a projective line over $\mathbb{K}$ in a natural way, like for instance in a symplectic polar space. For buildings of type F_4, the above would imply that only for the residues of type $\{2, 3, 4\}$, $\{2, 3\}$, and all rank 1 residues, the special and general projectivity groups generically coincide. In the split case, however, also residues of type $\{1, 2, 3\}$ show this behavior, whereas we have rank 1 residues for which the special and general projectivity groups do not coincide. The latter was, in fact, to be expected, as this phenomenon also exists in the non-simply laced rank 2 case; see [**Kna88**] and [**Van98**, Chapter 8]. Furthermore, Corollary B does not hold in the non-simply laced case, since opposition does not determine the direction of the arrow in the Dynkin diagram. Indeed, for types B_n, C_n, and F_4, there are each time two single types satisfying the given condition. This reflects the fact that, in characteristic 2, there are really two choices. For buildings of type B_n, C_n, and F_4, we have the following two results.

Theorem H. *Let Δ be a thick polar space of rank at least 3. Let U be a non-maximal singular subspace of odd dimension at least 1. Then $\Pi^+_{\geq}(U) \equiv \Pi_{\geq}(U)$. The same conclusion holds if Δ is not a separable orthogonal polar space and U has arbitrary dimension (but is non-maximal).*

Theorem I. *Let p be a point of a metasymplectic space Γ. Then $\Pi(p) = \Pi^+(p)$ as soon as there exists an imaginary line in Γ of size at least 3 containing p.*

General Reduction Theorems

Let Δ be a thick spherical building over the type set I and let $J \subseteq I$ be a self-opposite type.

Theorem J. *Suppose that for each quadruple of simplices of type J, there exists a simplex of type J opposite all the given simplices. Let F, F', F'' be three pairwise opposite simplices of type J and denote by θ_0 the projectivity $F \overline{\wedge} F' \overline{\wedge} F'' \overline{\wedge} F$. Denote by $\Pi_3(F)$ the set of all self-projectivities of F of length 3 and suppose that $\Pi_3(F)$ is geometric. Then $\Pi(F) = \langle \Pi_3(F) \rangle$ and $\Pi^+(F) = \langle \theta_0^{-1}\theta \mid \theta \in \Pi_3(F) \rangle$.*

Theorem K. *Suppose that for each pair of simplices F, F' of type J, the subgraph $\Gamma_J^{\{F,F'\}}$ of Γ_J induced by the vertices opposite both F and F', is connected. Suppose there exists a simplex of type J opposite any given set of three simplices of type J. Let F be a given simplex of type J. Denote by $\Pi_4(F)$ the set of all self-projectivities $F \overline{\wedge} F_2 \overline{\wedge} F_3 \overline{\wedge} F_4 \overline{\wedge} F$ of F of length 4 with $F \sim F_3$, $F_2 \sim F_4$. Suppose that $\Pi_4(F)$ is geometric. Then $\Pi^+(F) = \langle \Pi_4(F) \rangle$.*

The Projectivity Groups

Our last main results state the exact permutation representations of the special and general projectivity groups on the corresponding residues of the building.

Projectivity Groups of Irreducible Residues in Simply Laced Buildings

First, we consider the case left out in Theorem G above, where $I \setminus J$ has only connected components of rank 1. We reduce the action of $\Pi^+(F)$ on each panel to a case where $|I \setminus J| = 1$, and show:

Theorem L. *Let Δ be a thick irreducible spherical building of simply laced type with type set I. Let $J \subseteq I$ with $|I \setminus J| = 1$, and let P be a panel of type J. Then $\Pi^+(P)$ is permutation equivalent to the natural action of $\mathsf{PGL}_2(\mathbb{K})$ on the projective line $\mathsf{PG}(1, \mathbb{K})$, and equals $\Pi(P)$.*

In view of Theorem L, one could expect that the general and special projectivity groups of simplices whose residue is isomorphic to $\mathsf{A}_r(\mathbb{K})$ are isomorphic to $\mathsf{PGL}_{r+1}(\mathbb{K})$. This is indeed true in most cases, but not always. If it is not true,

then the residue in question is necessarily not contained in a larger residue of type A_{r+1}.

Theorem M. *Let Δ be a thick irreducible spherical building of simply laced type with type set I. Let $I \neq J \subseteq I$ with $I \neq I \setminus J$ connected, and let F be a simplex of type J. Then $\Pi^+(F)$ and $\Pi(F)$ are*

(i) isomorphic to $\mathsf{PGL}_n(\mathbb{L})$ in its natural action, if Δ has type A_r, $r \geq 2$, it is defined over the skew field $\mathbb{L}$, and $|I \setminus J| = n - 1$;
(ii) as in Table 7 *and* Table 8 *for $\mathsf{typ}(\Delta) \in \{\mathsf{D}_r, \mathsf{E}_m \mid r \geq 4, m = 6, 7, 8\}$.*

The notation used in Tables 7 and 8 is explained in Section 10.6, where Theorem M is proved.

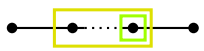
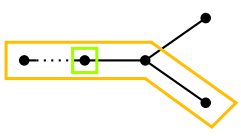
Summary 10.1.0.1. Tables 7 and 8 are exact and detailed. For a more schematic overview intended to give the reader a quick visual impression of the results, we also include a less precise version of these tables in Table 2. This table presents the same information in a simplified, graphical form. Its purpose is not exactness but intuitive orientation. We therefore briefly explain the conventions used and the sense in which the table should be understood as approximate.

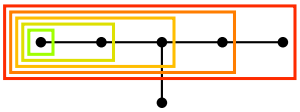
In the first column, each diagram highlights a residue by enclosing the corresponding subdiagram in a colored box. The associated projectivity groups are listed in the second and third columns, each enclosed in a box of the same color. The final column records additional specifications.

Whenever a marked subdiagram can be "shifted" within the larger diagram — i.e., whenever there exists another isomorphic subdiagram (that also satisfies the conditions listed in the fourth column) — the resulting projectivity groups are identical. In such cases, a single boxed position represents all possible shifts of that subdiagram; only one representative placement is drawn.

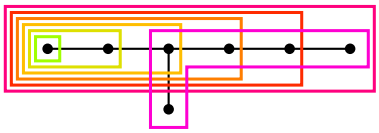
For example, in the first row the green box enclosing a single vertex could be positioned at any vertex of the diagram; in all cases the corresponding projectivity group is the same. In the figure, however, we display only one such placement (around the $(n-1)$-st vertex).

The diagrams should therefore be understood as illustrative rather than exhaustive. Precise statements and a complete, exact tabulation of the results are provided in 7 and 8.

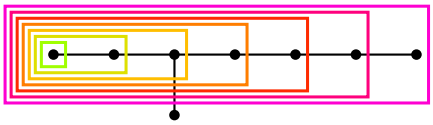
	$\Pi^+(F)$	$\Pi(F)$	
	$\mathsf{PGL}_2(\mathbb{K})$	$\mathsf{PGL}_2(\mathbb{K})$	
	$\mathsf{PGL}_{\ell+1}(\mathbb{K})$	$\mathsf{PGL}_{\ell+1}(\mathbb{K})$	for $\mathsf{cotyp}(F)$ of type A_ℓ, $\ell \in [2, r-1]$
	$\mathsf{PGL}_2(\mathbb{K})$	$\mathsf{PGL}_2(\mathbb{K})$	
	$\mathsf{PGL}_{\ell+1}(\mathbb{K})$	$\mathsf{PGL}_{\ell+1}(\mathbb{K}).2$	for $\mathsf{Res}(F)$ of type A_ℓ, $\ell \in [2, r-2]$, $\mathsf{cotyp}(F) \neq \{r-2, r-1, r\}$
	$\mathsf{PGL}_r(\mathbb{K}, 2)$	$\mathsf{PGL}_r(\mathbb{K}).2$	for $\mathsf{Res}(F)$ of type A_{r-1}, r even
	$\mathsf{PGL}_r(\mathbb{K})$	$\mathsf{PGL}_r(\mathbb{K})$	for $\mathsf{Res}(F)$ of type A_{r-1}, r odd



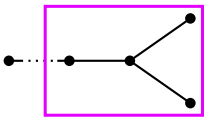
$\mathsf{PGL}_2(\mathbb{K})$	$\mathsf{PGL}_2(\mathbb{K})$
$\mathsf{PGL}_3(\mathbb{K})$	$\mathsf{PGL}_3(\mathbb{K}).2$
$\mathsf{PGL}_4(\mathbb{K})$	$\mathsf{PGL}_4(\mathbb{K})$
$\mathsf{PGL}_5(\mathbb{K})$	$\mathsf{PGL}_5(\mathbb{K})$
$\mathsf{PSL}_6(\mathbb{K},3)$	$\mathsf{PSL}_6(\mathbb{K},3)$



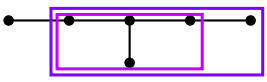
$\mathsf{PGL}_2(\mathbb{K})$	$\mathsf{PGL}_2(\mathbb{K})$	
$\mathsf{PGL}_3(\mathbb{K})$	$\mathsf{PGL}_3(\mathbb{K}).2$	
$\mathsf{PGL}_4(\mathbb{K})$	$\mathsf{PGL}_4(\mathbb{K}).2$	
$\mathsf{PGL}_5(\mathbb{K})$	$\mathsf{PGL}_5(\mathbb{K}).2$	
$\mathsf{PSL}_6(\mathbb{K},2)$	$\mathsf{PSL}_6(\mathbb{K},2).2$	for $\mathsf{cotyp}(F) = \{2,4,5,6,7\}$
$\mathsf{PGL}_6(\mathbb{K})$	$\mathsf{PGL}_6(\mathbb{K}).2$	for $2 \notin \mathsf{cotyp}(F)$
$\mathsf{PGL}_7(\mathbb{K})$	$\mathsf{PGL}_7(\mathbb{K}).2$	



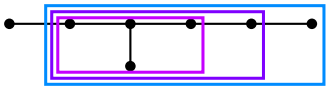
$\mathsf{PGL}_2(\mathbb{K})$	$\mathsf{PGL}_2(\mathbb{K})$
$\mathsf{PGL}_3(\mathbb{K})$	$\mathsf{PGL}_3(\mathbb{K}).2$
$\mathsf{PGL}_4(\mathbb{K})$	$\mathsf{PGL}_4(\mathbb{K}).2$
$\mathsf{PGL}_5(\mathbb{K})$	$\mathsf{PGL}_5(\mathbb{K}).2$
$\mathsf{PGL}_6(\mathbb{K})$	$\mathsf{PGL}_6(\mathbb{K}).2$
$\mathsf{PGL}_7(\mathbb{K})$	$\mathsf{PGL}_7(\mathbb{K}).2$
$\mathsf{PGL}_8(\mathbb{K})$	$\mathsf{PGL}_8(\mathbb{K}).2$



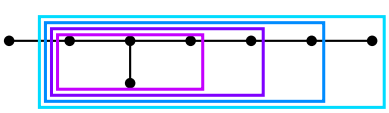
$\mathsf{PGO}^\circ_{2r-4\ell}(\mathbb{K})$	$\mathsf{PGO}^\circ_{2r-4\ell}(\mathbb{K})$	for $\mathsf{Res}(F)$ of type $\mathsf{D}_{r-2\ell}$, $4 \le r-2\ell \le r-1$
$\mathsf{PGO}^\circ_{2r-4\ell+2}(\mathbb{K})$	$\mathsf{PGO}_{2r-4\ell+2}(\mathbb{K})$	for $\mathsf{Res}(F)$ of type $\mathsf{D}_{r-2\ell+1}$, $3 \le r-2\ell+1 \le r-1$



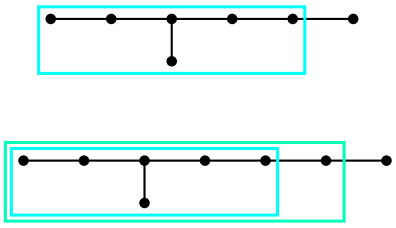
$\overline{\mathsf{PGO}^\circ_8}(\mathbb{K})$	$\overline{\mathsf{PGO}_8}(\mathbb{K})$
$\overline{\mathsf{PGO}^\circ_{10}}(\mathbb{K})$	$\overline{\mathsf{PGO}^\circ_{10}}(\mathbb{K})$



$\overline{\mathsf{PGO}^\circ_8}(\mathbb{K})$	$\overline{\mathsf{PGO}^\circ_8}(\mathbb{K})$
$\overline{\mathsf{PGO}^\circ_{10}}(\mathbb{K})$	$\overline{\mathsf{PGO}_{10}}(\mathbb{K})$
$\overline{\mathsf{P\Omega}}_{12}(\mathbb{K})$	$\overline{\mathsf{P\Omega}}_{12}(\mathbb{K})$



$\overline{\mathsf{PGO}^\circ_8}(\mathbb{K})$	$\overline{\mathsf{PGO}^\circ_8}(\mathbb{K})$
$\overline{\mathsf{PGO}^\circ_{10}}(\mathbb{K})$	$\overline{\mathsf{PGO}_{10}}(\mathbb{K})$
$\overline{\mathsf{PGO}^\circ_{12}}(\mathbb{K})$	$\overline{\mathsf{PGO}^\circ_{12}}(\mathbb{K})$
$\overline{\mathsf{PGO}^\circ_{14}}(\mathbb{K})$	$\overline{\mathsf{PGO}_{14}}(\mathbb{K})$



	$\mathsf{PGE}_6(\mathbb{K})$	$\mathsf{PGE}_6(\mathbb{K}).2$
	$\mathsf{PGE}_6(\mathbb{K})$	$\mathsf{PGE}_6(\mathbb{K}).2$
	$\mathsf{PGE}_7(\mathbb{K})$	$\mathsf{PGE}_7(\mathbb{K})$

TABLE 2

PROJECTIVITY GROUPS FOR IRREDUCIBLE RESIDUES IN THICK POLAR SPACES

Unfortunately, for buildings of type B_n, C_n, we can not summarize our results in a table like for the other buildings. For buildings corresponding to thick polar spaces, we have to make many case distinction and end up with this rather lengthy five-part theorem.

Theorem N. *Let Δ be a polar space of rank $r \geq 3$, and let U be a singular subspace of Δ. Then the following hold.*

(i) If U is a projective space of dimension $d \leq r-2$, then $\Pi^+_{\leq}(U)$ is the full linear type-preserving group of U, and $\Pi_{\leq}(U)$ is generated by $\Pi^+_{\leq}(U)$ and a duality, which is linear if the polar space is orthogonal or symplectic, but which is semi-linear with companion skew field automorphism σ if the polar space is associated to a σ-quadratic form.

(ii) If U is a maximal singular subspace, then,

- *if Δ is embeddable and associated to a pseudo-quadratic form, $\Pi^+(U)$ is the linear group generated by homologies with factors in a certain norm set (cf. 10.8.2.2), whereas $\Pi(U)$ extends $\Pi^+(U)$ with a duality having companion field automorphism the involution of the pseudo-quadratic form defining Δ.*
- *if Δ is a symplectic polar space, the same conclusions hold as in (i).*
- *if Δ is non-embeddable, then $\Pi^+(U)$ is the full projective group of the corresponding Cayley plane, and $\Pi(U)$ extends this group with a standard polarity.*

(iii) If U has dimension at most $r-3$, then $\Pi^+_{\geq}(U)$ is generated by products of two reflections, that is, collineations that pointwise fix a geometric hyperplane.

- *In the cases that $\Pi^+_{\geq}(R) \neq \Pi_{\geq}(R)$, we have that $\mathsf{Res}_\Delta(U)$ arises from a non-degenerate quadratic form with non-degenerate associated polarity and such that r minus the rank of $\mathsf{Res}_\Delta(U)$ is an odd integer.*
- *In that case, $\Pi_{\geq}(U)$ is generated by all reflections.*
- *If Δ is non-embeddable, then U is a point and $\Pi^+(U) = \Pi(U)$ is the full group of direct similitudes of the corresponding quadratic form.*

(iv) If U has dimension $r-2$, then,

- *if $\mathsf{Res}_\Delta(U)$ can be identified with an embeddable polar space of rank 1 in the natural way, the same conclusion holds as in the previous case.*

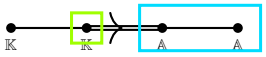
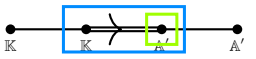
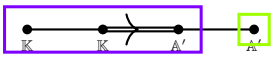
- *if Δ is non-embeddable, then $\Pi^+_{\geq}(U) = \Pi_{\geq}(U)$ is the full linear group of the corresponding projective line.*

(v)
- *For mixed residues, the same conclusions as in (i) hold for embeddable polar spaces.*
- *In the non-embeddable case, for a polar space Δ associated to the Cayley division algebra $\mathbb{O}$ over the field $\mathbb{K}$, a mixed residue is a planar line pencil P and the special projectivity group $\Pi^+(P)$ is permutation equivalent to the group of direct similitudes of the quadratic form of Witt index 1 defined by the norm form of $\mathbb{O}$ over $\mathbb{K}$. Also, $\Pi(P)$ is isomorphic to the group generated by $\Pi^+(P)$ and the permutation induced by the standard involution of $\mathbb{O}$, when considering P as a projective line over $\mathbb{O}$.*

Projectivity Groups for Irreducible Residues in Metasymplectic Spaces

Theorem O. *Let $\mathsf{F}_4(\mathbb{K}, \mathbb{A})$ be a building of type F_4, with $\mathbb{K}$ a field, and $\mathbb{A}$ a quadratic alternative division algebra over $\mathbb{K}$. Let F be a simplex with irreducible residue. Then $\Pi^+(F)$ and $\Pi(F)$ are as given in* Table 9.

Summary 10.1.0.2. We now provide a graphical overview of the results, and refer to Table 9 for a more detailed overview. The conventions here are the same as in Summary 10.1.0.1 (but note that, since everything is enumerated, there are no "shifts" here).

	$\Pi^+(F)$	$\Pi(F)$
	$\mathsf{PGL}_2(\mathbb{K})$ $\mathsf{PGL}_3(\mathbb{K})$	$\mathsf{PGL}_2(\mathbb{K})$ $\mathsf{PGL}_3(\mathbb{K}).2$
	$\mathsf{PGL}_2(\mathbb{K})$ $\mathsf{PGL}_3(\mathbb{A})$	$\mathsf{PGL}_2(\mathbb{K})$ $\mathsf{PGL}_3(\mathbb{A}).2$
	$\overline{\mathsf{PO}}^+_{d+2}(\mathbb{K}, \mathbb{A}')$ $\overline{\mathsf{PO}}^+_{d+4}(\mathbb{K}, \mathbb{A}')$	$\overline{\mathsf{PO}}_{d+2}(\mathbb{K}, \mathbb{A}')$ $\overline{\mathsf{PO}}^+_{d+4}(\mathbb{K}, \mathbb{A}')$
	$\overline{\mathsf{PO}}^+_{d+2}(\mathbb{K}, \mathbb{A}')$ $\overline{\mathsf{PO}}^+_{d+6}(\mathbb{K}, \mathbb{A}')$	$\overline{\mathsf{PO}}_{d+2}(\mathbb{K}, \mathbb{A}')$ $\overline{\mathsf{PO}}_{d+6}(\mathbb{K}, \mathbb{A}')$
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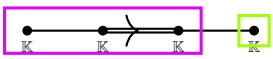
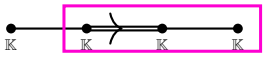
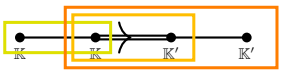
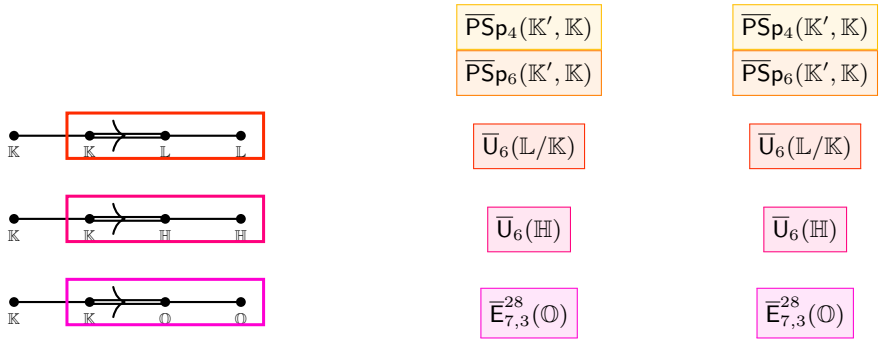
	$\mathsf{PGL}_2(\mathbb{K})$ $\mathsf{PO}_7(\mathbb{K})$	$\mathsf{PGL}_2(\mathbb{K})$ $\mathsf{PO}_7(\mathbb{K})$
	$\overline{\mathsf{PSp}}_6(\mathbb{K})$	$\overline{\mathsf{PSp}}_6(\mathbb{K})$
	$\mathsf{PGL}_3(\mathbb{K})$	$\mathsf{PGL}_3(\mathbb{K}).2$



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10.2. Special Versus General Projectivity Groups

In this section, we establish when the special and general projectivity groups generically coincide.

10.2.1. In Projective Spaces

We will first settle the case of type A_r, regarding the number $n(J)$. The main reason for treating this case separately is that we can provide an elementary proof, only using projective geometry, and independent from building-theoretic notions – we do refer to Proposition 2.5.0.4, but the result for projective spaces can be verified geometrically without needing creativity.

Theorem 10.2.1.1. *For buildings of type A_r with type set I and $J \subseteq I$, we have $n(J) = 1$ if and only if either $J^{\equiv} \neq J$, or $|J| = 2k$, for some $k \leq \frac{r-1}{2}$, and $J = \{1, 2, \ldots, k, r-k+1, r-k+2, \ldots, r\}$, that is, J is polar closed.*

Proof. Let F and F' be two opposite simplices of type J. First note that, for both the "if" and the "only if" parts, we may assume that J is self-opposite. First, suppose $n(J) = 1$. Let $j \in I \setminus J$ be minimal with respect to the Bourbaki labeling of the diagram and let v be a vertex of type j incident to F. Since J is self-opposite, $j \leq \frac{r}{2}$. Then, according to Proposition 2.5.0.4, the type j' of $\mathsf{proj}_{F'}^{F}(v)$ is the opposite type in $\mathsf{Res}_{\Delta}(F')$ of type $n+1-j$ (which belongs to $I \setminus J$ since $I \setminus J$ is self-opposite). If $n(J) = 1$, we should have $j = j'$. This is only possible if the integer interval $[j, r+1-j]$ belongs to $I \setminus J$. Putting $k = j-1$, we obtain the "only if" part of the statement.

Now we show the "if" part. We establish the identity projectivity as a product of three perspectivities. Let F be any simplex of type J. Suppose

$$F = \{U_i \mid i \in J,\ \dim U_i = i - 1\}.$$

Note that, since F is a simplex, $U_i \leq U_j$ for $i \leq j$, with $i, j \in J$. Select a simplex F' opposite F and set

$$F' = \{U'_i \mid i \in J,\ \dim U'_i = i - 1\}.$$

Choosing a basis $\{p_0, p_1, \ldots, p_r\}$ well, we may assume

$$U_i = \langle p_0, \ldots, p_{i-1}\rangle \quad \text{and} \quad U'_i = \langle p_r, p_{r-1}, \ldots, p_{r-i+1}\rangle.$$

Let, for $0 \leq i \leq k-2$, the point q_i be an arbitrary point on the line $\langle p_i, p_{r-i}\rangle$ distinct from both p_i and p_{r-i}. Define

$$U''_i = \langle q_0, \ldots, q_{i-1}\rangle \text{ for } 1 \leq i \leq k-1, \quad \text{and} \quad U''_i = \langle U_{r-i+1}, p_{r-i+1}, \ldots, p_{i-1}\rangle.$$

Then the simplex $F'' = \{U''_i \mid i \in J\}$ is easily checked to be opposite both F and F'. Let W be an arbitrary subspace of dimension k containing U_k and contained in U_{r-k+1}. Then W is generated by U_k and a point $p \in \langle p_k, \ldots, p_{r-k}\rangle$. The point p belongs to $U'_{r-k+1} \cap U''_{r-k+1}$. Consequently,

$$\mathsf{proj}^F_{F'}(W) = \langle U'_k, p\rangle =: W', \quad \mathsf{proj}^{F'}_{F''}(W') = \langle U''_k, p\rangle =: W'', \quad \text{and} \quad \mathsf{proj}^{F''}_{F}(W'') = W.$$

This implies that $\mathsf{proj}^{F''}_F \circ \mathsf{proj}^{F'}_{F''} \circ \mathsf{proj}^F_{F'}$ is the identity, and, by Observation 2.5.0.8, the assertion is proved. □

10.2.2. In the Simply Laced Case

The following lemma is basically the gate property of buildings.

Lemma 10.2.2.2. *Let Δ be a thick spherical building over the type set I and let F_J be a simplex of type $J \subseteq I$. Let $K \subseteq J$ and let $F_K \subseteq F_J$ be a simplex of type K. Let F^*_J be opposite F_J and let $F^*_K \subseteq F^*_J$ be opposite F_K. Set $F'_J := \mathsf{proj}_{F_K}(F^*_J)$. Let $C \supseteq F_J$ be a chamber. Then*

$$\mathsf{proj}_{F^*_J}(C) = \mathsf{proj}_{F^*_J}(\mathsf{proj}_{F'_J}(C)).$$

Proof. This follows from the gate property of residues. Since $F'_J = \mathsf{proj}_{F_K}(F^*_J)$,

$$F'_J \subseteq \mathsf{proj}_{F_K}(\mathsf{proj}_{F^*_J}(C)).$$

The latter is on every minimal gallery joining $\mathsf{proj}_{F^*_J}(C)$ with C and hence equals $\mathsf{proj}_{F'_J}(C)$. The assertion follows. □

In the next lemma, we use the following terminology. A triple of pairwise opposite simplices S_1, S_2, S_3 is called a *projective 3-cycle* if $\mathsf{proj}^{S_3}_{S_1} \circ \mathsf{proj}^{S_2}_{S_3} \circ \mathsf{proj}^{S_1}_{S_2} = \mathsf{id}$. Note that, if the triple S_1, S_2, S_3 is a projective 3-cycle, then so is the triple S_i, S_j, S_k, with (i, j, k) any permutation of $(1, 2, 3)$.

Lemma 10.2.2.3. *Let Δ be a thick spherical building over the type set I and let S_1, S_2, S_3 be a projective 3-cycle of type $J \subseteq I$. Let $K \subseteq I \setminus J$ be such that for each pair of S_3-opposite simplices $T_3, T'_3 \in \mathsf{Res}(S_3)$, there exists a simplex T''_3 such that*

T_3, T_3', T_3'' is a projective 3-cycle in $\mathsf{Res}_\Delta(S_3)$. Then $n(J \cup K) = 1$. More exactly, if T_1 is a simplex of type K adjacent to S_1, then there exist simplices $T_2' \sim S_2$ and $T_3'' \sim S_3$ of type K such that the triple $S_1 \cup T_1$, $S_2 \cup T_2'$, $S_3 \cup T_3''$ is a projective 3-cycle.

PROOF. Let T_1 be a simplex of type K adjacent to S_1. We want to write id in $\mathsf{Res}_\Delta(S_1 \cup T_1)$ as the product of three projections.

Since S_1, S_2, S_3 is a projective 3-cycle, $\mathsf{proj}_{S_2}^{S_1} T_1 = \mathsf{proj}_{S_2}^{S_3} T_3$, where $T_3 = \mathsf{proj}_{S_3}^{S_1} T_1$. Hence we have

$$\begin{aligned} T_2 &= \mathsf{proj}_{S_2}(T_1) = \mathsf{proj}_{S_2}(T_3), \\ T_3 &= \mathsf{proj}_{S_3}(T_2) = \mathsf{proj}_{S_3}(T_1), \\ T_1 &= \mathsf{proj}_{S_1}(T_3) = \mathsf{proj}_{S_1}(T_2). \end{aligned}$$

Let T_2' be a simplex locally opposite T_2 at S_2. Then, by Proposition 2.5.0.4, the simplices T_1 and T_2' are opposite in Δ. Set $T_3' = \mathsf{proj}_{S_3}(T_2')$. Then T_3' is opposite T_2 in Δ (again by Proposition 2.5.0.4). Since $T_3 = \mathsf{proj}_{S_3}(T_2)$, this implies, again using Proposition 2.5.0.4, that T_3' is locally opposite T_3 at S_2. Our assumption permits choosing a simplex $T_3'' \sim S_3$ of type K, such that T_3, T_3', T_3'' is a projective 3-cycle in $\mathsf{Res}_\Delta(S_3)$. Since, in particular, T_3'' is locally opposite both T_3 and T_3' at S_3, we have, similarly as before (using Proposition 2.5.0.4), the following opposite relations:

$$\begin{aligned} T_1 \equiv T_2' \equiv T_3'' \equiv T_1, \\ T_3 \equiv_{S_3} T_3' \equiv_{S_3} T_3'' \equiv_{S_3} T_3. \end{aligned}$$

Let v_1 be an arbitrary vertex adjacent to $S_1 \cup T_1$. We want to see that if we project v_1 first onto $S_2 \cup T_2'$, then onto $S_3 \cup T_3''$, and back to $S_1 \cup T_1$, then we get v_1 again. Define:

$$\begin{aligned} v_2 &:= \mathsf{proj}_{S_2 \cup T_2}(v_1), \\ v_3 &:= \mathsf{proj}_{S_3 \cup T_3}(v_1) = \mathsf{proj}_{S_3 \cup T_3}(v_2), \\ v_2' &:= \mathsf{proj}_{S_2 \cup T_2'}(v_2);\ \text{then } v_2' = \mathsf{proj}_{S_2 \cup T_2'}(v_3), \\ v_3' &:= \mathsf{proj}_{S_3}(v_2');\ \text{then } v_3' \sim T_3', \\ v_3'' &:= \mathsf{proj}_{S_3 \cup T_3''}(v_2'). \end{aligned}$$

We have that v_i is adjacent to T_i for $i \in \{1, 2, 3\}$, that v_j' is adjacent to T_j' for $j \in \{2, 3\}$, and that v_3'' is adjacent to T_3'', since incidences are preserved under projection.

By Lemma 10.2.2.2, the projection of v_2' from $S_2 \cup T_2'$ onto $S_3 \cup T_3''$ is the same as the projection onto T_3'' of the projection of v_2' from S_2 onto S_3, and this is the same as $\mathsf{proj}_{T_3''}(v_3')$ (namely v_3'').

If we project v_3' onto T_2, we get the vertex v_2'. If we project further onto T_2, we get the vertex v_2. The converse shows that v_3 maps to v_3' under the projection locally at S_3 from T_3 to T_3'.

Now the projection of v_1 onto $S_3 \cup T_3''$ is obtained by first projecting onto S_3 (and this is v_3), and then projecting v_3 locally at S_3 onto T_3''. But since the triple T_3, T_3', T_3''

is a projective 3-cycle, we have locally at S_3:

$$\mathsf{proj}_{T_3''}(v_3) = \mathsf{proj}_{T_3''}(\mathsf{proj}_{T_3'}(v_3)) = \mathsf{proj}_{T_3''}(v_3') = v_3'',$$

which shows that the triple $S_1 \cup T_1$, $S_2 \cup T_2'$, $S_3 \cup T_3''$ is a projective 3-cycle. This concludes the proof of the lemma. □

Proof of Theorem F

In view of Lemma 10.2.2.3, and in order to prove Theorem F, it suffices to show that, for any irreducible building Δ, there exists a triple of simplices of polar type which is a projective 3-cycle.

Proposition 10.2.2.4. *Let Δ be a thick spherical building. Let F and F' be two opposite simplices of polar type. Then F and F' are contained in a projective* 3*-cycle.*

Proof. Let C be a chamber containing F, let Σ be an apartment containing C and F', let α be the root in Σ with center F (and so containing C), and let $C' = \mathsf{proj}_{F'}(C)$. Then F' is the center of the opposite root $-\alpha$ of α in Σ. Let $\theta \in U_\alpha$ be a non-trivial root elation and set $F'' = F'^{\theta}$. Let $(C_0, C_1, \ldots, C_\ell)$ be a minimal path in the chamber graph of Δ, connecting $C = C_0$ with $C' = C_\ell$. By symmetry, $\ell = 2k$ is even and $C_0, \ldots, C_k$ all belong to α, whereas $C_{k+1}, \ldots, C_{2k}$ belong to $-\alpha$. The root $(-\alpha)^\theta$ has center F'' and contains $C_{k+1}^\theta, \ldots, C_{2k}^\theta =: C''$. Moreover, since θ fixes $\partial\alpha = \partial(-\alpha)$ pointwise, the union $(-\alpha) \cup (-\alpha)^\theta$ is an apartment and the chambers C_{k+1} and C_{k+1}^θ are adjacent. Hence, F'' is opposite F' and

$$\delta(C, C') = \delta(C, C'') = \delta(C', C'').$$

All this yields

$$\mathsf{proj}_{F'}^{F''}(C'') = C',$$

which shows that $\{F, F', F''\}$ is a projective 3-cycle. □

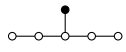
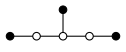
Proof of Theorem G

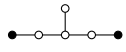
Let Δ be a thick spherical building over a type set I and let $J \subseteq I$. If there exists a simplex F of type J such that $\mathsf{Res}(F)$ contains a connected component of rank at least 2, then, by abuse of language, we also say that J *contains a connected component of rank at least* 2, for short.

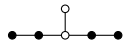
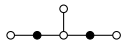
We prove Theorem G by verifying that, as soon as J is not polar closed and $I \setminus J$ contains a connected component of rank at least 2, then for some connected component of $I \setminus J$, Proposition 2.5.0.4 implies that a single perspectivity $\mathsf{proj}_{F'}^{F}$, with F, F' simplices of type J, does not preserve types. This implies that every member of $\Pi(F)$, which is the product of an odd number of perspectivities, is a duality in $\mathsf{Res}_\Delta(F)$ and hence cannot be the identity. Observation 2.5.0.8 then yields $n(J) = 2$. We first treat the exceptional cases and then the infinite class of type D_n. The case A_n follows from Theorem 10.2.1.1.

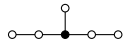
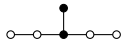
10.2.2.1. Type E_6

Out of the $2^6 - 2 = 62$ possible types of a non-empty non-maximal simplex, there are exactly $2^4 - 2 = 14$ self-opposite ones. Out of these 14, there are precisely seven for which $I \setminus J$ has a connected component of rank at least 2. We present the possibilities pictorially, coloring the vertices of types in J black. For the other seven, $I \setminus J$ is the union of isolated vertices.

 and  are polar closed.

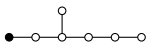
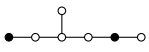
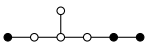
 : According to Proposition 2.5.0.4, types 3 and 5 are interchanged by a perspectivity.

 and  : According to Proposition 2.5.0.4, types 2 and 4 are interchanged by a perspectivity.

 and  : According to Proposition 2.5.0.4, types 1 and 5 are interchanged, as are types 3 and 6, by a perspectivity.

10.2.2.2. Type E_7

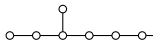
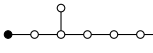
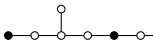
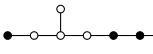
All of the $2^7 - 2 = 126$ possible types of non-empty non-maximal simplices are self-opposite, as opposition is trivial here. There are 18 polar closed types, of which only three admit a residue containing a connected component of rank at least 2. These components are of types D_4 and D_6; the three cases are

 ,  and  .

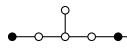
Now, the only connected subdiagrams of size at least 2, admitting trivial opposition, are precisely the ones of types D_4 and D_6. The above choices for J are the only ones for which $I \setminus J$ has a connected component of size at least 2 admitting trivial opposition. It follows from Proposition 2.5.0.4 that in all other cases, where a connected component of $I \setminus J$ has rank at least 2, the corresponding group of projectivities contains a duality and hence $n(J) = 2$.

10.2.2.3. Type E_8

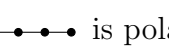
Here, opposition is trivial as well. There are 19 polar closed types, of which only four admit a residue containing a connected component of rank at least 2. These components are of types D_4, D_6 and E_7; the four cases are

 ,  ,  and  .

There is actually exactly one more type with a residue of rank 4 admitting trivial opposition:

 : Here the unique connected component $K = \{2, 3, 4, 5\}$ of

$$I \setminus J = \{2, 3, 4, 5, 8\}$$

has the property that $I \setminus K = \{1, 6, 7, 8\} =$  is polar closed.

Since all other connected subdiagrams of size at least 2 are either of type $\mathsf{A}_2, \ldots, \mathsf{A}_7$, D_5, D_7 or E_6, we see that for all other types J, such that $I \setminus J$ has a connected component of size at least 2, we have $n(J) = 2$.

10.2.2.4. Type D_n, $n \geq 4$

The only connected subdiagrams of size at least 2 of a diagram of type D_n, $n \geq 4$, where opposition agrees with the opposition in D_n, are of type D_{n-2k}, for $k \in \mathbb{N}$, such that $n - 2k \geq 3$. So a counterexample J to the assertion has

$$\max J = n - (n - 2k) = 2k$$

and the connected component K of size at least 2 of $I \setminus J$ is unique. We have that $I \setminus K$, which consists of the vertices of types $1, 2, \ldots, 2k$, is polar closed (indeed, consider the ordering $2, 1; 4, 3; \ldots; 2k, 2k-1$).

10.2.3. In Polar Spaces

Turning to polar spaces, it is clear that the general projectivity group of the lower residue of a singular subspace contains dualities and the special group does not. Thus, they are always different. However, for the projectivity groups of the upper residues, we see almost the opposite.

Proof of Theorem H

Theorem H. *Let Δ be a thick polar space of rank at least 3. Let U be a non-maximal singular subspace of odd dimension at least 1. Then $\Pi^+_{\geq}(U) \equiv \Pi_{\geq}(U)$. The same conclusion holds, if Δ is not a separable orthogonal polar space and U has arbitrary dimension (but is non-maximal).*

Proof. Pick a singular subspace U' opposite U. If Δ is non-embeddable, then U and U' are points or lines and $(U^\perp \cap U'^\perp)^\perp$ is a thick polar space Γ of rank 1 or 2, respectively (this follows from Proposition 5.11 of [**PV24**]). Hence, there exists a point or line U'' in Γ opposite both U and U'. The projectivity of upper residues $U \barwedge U' \barwedge U'' \barwedge U$ is the identity, showing the assertion in this case.

Now suppose that Δ is embedded in $\mathsf{PG}(V)$, with V minimal; that is, the associated polarity is non-degenerate. In this case, the subspace Γ of Δ, induced by the subspace of $\mathsf{PG}(V)$ spanned by U and U', is a polar space distinct from a hyperbolic space of odd rank. Therefore, Γ contains a singular subspace U'' opposite both U and U'. Since the associated polarity in $\mathsf{PG}(V)$ is non-degenerate, we have $U'' \subseteq (U^\perp \cap U'^\perp)^\perp$, and so $U \barwedge U' \barwedge U'' \barwedge U$ is the identity. Now the assertion follows from Observation 2.5.0.8. □

10.2.4. In Metasymplectic Spaces

Proof of Theorem I

Theorem I. *Let p be a point of a metasymplectic space Γ. Then $\Pi(p) = \Pi^+(p)$, as soon as there exists an imaginary line in Γ of size at least 3 which contains p.*

Proof. Let p, q, r be three points of the same imaginary line. Then all points symplectic to both p and q are also symplectic to r, which implies that $p \overline{\wedge} q \overline{\wedge} r \overline{\wedge} p$ fixes each symplecton through p and is hence the identity. Now apply Observation 2.5.0.8. □

From Lemma 10.2.2.3 it follows now, for a simplex S of type $\{1\}, \{1,4\}$, or $\{1,3,4\}$ of $\mathsf{F}_4(\mathbb{K}, \mathbb{A})$ — with $\mathbb{K}$ a field and $\mathbb{A}$ an alternative quadratic division algebra over $\mathbb{K}$ — that $\Pi(F) = \Pi^+(F)$. For simplices of type $\{1,4\}$, we will come back to this in an explicit way in the proof of Lemma 10.10.0.1.

Note that the condition stated in Theorem I is not necessary, as for split buildings the conclusion will hold without the condition being satisfied (see the first line of row (B3) in Table 9).

10.3. General Reduction Theorems

In principle, to determine the projectivity groups, one has to consider arbitrarily long sequences of perspectivities. However, the following results will lead to the fact that projectivity groups are generated by self-projectivities, which are products of at most four perspectivities in a particular sequence. We will need the following definition.

Definition 10.3.0.1 (**Geometric sets of automorphisms**)**.** The *fix set* of an automorphism ρ of a building Δ is the set of simplices fixed under ρ. Two automorphisms φ_1 and φ_2 are called *congruent* if their fix sets are isomorphic (using a type-preserving automorphism of the building). Conjugate automorphisms (with respect to a type-preserving automorphism) are examples of congruent automorphisms. Congruence is an equivalence relation, and an equivalence class Π is called a *geometric set of automorphisms*.

We are going to use this notion only for rather large fix sets, and give some examples in the following.

Examples 10.3.0.2.

- Let U and U' be two complementary subspaces of the projective space $\mathsf{PG}(r, \mathbb{K})$ (that is, U and U' generate the whole space and are disjoint). Then the set of nontrivial collineations pointwise fixing $U \cup U'$, together with all their conjugates, is a geometric set of automorphisms of $\mathsf{PG}(r, \mathbb{K})$.

- The set of all symplectic polarities of a given projective space is a geometric set of dualities. This follows from the fact that such polarities are characterized by the property that each point is mapped onto a hyperplane it is contained in.
- Since a parabolic subquadric of Witt index $r-1$ of a given hyperbolic quadric of Witt index r defines a geometric hyperplane, one deduces that the set of parabolic polarities of a given hyperbolic quadric is geometric.
- Let U and U' be two opposite maximal singular subspaces of a hyperbolic quadric. Then it is well known that every point not contained in either U or U' is contained in a unique line joining a point of U and a point of U'. One can use this property to deduce that if a non-trivial homology pointwise fixing $U \cap U'$ fixes a subspace S, then either $S \cap U$ and $S \cap U'$ generate S. The set of all non-trivial homologies of a hyperbolic quadric pointwise fixing two opposite maximal singular subspaces of a given type, is geometric.
- Recall from Definition 5.7.2.5 that for buildings of type E_6, there is a special type of diagram automorphism, called a *symplectic polarity*, which is an involution centralizing a split group of type F_4 over the same field $\mathbb{K}$. All symplectic polarities are conjugate (see [**DSV18**]), and as a consequence of the main results of [**Van12**], every graph automorphism having a fix set isomorphic to the fix set of a symplectic polarity is a symplectic polarity. The set of symplectic polarities of a building of type E_6 is geometric.
- The Chevalley group $\mathsf{PGE}_7(\mathbb{K})$ contains elements that pointwise fix the union of two opposite paras in the associated Lie incidence geometry of type $\mathsf{E}_{7,1}$, see Lemma 5.7.3.10. The set of such automorphisms distinct from the identity is a geometric set.

10.3.1. Reduction to the Product of Three Perspectivities

10.3.1.1. For Thick Spherical Buildings

Proof of Theorem J

Theorem J. *Let Δ be a thick spherical building over the type set I and let $J \subseteq I$ be a self-opposite type. Suppose that for each quadruple of simplices of type J, there exists a simplex of type J opposite all the given simplices. Let F, F', F'' be three pairwise opposite simplices of type J and denote by θ_0 the projectivity $F \barwedge F' \barwedge F'' \barwedge F$. Denote by $\Pi_3(F)$ the set of all self-projectivities of F of length 3, and suppose that $\Pi_3(F)$ is geometric. Then $\Pi(F) = \langle \Pi_3(F) \rangle$ and $\Pi^+(F) = \langle \theta_0^{-1}\theta \mid \theta \in \Pi_3(F) \rangle$.*

Proof. It is clear that said groups are subgroups of the respective projectivity groups. Now we claim that every self-projectivity of F of length ℓ is the product of an odd or even number of members of $\Pi_3(F)$, according to whether ℓ is odd or even. First note that if F^* is a simplex of type J and $\theta\colon \mathsf{Res}(F) \to \mathsf{Res}(F^*)$ is an isomorphism, then $\theta\Pi_3(F^*)\theta^{-1} = \Pi_3(F)$, by the fact that $\Pi_3(F)$ is geometric.

Now let $F = F_0 \overline{\wedge} F_1 \overline{\wedge} \cdots \overline{\wedge} F_\ell = F$ be a self-projectivity of length ℓ. Suppose $\ell \geq 5$. Let G be opposite all of F_0, F_1, F_2 and F_3. Put $\theta_i := F_0 \overline{\wedge} G \overline{\wedge} F_i$ and $\rho_i := F_i \overline{\wedge} G \overline{\wedge} F_{i-1} \overline{\wedge} F_i$, for $i \in \{1, 2, 3\}$. Then we have

$$F_0 \overline{\wedge} F_1 \overline{\wedge} F_2 \overline{\wedge} F_3 = \theta_1 \rho_1 \theta_1^{-1} \cdot \theta_2 \rho_2 \theta_2^{-1} \cdot \theta_3 \rho_3 \theta_3^{-1} \cdot \theta_3.$$

Hence, we can replace $F_0 \overline{\wedge} F_1 \overline{\wedge} F_2 \overline{\wedge} F_3$ by the product of three members of $\Pi_3(F)$ and the projectivity $\theta_3 = F \overline{\wedge} G \overline{\wedge} F_3$ of length 2. So, the claim will follow inductively if we show it for $\ell = 4$. If $\ell = 4$, then we have the additional perspectivity $F_3 \overline{\wedge} F_0$. We put $\rho_4 := F_0 \overline{\wedge} G \overline{\wedge} F_3 \overline{\wedge} F_0$. Then we have

$$F_0 \overline{\wedge} F_1 \overline{\wedge} F_2 \overline{\wedge} F_3 \overline{\wedge} F_0 = \theta_1 \rho_1 \theta_1^{-1} \cdot \theta_2 \rho_2 \theta_2^{-1} \cdot \theta_3 \rho_3 \theta_3^{-1} \cdot \rho_4,$$

which is a product of four, hence an even number of, elements of $\Pi_3(F)$. Now Theorem J follows, noting that every product $\theta_1 \theta_2$ of members of $\Pi_3(F)$ can be written as the product $(\theta_0 \theta_1^{-1})^{-1} \cdot (\theta_0 \theta_2)$ of two automorphisms of the form $\theta_0 \theta$, where $\theta \in \Pi_3(F)$. □

We will usually apply this theorem when all members of $\Pi_3(F)$ are type-interchanging involutions. In that case, $\Pi^+(F)$ will be the intersection of $\Pi(F)$ and the group of type-preserving collineations.

In Theorem J, the condition that we find a simplex opposite four given simplices is automatically given for buildings of thickness at least 5 with Proposition 7.1.0.1. For the simply laced case, this just means that the building is not defined over the fields $\mathbb{F}_2$ or $\mathbb{F}_3$. Over $\mathbb{F}_2$, the projectivity groups will always already be determined by Proposition 2.10.0.2 (or Theorem 2.10.0.3). For the case of $\mathbb{F}_3$, we proved more generally that if the simply laced thick spherical building is defined over the finite field $\mathbb{F}_q$, then we can find a simplex opposite $q + 1$ given simplices of certain given types — all cases were covered in Theorem B and Theorem D.

10.3.1.2. For Polar Spaces

We will give a translation for Theorem J for polar spaces.

Theorem J'. *Let Δ be a polar space of rank r and let j, with $0 \leq j < r$, be an arbitrary natural number.*

- *If $j > 0$, then suppose that for each quadruple of singular subspaces of dimension j in Δ, containing at least one opposite pair, there exists a singular subspace of dimension j that is opposite all the members of the given quadruple.*
- *If $j = 0$, suppose the same conclusion holds for each quadruple of points, with the property that the pairwise intersections of the perps are not all the same.*
- *Let F, F', F'' be three pairwise opposite singular subspaces of dimension j and denote by θ_0 the projectivity $F \overline{\wedge} F' \overline{\wedge} F'' \overline{\wedge} F$ of upper residues, if $j < r-1$, and of lower residues, if $j = r-1$.*
- *Denote by $\Pi_3(F)$ the set of all corresponding self-projectivities of F of length 3 and suppose that $\Pi_3(F)$ is geometric.*

Then $\Pi(F) = \langle \Pi_3(F) \rangle$ *and* $\Pi^+(F) = \langle \theta_0^{-1}\theta \mid \theta \in \Pi_3(F) \rangle$.

The conditions on the quadruple of singular subspaces do not appear in Theorem J. However, in the proof, each considered quadruple (F_0, F_1, F_2, F_3) is part of a chain of perspectivities

$$F_0 \mathbin{\overline{\wedge}} F_1 \mathbin{\overline{\wedge}} F_2 \mathbin{\overline{\wedge}} F_3.$$

Hence, F_0 is opposite F_1, which is opposite F_2, and F_2 is opposite F_3. Also, if $j = 0$, and if p_0, p_1, p_2 are points such that

$$p_0^\perp \cap p_1^\perp = p_1^\perp \cap p_2^\perp, \quad \text{then} \quad p_0 \mathbin{\overline{\wedge}} p_1 \mathbin{\overline{\wedge}} p_2 = p_0 \mathbin{\overline{\wedge}} p_2.$$

So we may shorten the chain without altering the projectivity defined by the chain.

10.3.2. Reduction to the Product of Four Perspectivities

One might wonder why one would bother to reduce the computation of the special projectivity groups to the computation of self-projectivities $F \mathbin{\overline{\wedge}} F_2 \mathbin{\overline{\wedge}} F_3 \mathbin{\overline{\wedge}} F_4 \mathbin{\overline{\wedge}} F$ of length 4, when we can reduce it to the computation of those of length 3 with Theorem J. The reason is, firstly, that in general, a generic subspace opposite two given subspaces has an algebraically complicated form with many parameters (especially when the dimension of F is large). In the length 4 case (see Remark 10.3.2.4), there are essentially only two single parameters: one to fix F_3, and one to fix F_4, whereas $F, F_2, F \cap F_3$ and $F_2 \cap F_4$ can be chosen freely. Secondly, the conditions are different, and sometimes those of Theorem K are easier to meet than those of Theorem J.

10.3.2.1. For Thick Spherical Buildings

Proof of Theorem K

Theorem K. *Let* Δ *be a thick spherical building over the type set* I *and let* $J \subseteq I$ *be a self-opposite type. Suppose that for each pair of simplices* F, F' *of type* J*, the subgraph* $\Gamma_J^{\{F,F'\}}$ *of* Γ_J*, induced by the vertices opposite both* F *and* F'*, is connected. Suppose that there is a simplex of type* J *opposite any given set of three simplices of type* J*. Let* F *be a given simplex of type* J*. Denote by* $\Pi_4(F)$ *the set of all self-projectivities* $F \mathbin{\overline{\wedge}} F_2 \mathbin{\overline{\wedge}} F_3 \mathbin{\overline{\wedge}} F_4 \mathbin{\overline{\wedge}} F$ *of* F *of length* 4 *with* $F \sim F_3$*,* $F_2 \sim F_4$*. Suppose that* $\Pi_4(F)$ *is geometric. Then* $\Pi^+(F) = \langle \Pi_4(F) \rangle$.

Proof. We first prove the following property for four simplices F_1, F_2, F_3, F_4, where $\mathsf{typ}(F_1) = \mathsf{typ}(F_3) = J$, and both F_2 and F_4 are opposite both F_1 and F_3.

(*) *The projectivity* $\rho\colon F_1 \mathbin{\overline{\wedge}} F_2 \mathbin{\overline{\wedge}} F_3 \mathbin{\overline{\wedge}} F_4$ *can be written as a product of a perspectivity* $F_1 \mathbin{\overline{\wedge}} F_4$ *and conjugates of members of* $\Pi_4(F_1)$.

Indeed, let

$$F_1 = F_1' \sim F_2' \sim \cdots \sim F_n' = F_3$$

be a path in $\Gamma_J^{\{F_2,F_4\}}$. Define

$$\rho_i\colon F_4 \mathbin{\overline{\wedge}} F_i' \mathbin{\overline{\wedge}} F_2 \mathbin{\overline{\wedge}} F_{i+1}' \mathbin{\overline{\wedge}} F_4, \quad i \in \{1, 2, \ldots, n-1\}.$$

Denote by ρ_0 the perspectivity $F_1 \barwedge F_4$. Then it is elementary to see that

$$\rho = \rho_0\rho_1\rho_2\cdots\rho_{n-1}.$$

So, since $\Pi_4(F)$ is geometric, it suffices to show that each ρ_i can be written as the product of conjugates of members of $\Pi_4(F_1)$. It follows from letting $(F_4, F_i', F_2, F_{i+1}')$ play the role of (F_1, F_2, F_3, F_4) in the previous argument that ρ_i is a product of conjugates of members of $\Pi_4(F_{i+1}')$. Hence, (*) is proved.

Now let

$$\rho\colon F \barwedge F_2 \barwedge F_3 \barwedge \cdots \barwedge F_{2\ell-1} \barwedge F_{2\ell} \barwedge F$$

be an arbitrary even projectivity. We prove by induction on $\ell \in \{1, 2, \ldots\}$ that ρ is the product of conjugates of members of $\Pi_4(F)$. This is trivial for $\ell = 1$, and it is equivalent to (*) for $\ell = 2$. So let $\ell \geq 3$. Select a simplex F_2' opposite each of F, F_3, and F_5 (since these all have the same type, this is still possible if J is not self-opposite). Setting

$$\begin{cases} \rho_1^*\colon F \barwedge F_2 \barwedge F_3 \barwedge F_2' \barwedge F, \\ \rho_2^*\colon F_2' \barwedge F_3 \barwedge F_4 \barwedge F_5 \barwedge F_2', \\ \rho'\colon F \barwedge F_2' \barwedge F_5 \barwedge F_6 \cdots \barwedge F_{2\ell} \barwedge F, \end{cases}$$

we see that, if $\rho_0\colon F \barwedge F_2'$, we have

$$\rho = \rho_1^* \cdot (\rho_0\rho_2^*\rho_0^{-1}) \cdot \rho',$$

where we know by the induction hypothesis that all factors are products of members of $\Pi_4(F)$, using the fact that $\Pi_4(F)$ is geometric (and hence closed under conjugation). □

This method of determining $\Pi^+(F)$ "explains" these groups in a way equivalent to the algebraic approach of Section 2.11. Of course, in order to be able to apply Theorem K, we have to check the conditions in the various cases in which we want to use Theorem K.

10.3.2.2. For Polar Spaces

Definition 10.3.2.3 (***s*-space-graph**)**.** Let Δ be a polar space of rank $n \geq 2$. Let S be a set of singular subspaces of dimension s in Δ. We define the *s-space-graph* $\Gamma(S) = (S, \sim)$, where $\sim$ denotes adjacency, as follows:

(V) The vertices are the elements of S.

(E)
- For $s = 0$, we draw an edge between two vertices in $\Gamma(S)$ if the corresponding points are collinear in Δ.
- For $s \in [1, n-2]$, we draw an edge between two vertices in $\Gamma(S)$ if the corresponding subspaces in Δ intersect in a subspace of dimension $s-1$ and are both contained in a common subspace of dimension $s+1$.
- For $s = n-1$, we draw an edge between two vertices in $\Gamma(S)$ if the corresponding maximal subspaces in Δ intersect in a subspace of dimension $s-1$.

Theorem K'. *Let Δ be a polar space of rank r and let j, with $0 \leq j < r$, be an arbitrary natural number. Suppose that for each pair of singular subspaces F, F' of dimension j, the graph $\Gamma(S)$, where S is the set of singular subspaces opposite both F and F', is connected. Suppose also that there exists a singular subspace opposite any given set of three singular subspaces of dimension j. Let F be a given singular subspace of dimension j. Denote by $\Pi_4(F)$ the set of all self-projectivities $F \barwedge F_2 \barwedge F_3 \barwedge F_4 \barwedge F$ of F of length 4 with $F \sim F_3$, $F_2 \sim F_4$. Suppose that $\Pi_4(F)$ is geometric. Then $\Pi^+(F) = \langle \Pi_4(F) \rangle$.*

Remark 10.3.2.4. In Theorem K', we may also assume that $F \cap F_3$ and $F_2 \cap F_4$ are opposite. Indeed, if not, then we can select a subspace F_3' of dimension j in $\langle F, F_3 \rangle$, disjoint from $\mathsf{proj}_F(F_2 \cap F_4)$. We then write

$$F \barwedge F_2 \barwedge F_3 \barwedge F_4 = (F \barwedge F_2 \barwedge F_3' \barwedge F_4 \barwedge F) \cdot (F \barwedge F_4) \cdot (F_4 \barwedge F_3' \barwedge F_2 \barwedge F_3 \barwedge F_4) \cdot F_4 \barwedge F.$$

Likewise, if $j \leq r-2$, we may also assume that $\langle F, F_3 \rangle$ and $\langle F_2, F_4 \rangle$ are opposite. Indeed, noting that a singular subspace U_1 through $F \cap F_3$ is opposite a singular subspace U_2 through $F_2 \cap F_4$ if and only if $U_1 \cap (F \cap F_3)^\perp \cap (F_2 \cap F_4)^\perp$ is opposite $U_2 \cap (F \cap F_3)^\perp \cap (F_2 \cap F_4)^\perp$ in the polar space $(F \cap F_3)^\perp \cap (F_2 \cap F_4)^\perp$ (where we may assume, due to the previous paragraph, that $F \cap F_3$ and $F_2 \cap F_4$ are opposite), it suffices to verify the claim for $i = 0$.

First, suppose the line $L_1 := \langle F, F_3 \rangle$ intersects the line $L_2 := \langle F_2, F_4 \rangle$ in a point p. Select a plane π through L_1 not in a 3-space with L_2, and choose a point F_3' in π not on L_1 and not collinear to L_2. As above, we may substitute F_3 by F_3' in our sequence of perspectivities. Now, neither $\langle F, F_3' \rangle$ nor $\langle F_3, F_3' \rangle$ intersects L_2, and we may assume now that L_1 does not intersect L_2. Then there is a unique point p_2 on L_2 collinear to all points of L_1, defining the plane $\pi := \langle p_2, L_1 \rangle$. We select a plane α through L_1 not in a 3-space with π. It follows that $\mathsf{proj}_\alpha L_2 \in L_1$. Thus, we may consider any point F_3' of $\alpha \setminus L_1$, and we find that L_2 is opposite both $\langle F, F_3' \rangle$ and $\langle F_3, F_3' \rangle$. This completes the argument.

10.3.2.3. For Metasymplectic Spaces

We define the *s-space-graph* $\Gamma(S)$ of a metasymplectic space in exactly the same way as in Definition 10.3.2.3.

Theorem K". *Let Δ be a metasymplectic space. Suppose that for each pair of points p, p', the graph $\Gamma(S)$, where S is the set of points opposite both p and p', is connected. Suppose that there exists a point opposite any given triple of points. Let p be a given point. Denote by $\Pi_4(p)$ the set of all self-projectivities $p \barwedge p_2 \barwedge p_3 \barwedge p_4 \barwedge p$ of p of length 4 with $p \perp p_3$, $p_2 \perp p_4$, and pp_3 opposite p_2p_4. Suppose that $\Pi_4(F)$ is geometric. Then $\Pi^+(F) = \langle \Pi_4(F) \rangle$.*

Proof. The statement, without the condition that pp_3 is opposite p_2p_4, is Theorem K, specialized to buildings of type F_4 and vertices of type 1 or 4.

Now let $p \barwedge p_2 \barwedge p_3 \barwedge p_4 \barwedge p$ be a self-projectivity of p with $p \perp p_3$ and $p_2 \perp p_4$. Let L_3 be the projection of pp_3 onto the residue of p_2. Set $L_4 := p_2p_4$. In $\mathsf{Res}_\Delta(p_2)$, the elements L_3 and L_4 have distance $0, 1, 2$, or 3 from each other. If they have distance

3, then [**Tit74**, Proposition 3.29] implies that pp_3 and p_2p_4 are opposite. Hence, we may assume that their distance d is $0, 1$, or 2. Then there exists a line L_2 through p_2 at distance $d+1$ from L_3 in $\mathsf{Res}_\Delta(p)$ and at distance 1 from L_4 in that residue. That means L_2 is coplanar with L_4 in Δ. Let π_0 be the plane spanned by L_2 and L_4. Let L_1 be the projection of L_2 onto (the residue of) p, and let q_2 be a point on L_2 opposite p. The projection of the line p_4q_2 from p_4 onto p is contained in the projection of π_0 onto p, is different from the projection of p_4p_2 onto p, and hence has distance $d+1$ to L_3. If we define the *width of* $p \barwedge p_2 \barwedge p_3 \barwedge p_4 \barwedge p$ to be the distance d in $\mathsf{Res}_\Delta(p)$ between the line pp_3 and the projection of p_2p_4 onto p, then we can write

$$p \barwedge p_2 \barwedge p_3 \barwedge p_4 \barwedge p = (p \barwedge p_2 \barwedge p_3 \barwedge q_2 \barwedge p) \cdot (p \barwedge q_2 \barwedge p_3 \barwedge p_4 \barwedge p),$$

where the width of both self-projectivities of p on the right-hand side is $d+1$ (look at the inverse of the second to see this). An induction argument on d proves the assertion. □

10.4. Preparations and Auxiliary Results

In this section, we check the conditions of Theorem J', Theorem J", Theorem K, Theorem K' and Theorem K", in the cases where we shall apply them.

10.4.1. Conditions for Reduction for Polar Spaces

Regarding Theorem J', we can be brief. The following is a direct consequence of Theorem B.

Proposition 10.4.1.1. *Let Δ be a polar space of rank $r \geq 2$. First suppose that each line contains at least four points and let $0 \leq j < r-1$. Then, if $j > 0$, for each quadruple of singular subspaces of dimension j containing at least one opposite pair, there exists a singular subspace of dimension j opposite all the members of the given quadruple. If $j = 0$, the same conclusion holds for each quadruple of points with the property that the pairwise intersections of the perps are not all the same. Secondly, suppose that each submaximal subspace is contained in at least four maximal singular subspaces and $r \geq 3$, then each quadruple of maximal singular subspaces containing at least one opposite pair admits a common opposite singular subspace.*

Now we consider the main condition in Theorem K'. First, we treat the rank 2 case, although we only consider the projectivity groups for rank at least 3; we need the rank 2 case for induction purposes. Strictly speaking, we could assume that the rank 2 polar spaces we deal with are Moufang, but we prove the result for all generalized quadrangles (with at least four points per line and four lines through a point, respectively).

Proposition 10.4.1.2. *Let Δ be a thick polar space of rank 2. Let p_1 and p_2 be two points. Let S be the set of points opposite both p_1 and p_2. Then $\Gamma(S)$ is connected if each line contains at least four points.*

PROOF. Assume each line contains at least four points. Let q_1 and q_2 be two points in $\Delta \setminus (p_1^\perp \cup p_2^\perp)$. If q_1 and q_2 are collinear in Δ, then they are also collinear in $\Delta \setminus (p_1^\perp \cup p_2^\perp)$.

Suppose q_1 and q_2 are opposite in Δ. Let L be a line through q_1 in Δ. If $\mathsf{proj}_L(q_2)$ is not equal to $\mathsf{proj}_L(p_1)$ or $\mathsf{proj}_L(p_2)$, then $\mathsf{proj}_L(q_2)$ is a vertex of $\Delta \setminus (p_1^\perp \cup p_2^\perp)$, and the lines $q_2\mathsf{proj}_L(q_2)$ and $q_1\mathsf{proj}_L(q_2)$ give rise to edges in $\Gamma(\Delta \setminus (p_1^\perp \cup p_2^\perp))$ that form a path between the vertices corresponding to q_1 and q_2 via the vertex corresponding to $\mathsf{proj}_L(q_2)$.

Only if for every line L through q_1 it is the case that $\mathsf{proj}_L(q_2)$ is equal to either $\mathsf{proj}_L(p_1)$ or $\mathsf{proj}_L(p_2)$, we cannot immediately find such a path. So assume this is the case. Then every point in $q_1^\perp \cap q_2^\perp$ is either collinear to p_1 or to p_2.

Suppose $p_1^\perp \cap p_2^\perp \cap q_1^\perp \cap q_2^\perp \neq \varnothing$ and let x be collinear to all of p_1, p_2, q_1, q_2. Let L be a line through q_1 not containing x. Without loss of generality, we may assume that the unique point y on L collinear to q_2 is collinear to p_1. Let s_1 be a point on L not collinear to p_2, and distinct from q_1 and y. Let s_2 be the point on q_2x collinear to s_1. Neither s_1 nor s_2 is collinear to either p_1 or p_2, and we have the path $q_1 \sim s_1 \sim s_2 \sim q_2$.

Hence, we may suppose that each point of $q_1^\perp \cap q_2^\perp$ is collinear to either p_1 or p_2, but never to both. Without loss of generality, we may assume that p_1 is collinear to at least 2 points o_1, o_2 of $q_1^\perp \cap q_2^\perp$. If there are at least 5 points on a line, then there is a point r_1 on q_1o_1 distinct from all of $q_1, o_1, \mathsf{proj}_{q_1o_1}p_2$ and $\mathsf{proj}_{q_1o_1}\mathsf{proj}_{q_2o_2}p_2$. Then we have the path $q_1 \sim r_1 \sim \mathsf{proj}_{q_2o_2}r_1 \sim q_2$. Consequently, we may assume that there are exactly 4 points per line, and hence, in view of the main result of [**Bro91**], there is a finite number of lines through each point, say n. By assumption, p_1 is collinear to $m \geq 2$ points $o_1, o_2, \ldots, o_m$ of $q_1^\perp \cap q_2^\perp$. Hence, there are $n - m$ lines through p_2 containing a point of $q_1^\perp \cap q_2^\perp$. The other m lines through p_2 each have to meet at least one of the lines q_1o_i, q_2o_i, $1 \leq i \leq m$, and each such line must meet one of those m lines through p_2. It follows that there is a line L through p_2 meeting, without loss of generality, the line q_1o_1 and q_2o_2. Then the unique points r_1 and r_2 on q_1o_1 and q_2o_2, respectively, distinct from $q_1, q_2, o_1, o_2, q_1o_1 \cap L$ and $q_2o_2 \cap L$, are collinear and opposite both p_1 and p_2. Hence, we have the path $q_1 \sim r_1 \sim r_2 \sim q_2$. □

Dually, we have:

Corollary 10.4.1.3. *Let Δ be a thick polar space of rank 2. Let L_1 and L_2 be two lines. Let S be the set of lines opposite both L_1 and L_2. Then $\Gamma(S)$ is connected if each point is on at least four lines.*

Now we treat the higher rank cases.

Proposition 10.4.1.4. *Let Δ be a thick polar space of rank $n \geq 3$. Let U_1 and U_2 be two singular subspaces of dimension s. Let S_{U_1,U_2} be the set of all singular subspaces of dimension s opposite both U_1 and U_2. Then $\Gamma(S_{U_1,U_2})$ is connected for $s \leq n-2$ if each line has at least four points (this additional condition is not needed for $s = 0$) and for $s = n-1$, if either each submaximal singular subspace is contained in at least four maximal singular subspaces or each line contains at least four points.*

PROOF. The proof goes with induction on the rank n, the base case being Proposition 10.4.1.2 and Corollary 10.4.1.3. However, for $s = 0$, we provide an independent proof, neglecting the additional conditions on the sizes of the lines.

Case 1: $s = 0$**.** Let p_1 and p_2 be two points. Let $\Delta \setminus (p_1^\perp \cup p_2^\perp)$ be the point-line geometry containing all points of Δ that are not collinear to either p_1 or p_2 and all lines of Δ between these points. These lines are exactly the lines L that are not in a plane with either p_1 or p_2 and for which $L \setminus (\mathsf{proj}_L(p_1) \cup \mathsf{proj}_L(p_2))$ contains more than one point. We have to show that the point graph of $\Delta \setminus (p_1^\perp \cup p_2^\perp)$ is connected.

Let q_1 and q_2 be two arbitrary points of $\Delta \setminus (p_1^\perp \cup p_2^\perp)$. If q_1 and q_2 are collinear in Δ, then they are also collinear in $\Delta \setminus (p_1^\perp \cup p_2^\perp)$. Suppose q_1 and q_2 are opposite in Δ. As in the proof of Proposition 10.4.1.2, we may assume that every point in $q_1^\perp \cap q_2^\perp$ is either collinear to p_1 or to p_2. Since q_1 and q_2 are opposite, $q_1^\perp \cap q_2^\perp =: \Lambda$ defines a thick polar space of rank $n - 1 \geq 2$. Both $p_1^\perp \cap \Lambda$ and $p_2^\perp \cap \Lambda$ define geometric hyperplanes that we will denote by H_1 and H_2, and we have $\Lambda = H_1 \cup H_2$.

Suppose H_1 and H_2 are proper geometric hyperplanes. Then we obtain a contradiction, because a polar space can never be the union of two proper geometric hyperplanes (see Exercise 2.5 in [**Van24**]): Let x be a point of $H_1 \setminus H_2$ such that $x^\perp \neq H_1$. Then H_1 induces a proper geometric hyperplane in $\mathsf{Res}_\Lambda(x)$ and H_2 has to contain the complement. Let M be a line through x that is not contained in H_1. Then $M \setminus \{x\}$ has to be contained in H_2, and with that, x has to be contained in H_2, which is a contradiction.

So we may assume, without loss of generality, $p_1^\perp \cap \Lambda = \Lambda$. Since H_2 is a hyperplane, it contains two opposite points o_1 and o_2, and since $H_2 \subseteq H_1 = \Lambda$, the points o_1 and o_2 are both collinear to q_1, q_2, p_1 and p_2, and the lines q_1o_1 and q_2o_2 are opposite. Let x_1 be a point on $q_1o_1 \setminus \{q_1\}$ that is not in Λ. The projection of x_1 onto q_2o_2 is a point x_2 that is not in Λ and that does not coincide with q_2. Furthermore, x_1 and x_2 are not collinear to either p_1 or p_2 and thus are contained in $\Delta \setminus (p_1^\perp \cup p_2^\perp)$. The lines q_1x_1, x_1x_2 and q_2x_2 give rise to edges in $\Gamma(S_{p_1,p_2})$ that form a path $q_1 \sim x_1 \sim x_2 \sim q_2$ between the vertices corresponding to q_1 and q_2.

This concludes the proof for the case $s = 1$. For Cases 2 and 3 below, we assume that each line has at least four points. But before moving to these other cases, we prove a common claim under our general assumptions.

Claim: *If* $1 \leq s \leq n-1$*, then different components of* $\Gamma(S_{U_1,U_2})$ *only contain vertices corresponding to disjoint subspaces.* Indeed, let V_1 and V_2 belong to $\Gamma(S_{U_1,U_2})$, with $x \in V_1 \cap V_2$. The projection of x onto U_i is a hyperplane H_i of U_i, $i \in \{1, 2\}$. With Proposition 2.5.0.4, it follows that every subspace X through x that is opposite $\langle x, H_i \rangle$ in $\mathsf{Res}(x)$ is opposite U_i in Δ. Since $\mathsf{Res}(x)$ is a polar space of rank $n - 1$, it follows from the induction hypothesis that we can find a path between the vertices corresponding to V_1 and V_2 in $\Gamma(S_{U_1,U_2})$. The claim is proved.

Case 2: $s = 1$**.** Remember the rank of Δ is at least 3, so Δ contains planes. Let L_1 and L_2 be two lines in Δ and let S_{L_1,L_2} be the set of all lines opposite both L_1 and L_2. Let J_1 and J_2 be two lines of S_{L_1,L_2}.

(I) If J_1 and J_2 intersect in a point, then by the above claim, we can find a path in $\Gamma(S_{L_1,L_2})$ connecting J_1 and J_2.

(II) Now suppose J_1 and J_2 do not intersect in a point, but there is some point $x \in J_1$, collinear to all points of J_2. Set $\pi := \langle x, J_2 \rangle$. Since J_j is opposite L_i, $i, j = 1, 2$, the projection of L_i onto π is a point $u_i \in \pi \setminus (J_2 \cup \{x\})$. Select a line J_1' through x avoiding both u_1 and u_2, then J_1' is opposite both L_1 and L_2 and intersects both J_1 and J_2. By (I) we can connect J_1 with J_1' and J_1' with J_2.

(III) Finally, suppose that J_1 and J_2 are opposite. Let α_1 and β_1 be two arbitrary planes containing J_1, and let α_2 and β_2 be the planes through J_2, intersecting α_1 and β_1, respectively. Set $a := \alpha_1 \cap \alpha_2$ and $b := \beta_1 \cap \beta_2$. If either a or b is not collinear to either L_1 or L_2, say a, then we find a line J_1' through a in α_1 opposite both L_1 and L_2, and we are reduced to Cases (I) and (II). In the other case, say $a \perp L_1$ and $b \perp L_i$, $i \in \{1, 2\}$, there is a unique point a' in α_1 collinear to L_2, and there is a unique point b' in β_2 collinear to L_j, $\{j\} = \{1, 2\} \setminus \{i\}$. Since lines have at least four points, we can now find a line L_2' in β_2 distinct from both J_2 and $\mathsf{proj}_{\beta_2} a'$, and avoiding both b and b'. Then we can find a line J_1' in α_1 through $\mathsf{proj}_{\alpha_1} J_2'$ avoiding a and a'. Hence, both J_1' and J_2' are opposite both L_1 and L_2 and by (I), L_1 and L_1' are connected in $\Gamma(S_{L_1,L_2})$ and L_2 and L_2' are connected in $\Gamma(S_{L_1,L_2})$; by (II) J_1' and J_2' are also connected in $\Gamma(S_{L_1,L_2})$. Hence, L_1 and L_2 are connected in $\Gamma(S_{L_1,L_2})$.

Case 3: $2 \leq s \leq n-1$**.** Let U_1 and U_2 be two singular subspaces of dimension s and V_1 and V_2 both opposite both U_1 and U_2. On top of the induction on the rank of Δ, we also perform an induction on the dimension s of U_1 and U_2. The base cases here are Case 1 and Case 2.

(I) If V_1 and V_2 are not disjoint, then by the above claim, we can find a path in $\Gamma(S_{U_1,U_2})$ connecting V_1 and V_2.

(II) Secondly, assume $V_1 \cap V_2 = \varnothing$. Let A_1 be some $(s-1)$-dimensional subspace in V_1. Then A_1 is opposite some $(s-1)$-dimensional subspaces B_1 of U_1 and B_2 of U_2. Let A_2 be some $(s-1)$-dimensional subspace in V_2 that is opposite both B_1 and B_2. Then we can find a path between A_1 and A_2 in $\Gamma(S_{B_1,B_2})$. Let $A_1 = X_1, X_2, \ldots, X_r = A_2$ be the $(s-1)$-dimensional subspaces corresponding to all vertices of that path, such that X_j is adjacent to X_{j+1} for $j \in \{1, 2, \ldots, r-1\}$. Then X_j is an $(s-1)$-dimensional subspace that intersects both X_{j-1} and X_{j+1} in $(s-2)$-dimensional subspaces and that is opposite both B_1 and B_2.

The projection of U_i onto X_j (for $j \in \{2, 3, \ldots, r-1\}$ and $i \in \{1, 2\}$) is a singular subspace W_{ij} of dimension s containing X_j. Let W_j be an s-dimensional subspace containing X_j, opposite both W_{1j} and W_{2j} in $\mathsf{Res}(X_j)$. Then, again using Proposition 2.5.0.4, W_j is opposite both U_1 and U_2. Set $W_1 := V_1$ and $W_r = V_2$. Since X_j intersects X_{j+1}, $j \in \{1, 2, \ldots, r-1\}$, and $X_j \subseteq W_j$, we find that W_j intersects W_{j+1} for all $j \in \{1, 2 \ldots, r-1\}$. Now we can use (I) again to find paths between W_j and W_{j+1} in $\Gamma(S_{U_1,U_2})$, for $j \in \{1, 2 \ldots, r-1\}$, which, taken together, form one big path between $W_1 = V_1$ and $W_r = V_2$ in $\Gamma(S_{U_1,U_2})$. This concludes the proof of Case 3.

For the final case, we assume that each submaximal singular subspace is contained in at least four maximal singular subspaces.

Case 4: $s = n-1$**.** Let M_1 and M_2 be two maximal singular subspaces and let N_1, N_2 be two maximal singular subspaces opposite both M_1 and M_2. As in the previous case, by the Claim above, we may assume that $N_1 \cap N_2 = \varnothing$, hence N_1 and N_2 are themselves opposite. We again proceed by induction on n, the case $n = 2$

being Corollary 10.4.1.3. So we assume $n > 2$. Select $x \in N_1$, and let N be the unique maximal singular subspace containing x and intersecting N_2 in a hyperplane of N_2. Then $M_i \cap N$ is at most a point, $i = 1, 2$, and we find a hyperplane H of N containing x and disjoint from both M_1 and M_2. Through H, there are at most two maximal singular subspaces not opposite either M_1 or M_2, and so there exists a maximal singular subspace N' through H, opposite both M_1 and M_2. Applying the claim above to N_1 and N', and to N' and N_2 (which meet in $H \cap N_2 \neq \varnothing$), Case 4 is proved. □

10.4.2. Conditions for Reduction for some Simply Laced Exceptional Cases

We now check the conditions of Theorem K in the three cases where we shall need it. In two of these cases, we can more generally prove that the graph induced on the complement of the union of two geometric hyperplanes is connected. Although we will not need this in the thesis, we thought it might have potential to be applied elsewhere, and so we include the (rather short) proof.

Lemma 10.4.2.5. *Let Δ be the thick spherical building over the field $\mathbb{K}$ ($\mathbb{K}$ of order $|\mathbb{K}| > 2$) of type either E_6 or E_7. Let $J = \{2\}$ if $\mathsf{typ}\,\Delta = \mathsf{E}_6$, and $J \in \{\{1\}, \{3\}\}$ if $\mathsf{typ}\,\Delta = \mathsf{E}_7$. Let v and v' be vertices of type J. Then the subgraph Γ'_J of Γ_J, induced on the vertices opposite both v and v', is connected. If $J \neq \{3\}$, then, more generally, the complement of the union of two geometric hyperplanes of the corresponding geometries of type $\mathsf{E}_{n,J}$, $n = 6, 7$, is connected.*

Proof. We start with the first part of the lemma. Select chambers C and C' of Δ containing v and v', respectively. Let u be a vertex of Γ'_J. In $\mathsf{Res}_\Delta(u)$, we can find a chamber opposite the projections of C and C' onto $\mathsf{Res}_\Delta(u)$ (use 7.1.0.1). That chamber is, by Proposition 2.5.0.4, opposite both C and C'. Hence, we have found a chamber C_u containing u opposite both C and C'. If u' is another vertex of $\Gamma_J^{v,v'}$, then we can also find a chamber $C_{u'}$ containing u' and opposite both C and C'. Now Proposition 2.3.1.13 implies that we can find a sequence of consecutively adjacent chambers, all opposite both C and C', connecting C_u with $C_{u'}$. The vertices of type J of two consecutive such chambers are either equal or adjacent in Γ_J. Moreover, since the chambers are opposite both C and C', their vertices of type J are opposite both v and v', as J is a self-opposite type. The first part of the lemma is proved.

For the second part, we start by proving the lemma for $J = \{2\}$ if $\mathsf{typ}\,\Delta = \mathsf{E}_6$, and for $J = \{1\}$ if $\mathsf{typ}\,\Delta = \mathsf{E}_7$. Let C be the complement of two proper geometric hyperplanes H_1, H_2 of the long root geometry Γ of type E_6 or E_7 over the field $\mathbb{K}$, with $|\mathbb{K}| > 2$. Let p and q be two points of C. A moment's thought reveals that we may assume that p and q are opposite in Γ. The equator geometry $E(p, q)$ is not the union of two geometric hyperplanes because lines contain at least 3 points, and hence, if $E(p, q)$ is contained in neither H_1 nor H_2, then there is some point $r \in E(p, q) \cap C$. Lemma 5.2.0.3 yields paths in C, connecting p with r and r with q. So, we may assume that $E(p, q) \subseteq H_1$. Since $E(p, q)$ contains planes, we find a point $x \in E(p, q) \cap H_1 \cap H_2$. Let ξ be the symp through p and x, and let ζ be a

symp opposite ξ through q. Set $H_{\xi,1} := H_1 \cap \xi$, $H_{\xi,2} := H_2 \cap \xi$, and let $H_{\xi,3}$ be the (pointwise) projection of $\zeta \cap H_1$ onto ξ; likewise, let $H_{\xi,4}$ be the pointwise projection of $\zeta \cap H_2$ onto ξ (projecting here means mapping to the symplectic points). Since the projection of q is x, we see that $H_{\xi,1} \cap H_{\xi,2} \ni x \notin H_{\xi,3} \cap H_{\xi,4}$. Hence, Lemma 5.2.0.2 yields a point $z \in C \cap \xi$, such that the unique point z' of ζ symplectic to z also belongs to C. Now Lemma 5.2.0.3 yields paths $p \perp\!\!\!\perp z \perp\!\!\!\perp z' \perp\!\!\!\perp q$, and the proof for these two cases is finished. □

10.4.3. Conditions for Reduction for Metasymplectic Polar Spaces

We now check the conditions of Theorem K". We will only need the result for points.

First, we present two lemmata for dual polar spaces. We only need them in rank 3, but the proofs are the same for general rank.

Lemma 10.4.3.6. *A dual polar space Γ is not the union of two proper geometric hyperplanes.*

Proof. Let H and H' be two geometric hyperplanes of Γ with $H \cup H' = X$, where X is the point set of Γ. We say that a *deep point* of H is a point of H such that $x^\perp \subseteq H$. Each point outside H belongs to H', as well as all points of H that are not deep. We now claim that a deep point x of H belongs to H', as soon as $x^\perp$ contains a point that is not deep in H. Indeed, let $y \perp x$ be a point that is not deep in H, and let L be a line through y not contained in H. Let ξ be a symp containing x and L.[2] Since x is deep, $H \cap \xi = x^\perp \cap \xi$. Hence, no point of $xy \setminus \{x\}$ is deep. We conclude that $xy \subseteq H'$, proving the claim. Hence, if H' is proper, then the set of deep points of H is closed under taking perps. Since Γ is connected, only X is a non-empty subset of points closed under taking perps. The lemma now follows. □

Lemma 10.4.3.7. *Let H and H' be two proper geometric hyperplanes of a dual polar space Γ. Then H contains a point x opposite some point $x' \in H'$.*

Proof. Let $y \in H$ be arbitrary. Suppose no point of H' is opposite y. Then $H' \subseteq y^{\not\equiv}$. Since both H' and $y^{\not\equiv}$ are hyperplanes, and since we cannot remove any point from $y^{\not\equiv}$ without losing the hyperplane property, it follows that $H' = y^{\not\equiv}$. Now let $x \in H$ be any other point of H different from y. Lemma 8.2.1.5 yields a point x' not opposite y, but opposite x. Then $x' \in H'$ and the assertion is proved. □

We are now ready to deal with the condition in Theorem K" about the connectivity of $\Gamma(S)$.

Lemma 10.4.3.8. *Let Δ be a metasymplectic space, in which either every line has at least four points, or each plane is contained in at least four symps. For each pair of points p_0, p_1, the graph $\Gamma(S)$, where S is the set of points opposite both p_0 and p_1, is connected.*

[2]Such a symp exists, since there must exist at least a point z on L that x is symplectic to. But then the symp $\xi(x, z)$ contains L, since x and z are both collinear to y.

PROOF. (I) Let q_0 and q_1 be two points opposite both p_0 and p_1. If q_0 and q_1 are collinear in Δ, then they are connected in $\Gamma(S)$. Suppose q_0 and q_1 are symplectic in Δ and denote the symp containing both of them by $\xi(q_0, q_1)$. Then no p_i, for $i \in \{0, 1\}$, can be contained in $\xi(q_0, q_1)$, and can also not be close to $\xi(q_0, q_1)$, since otherwise both q_0 and q_1 would not be opposite p_i. That means $p_i^{\perp\!\!\perp} \cap \xi(q_0, q_1)$ has to be a unique point r_i. We can find a path in the graph corresponding to $\xi(q_0, q_1) \setminus (r_0^\perp \cup r_1^\perp)$ between q_0 and q_1.

(II) Suppose q_0 and q_1 are special in Δ. Let q be the unique point collinear to both q_0 and q_1. Then we cannot find a path in $\Gamma(S)$ between q_0 and q_1 immediately if q is collinear to a point of $p_0^\perp$ or $p_1^\perp$. Let p be a point in $p_0^\perp \cup p_1^\perp$ collinear to q. We may assume $p \perp p_0$.

We have paths $p_0 \perp p \perp q \perp q_0$ and $p_0 \perp p \perp q \perp q_1$, and know that p_0 is opposite both q_0 and q_1. Therefore, we know that the pairs (p, q_0), (p, q_1), and (q, p_0) are special, and if p_1 has distance 2 to q, then the pair (q, p_1) is also special. We first consider this case and define p' as the unique point of $p_1^\perp \cap q^\perp$.

In the residue of q, we can see the lines pq, q_0q, q_1q, and $p'q$ as points. Since $\mathsf{Res}(q)$ is a dual polar space, we can apply Proposition 10.4.1.4 and find a path between the points q_0q and q_1q in $\mathsf{Res}(q)$, all members of which are locally opposite pq and $p'q$. In Δ we can see that path as a chain of planes, such that consecutive planes intersect in lines L locally opposite both pq and $p'q$. On each such line L, we choose a point distinct from both q and the projection of p_1 onto L. The thus-obtained path in Δ connects q_0 with q_1 within $p_0^{\equiv} \cap p_1^{\equiv}$.

(III) Suppose q_0 and q_1 are opposite in Δ. Set $Z_0 = q_0^\perp \cap q_1^{\bowtie}$. If $q \in Z_0$ is opposite both p_0 and p_1, then we can find paths from q_0 to q and, by Case (II), from q to q_1 in $\Gamma(S)$, and the concatenation is a path between q_0 and q_1. Suppose such a $q \in Z_0$ does not exist; in other words, $(p_0^{\not\equiv} \cup p_1^{\not\equiv}) \cap Z_0 = Z_0$. Now both $p_0^{\not\equiv}$ and $p_1^{\not\equiv}$ are geometric hyperplanes of Z_0. By Lemma 10.4.3.6, we may assume $Z_0 \subseteq p_0^{\not\equiv}$. Set $Y_0 = Z_0 \cap p_1^{\not\equiv}$. Likewise, defining $Z_1 = q_1^\perp \cap q_0^{\bowtie}$, one of $p_0^{\not\equiv}$ and $p_1^{\not\equiv}$ contains Z_1. The intersection of Z_1 with the other is denoted by Y_1. Let Y_1' be the projection of Y_1 onto Z_0. Then Lemma 10.4.3.7 yields a point $x_0 \in Y_0$ and a point $x_1' \in Y_1'$ with q_0x_0 locally opposite q_0x_1'. Projecting x_1' back onto Z_1, we obtain a point $x_1 \in Y_0$. Now pick $q_1' \in q_1x_1 \setminus \{x_1, q_1\}$ and note that, by this choice, $q_1' \in \Gamma(S)$. Then the projection q_0' of q_1' onto q_0x_0 is a point distinct from x_0 and hence belongs to $\Gamma(S)$. Since x_0 is special to x_1, Case (II) implies that x_0 and x_1 are connected in $\Gamma(S)$. Hence, $q_0 \perp x_0$ is connected to $q_1 \perp x_1$ in $\Gamma(S)$ and the lemma is proved. □

Now, we handle the condition in Theorem K" regarding the existence of a point opposite three given points. It follows from Proposition 7.1.0.1 that this condition is automatically satisfied whenever the building has no residues isomorphic to the unique projective plane of order 2 (with 3 points per line). It follows from Section 9.8.3 that only triples of points that form a geometric line have no common opposite. But such a triple is determined by any pair of its elements. Furthermore, it follows from the proof of Theorem K that the conclusion of Theorem K", possibly without the claim of pp_3 being opposite p_2p_3, holds for all projectivities that can be

written as a product

$$p_1 \overline{\wedge} p_2 \overline{\wedge} p_3 \overline{\wedge} \ldots \overline{\wedge} p_{2\ell-1} \overline{\wedge} p_{2\ell} \overline{\wedge} p_1,$$

such that we can find a common opposite for the triples

$$\{p_1, p_{2k-1}, p_{2k+1}\} \quad \text{for all} \quad k = 2, \ldots, \ell - 1.$$

Hence, if we were able to replace every subsequence

$$p_{2k-1} \overline{\wedge} p_{2k} \overline{\wedge} p_{2k+1}, \quad k = 2, \ldots, \ell - 1,$$

for which $\{p_1, p_{2k-1}, p_{2k+1}\}$ is a geometric line, with a sequence

$$p_{2k-1} \overline{\wedge} p_{2k} \overline{\wedge} q_{2k} \overline{\wedge} p_{2k} \overline{\wedge} p_{2k+1},$$

where $q_{2k} \equiv p_{2k}$ and none of $\{p_1, p_{2k-1}, q_{2k}\}$ or $\{p_1, q_{2k}, p_{2k+1}\}$ is a geometric line, then the conclusion of Theorem K' would still hold. But this can be achieved since, by Lemma 8.2.1.5, there exists a point q_{2k} opposite p_{2k} and distinct from p_1. Since the proof of Theorem K' does not use the condition of the existence of a point opposite three given points, we conclude:

Lemma 10.4.3.9. *Let Δ be a metasymplectic space. Let p be a given point. Denote by $\Pi_4(p)$ the set of all self-projectivities $p \overline{\wedge} p_2 \overline{\wedge} p_3 \overline{\wedge} p_4 \overline{\wedge} p$ of p of length 4 with $p \perp p_3$, $p_2 \perp p_4$, and pp_3 opposite p_2p_4. Suppose that $\Pi_4(p)$ is geometric. Then*

$$\Pi^+(p) = \langle \Pi_4(p) \rangle.$$

10.4.4. Collineations Pointwise Fixing a (Sub)Hyperplane

We will see that self-projectivities of length 3 in $\Pi(x)$, for x a point in a polar space Δ, pointwise fix a hyperplane of $\mathsf{Res}_\Delta(x)$. Therefore, we prove some results about collineations of polar spaces pointwise fixing a hyperplane.

Definition 10.4.4.10 (Reflection). We call a collineation of a polar space Δ, that pointwise fixes a geometric hyperplane of Δ, a *reflection* (in [**Die43b**] it is called a *symmetry*).

Lemma 10.4.4.11. *Let Δ be a separable quadric of Witt index $r \geq 1$ in $\mathsf{PG}(V)$, for some vector space V over the field $\mathbb{K}$.*

We extend the perp-relation to all subspaces of $\mathsf{PG}(V)$, according to the non-degenerate polarity defined by the quadratic form defining Δ.

Let θ be a central collineation of $\mathsf{PG}(V)$ with center c and axis the hyperplane $H := c^\perp$. If θ maps some point x of Δ to a distinct point x^θ of Δ, then θ preserves Δ and, consequently, defines a collineation of Δ.

Proof. If $\mathsf{char}\,\mathbb{K} \neq 2$, then this is a symmetry as defined by Dieudonné in [**Die43b**, §10], and the result follows from that reference.

Now suppose $\mathsf{char}\,\mathbb{K} = 2$. This situation is less standard, especially when $\mathbb{K}$ is not perfect. We have to show that, if y is an arbitrary point of Δ, then θ maps y to a point of Δ. Considering the plane $\langle x, c, y \rangle$, this boils down to showing, given a conic $\mathscr{C}$ in a plane $\mathsf{PG}(2, \mathbb{K})$ and a line L through the nucleus n of $\mathscr{C}$, the existence of a non-trivial central elation with a given arbitrary center c on L, with $c \neq n$.

We can take the conic with equation $Y^2 = XZ$, and L given by the equation $X = kZ$, for some $k \in \mathbb{K}$.[3] The center c has coordinates $(\ell, 1, k\ell)$. Then the non-trivial elation with center c, pointwise fixing L, and preserving $\mathscr{C}$, is described by the matrix

$$\begin{pmatrix} (1+ab^2)^{-1} & 0 & b^2(1+ab^2)^{-1} \\ ab(1+ab^2)^{-1} & 1 & b(1+ab^2)^{-1} \\ a^2b^2(1+ab^2)^{-1} & 0 & (1+ab^2)^{-1} \end{pmatrix},$$

as one can check with elementary calculations. This concludes the proof. □

Lemma 10.4.4.12. *Let Δ be a separable quadric of Witt index $r \geq 1$ in $\mathsf{PG}(V)$, for some vector space V over the field $\mathbb{K}$.*

We extend the perp-relation to all subspaces of $\mathsf{PG}(V)$, according to the non-degenerate polarity defined by the quadratic form defining Δ.

Let θ be a collineation pointwise fixing a subhyperplane G of $\mathsf{PG}(V)$. Set $L = G^\perp$. We assume that L is not a tangent. That means, we assume that either $L \subseteq G^\perp$, with $L \cap \Delta = \varnothing$, or $L \cap G = \varnothing$, and the former only occurs in characteristic 2.

Then θ is the product of two reflections.

Proof. First suppose $L \cap G = \varnothing$. Then L is either disjoint from Δ or intersects it in exactly two points. Let x be a point of Δ, not contained in $G \cup L$. We also assume that, if L contains two points x_1, x_2 of Δ, then x is not contained in $x_1^\perp \cup x_2^\perp$.

As θ fixes every subspace of G, it stabilizes every subspace through L and hence it stabilizes the plane $\langle x, L\rangle$. So we can define $\{a\} = xx^\theta \cap L$. If $x \perp x^\theta$, then a belongs to Δ and so a is one of x_1, x_2, contradicting our choice of x.

Hence, a does not belong to Δ. Also, $a \notin x^\perp$ and $a \notin (x^\theta)^\perp$, since a is on a secant through x and x^θ. Since $L^\perp = G$, $a \in L$ implies $G \subseteq a^\perp$.

Let θ_1 be the unique central collineation with center a that maps x to x^θ and fixes a^ρ pointwise. By Lemma 10.4.4.11, θ_1 defines a collineation of Δ.

Define θ_2 to be $\theta\theta_1^{-1}$.

Let c be the point $\langle x, G\rangle \cap L$. Then c maps to

$$\langle x^\theta, G\rangle \cap L = \langle x^{\theta_1}, H\rangle \cap L$$

under θ_1. Consequently, $c^\theta = c^{\theta_1}$ and so c is fixed under $\theta\theta_1^{-1} = \theta_2$.

With that, $\theta\theta_1^{-1}$ fixes G pointwise and also fixes x and c. Hence, it pointwise fixes $\langle x, H\rangle$, and therefore $\theta = (\theta\theta_1^{-1})\theta_1$ is the product of two central collineations.

Secondly, suppose $L \subseteq H$. Now L does not contain any point of Δ, and we may pick x in Δ, but not in G, arbitrarily. As before, we define a as the intersection of $xx^\theta \cap L$. We also know that x is not collinear to x^θ.

Define θ_1 as the elation that fixes $a^\perp$ pointwise, has center a, and maps x to x^θ. Then, again by Lemma 10.4.4.11, θ_1 induces a collineation in Δ.

Since two points of the line xx^θ are already contained in the quadric, and there are no more points of the quadric on xx^θ, it must be the case that x^θ maps back to x.

[3]Meaning L is defined by the points of the plane whose coordinates satisfy the equation $X = kZ$, for some $k \in \mathbb{K}$.

That means θ is an involution (which we can of course also deduce from the fact that $\mathsf{char}\,\mathbb{K} = 2$).

The map $\theta\theta_1^{-1}$ fixes H pointwise and fixes both x and x^θ. Hence, it pointwise fixes $\langle H, x\rangle$. Consequently, $\theta = (\theta\theta_1^{-1})\theta_1$ is the product of two central collineations. □

10.5. Projectivity Groups of Panels in Simply Laced Buildings

The next lemma will enable us to pin down the special and general projectivity groups for residues, which have the full linear group as respective projectivity group in a residue.

Lemma 10.5.0.1. *Let Δ be a thick spherical building over the type set I. Let K and J be two proper subsets of I, such that $K \subset J \subset I$. Let F_K be a simplex of type K and let F_J be a simplex of type J containing F_K. Let $\Pi_K^+(F_J)$ be the special projectivity group of $F_J \setminus F_K$ in $\mathsf{Res}_\Delta(F_K)$. Then $\Pi_K^+(F_J) \leq \Pi^+(F_J)$.*

Proof. Let F_J' and F_J'' be two simplices containing F_K, such that $F_J' \setminus F_K$ is opposite both $F_J \setminus F_K$ and $F_J'' \setminus F_K$ in $\mathsf{Res}_\Delta(F_K)$. We have to show that the product of the two perspectivities in $\mathsf{Res}_\Delta(F_K)$ from $\mathsf{Res}_\Delta(F_J)$ to $\mathsf{Res}_\Delta(F_J')$, subsequently to $\mathsf{Res}_\Delta(F_J'')$, coincides with the product of two perspectivities in Δ. To that aim, let F_K^* be a simplex in Δ opposite F_K, and let F_J^* be the projection of F_J' onto F_K^* (hence $F_J^* = \mathsf{proj}_{F_K^*}^{F_K'}(F_J')$). Let C be any chamber containing F_J. Set

$$\begin{aligned} C' &= \mathsf{proj}_{F_J'}(C), \\ C'' &= \mathsf{proj}_{F_J''}(C') = \mathsf{proj}_{F_J''}(\mathsf{proj}_{F_J'}(C)), \\ C^* &= \mathsf{proj}_{F_J^*}(C'). \end{aligned}$$

Then, according to Lemma 10.2.2.2, we have

$$C^* = \mathsf{proj}_{F_J^*}(C) \text{ and } C'' = \mathsf{proj}_{F_J''}(C^*),$$

which implies that C'' is indeed equal to the image of C under the product of two perspectivities in Δ. □

Definition 10.5.0.2. An automorphism of a thick spherical building Δ of simply laced type is called *linear* if it belongs to $\mathsf{PGL}_{r+1}(\mathbb{L})$, in case Δ corresponds to $\mathsf{PG}_r(\mathbb{L})$, or if it belongs to the linear algebraic group corresponding to the building, if Δ has type D_r, $r \geq 4$, or E_6, E_7, E_8. The next result is an immediate consequence of Lemma 10.5.0.1.

Corollary 10.5.0.3. *Let Δ be a thick spherical building over the type set I. Let K and J be two proper subsets of I, such that $K \subset J \subset I$. Let F_K be a simplex of type K and let F_J be a simplex of type J containing F_K. Let $\Pi_K^+(F_J)$ be the special projectivity group of $F_J \setminus F_K$ in $\mathsf{Res}_\Delta(F_K)$. Suppose that $\Pi_K^+(F_J)$ is the full linear type-preserving automorphism group of $\mathsf{Res}_\Delta(F_J)$. Then $\Pi^+(F_J)$ also coincides with the full linear type-preserving automorphism group of $\mathsf{Res}_\Delta(F_J)$.*

Proof of Theorem L

Now, Theorem L follows from Corollary 10.5.0.3, because every vertex of the Coxeter diagram of a simply laced thick irreducible spherical building of rank at least 3 is contained in a residue isomorphic to the building of a projective plane over some skew field $\mathbb{L}$, and in a projective plane, the special projectivity group of a line is $\mathsf{PGL}_2(\mathbb{L})$, acting naturally on $\mathsf{PG}(1,\mathbb{L})$.

10.6. Projectivity Groups of Irreducible Residues in Simply Laced Buildings

In this section, we determine the projectivity groups of non-maximal simplices with irreducible residues in thick spherical buildings of simply laced type.

10.6.1. Projective Spaces

We first consider the case that Δ is a projective space over a skew field $\mathbb{L}$. We will show that the special projectivity groups of any irreducible residue of rank ℓ are isomorphic to $\mathsf{PGL}_{\ell+1}(\mathbb{L})$. The general group always coincides with the special group, either because the type of the simplex is not self-opposite, or the type is polar closed.

Notation 10.6.1.1. Recall that $\mathsf{PGL}_n(\mathbb{K}) = {}^{\mathsf{GL}_n(\mathbb{K})}\!/\!_{\mathsf{Sc}_n(\mathbb{K})}$ (see Appendix A.2) and

$$\begin{aligned}\mathsf{PSL}_n(\mathbb{K}) &= {}^{\mathsf{SL}_n(\mathbb{K})}\!/\!_{(\mathsf{Sc}_n(\mathbb{K})\cap\mathsf{SL}_n(\mathbb{K}))}\\ &= {}^{\{M\in\mathsf{GL}_n(\mathbb{K})|\det M=1\}}\!/\!_{(\mathsf{Sc}_n(\mathbb{K})\cap\mathsf{SL}_n(\mathbb{K}))}.\end{aligned}$$

We want to consider the group $T := \{M \in \mathsf{GL}_n(\mathbb{K}) | \det M = k^a,\ k \in \mathbb{K}\}$ and factor out all scalar matrices. As for $\mathsf{PSL}_n(\mathbb{K})$, we cannot assume that $\mathsf{Sc}_n(\mathbb{K}) \subseteq T$. Therefore, we first multiply with all scalar matrices and factor them out afterwards. We set

$$\mathsf{PSL}_n(\mathbb{K},a) := {}^{T.\mathsf{Sc}_n(\mathbb{K})}\!/\!_{\mathsf{Sc}_n(\mathbb{K})},$$

where ${}^{T.\mathsf{Sc}_n(\mathbb{K})}\!/\!_{\mathsf{Sc}_n(\mathbb{K})}$ is isomorphic to

$${}^{\{M\in\mathsf{GL}_n(\mathbb{K})|\det M=k^a,\ k\in\mathbb{K}\}}\!/\!_{\{M\in\mathsf{GL}_n(\mathbb{K})|\det M=k^a,\ k\in\mathbb{K}\}\cap\mathsf{Sc}_n(\mathbb{K})}.$$

Note that $\mathsf{PSL}_n(\mathbb{K},1) \cong \mathsf{PGL}_n(\mathbb{K})$ and $\mathsf{PSL}_n(\mathbb{K},n) \cong \mathsf{PSL}_n(\mathbb{K})$.

Proof of Theorem M – for type A_r

Theorem 10.6.1.2. *Let Δ be a building of type A_r, defined over the skew field $\mathbb{L}$. Let F be any simplex, such that $I \setminus \mathsf{typ}(F)$ is connected in the Coxeter diagram (say of type A_ℓ). Then both $\Pi^+(F)$ and $\Pi(F)$ are permutation equivalent to $\mathsf{PGL}_{\ell+1}(\mathbb{L})$.*

Proof. Applying Proposition 2.10.0.2 and Corollary 10.5.0.3, it suffices to show that the stabilizer G of a hyperplane H of $\mathsf{PG}(r,\mathbb{L})$ in $\mathsf{PSL}_{r+1}(\mathbb{L})$ acts on H as $\mathsf{PGL}_r(\mathbb{L})$. Let g be an arbitrary element of $\mathsf{PGL}_r(\mathbb{L})$ acting on H. Then we can represent g, with respect to an arbitrarily chosen basis B in H, by an $r \times r$ matrix M. We have to find a member $g^* \in \mathsf{PSL}_{r+1}(\mathbb{L})$ that induces g in H. We can extend B to a basis B^* of $\mathsf{PG}(r,\mathbb{L})$, by adding one point $p_0 \notin H$ and a suitable unit point. Let d belong to the coset of the (multiplicative) commutator subgroup C of $\mathbb{L}^\times$, given by the Dieudonné determinant of M (cf. [**Die43a**]). Then the block matrix

$$M^* := \begin{pmatrix} d^{-1} & 0 \\ 0 & M \end{pmatrix}$$

represents a member g^* of $\mathsf{PGL}_{r+1}(\mathbb{L})$, fixing p_0, stabilizing H and inducing g in H. Moreover, by the properties of the Dieudonné determinant, in particular those established in the proof of [**Die43a**, Theorem 1], the determinant of M^* is equal to the product of the coset $d^{-1}C$ and the coset $\det M$. By the definition of d, this product is exactly C, and so $g^* \in \mathsf{PSL}_{r+1}(\mathbb{L})$. The proof is complete. □

10.6.2. Top-thin Polar Spaces

Here, when we consider residues of vertices of type 1, that is, the points of the corresponding polar space, we will aim to apply Theorem J. Proposition 7.1.0.1 already tells us that we can find a point opposite 4 arbitrarily given points if the underlying field has order at least 4. To handle the case with the field $\mathbb{F}_3$, we will use the slightly more general results for hyperbolic quadrics that we have proved in Lemma 9.3.0.1 and Lemma 9.3.0.2.

Proof of Theorem M – for type D_r

Theorem 10.6.2.3. *Let Δ be the building (of rank $r \geq 4$[4]) associated to a hyperbolic quadric Q of Witt index $r \geq 4$ over the field $\mathbb{K}$. Let F be a simplex of Δ, such that $\mathsf{Res}_\Delta(F)$ is irreducible. Then $\Pi(F)$ and $\Pi^+(F)$ are given as in* Table 7. *In Case* (A$*$)*, the permutation group $\mathsf{PGL}_r(\mathbb{K},2).2$ denotes the extension of $\mathsf{PGL}_r(\mathbb{K},2)$ by a symplectic polarity, acting on $\mathsf{PG}(r-1,\mathbb{K})$ (and coincides with the group generated by all symplectic polarities). A long hyphen in the table in the column of $\Pi(F)$ means that $\mathsf{typ}(F)$ is not self-opposite and so $\Pi(F)$ is trivially isomorphic to $\Pi^+(F)$ — it must be read as a "bysame" symbol. Grey rows correspond to projectivity groups that are not necessarily the full linear groups.*

[4]Recall that the D_3 diagram is the same as the A_3 diagram.

Cf.	$\mathsf{Res}_\Delta(F)$	$\mathsf{cotyp}(F)$	$\Pi^+(F)$	$\Pi(F)$
(A1)	A_1		$\mathsf{PGL}_2(\mathbb{K})$	$\mathsf{PGL}_2(\mathbb{K})$
(A3)	A_3	$\{r-2, r-1, r\}$	$\mathsf{PGO}^\circ_6(\mathbb{K})$	$\mathsf{PGO}_6(\mathbb{K})$
(A)	A_ℓ, $2 \leq \ell \leq r-2$	$\neq \{r-2, r-1, r\}$	$\mathsf{PGL}_{\ell+1}(\mathbb{K})$	$\mathsf{PGL}_{\ell+1}(\mathbb{K}).2$
(A*)	A_{r-1}, $r \in 2\mathbb{Z}$		$\mathsf{PGL}_r(\mathbb{K}, 2)$	$\mathsf{PGL}_r(\mathbb{K}, 2).2$
(A**)	A_{r-1}, $r \in 2\mathbb{Z}+1$		$\mathsf{PGL}_r(\mathbb{K})$	——
(D)	$\mathsf{D}_{r-2\ell}$, $4 \leq r-2\ell \leq r-1$		$\mathsf{PGO}^\circ_{2r-4\ell}(\mathbb{K})$	$\mathsf{PGO}^\circ_{2r-4\ell}(\mathbb{K})$
(D$'$)	$\mathsf{D}_{r-2\ell+1}$, $4 \leq r-2\ell+1 \leq r-1$		$\mathsf{PGO}^\circ_{2r-4\ell+2}(\mathbb{K})$	$\mathsf{PGO}_{2r-4\ell+2}(\mathbb{K})$

TABLE 7. Projectivity groups in buildings of type D_r over $\mathbb{K}$

PROOF. First, we notice that if $\mathbb{K} = \mathbb{F}_2$, then all groups are universal and adjoint (simple) at the same time, so the results follow from Theorem 2.10.0.3. Hence, we may assume $|\mathbb{K}| \geq 3$. Case (A1) follows from Theorem L, whereas Case (A) follows from Lemma 10.5.0.1 and Theorem 10.6.1.2. We now handle the other, less straightforward, cases. For ease of notation and language, we will speak of *plus-type* and *minus-type* to distinguish the two different classes of maximal singular subspaces of Q.

Case (A*) Let M_1, M_2, M_3 be three mutually opposite maximal singular subspaces of plus-type. Let $p_1 \in M_1$ be arbitrary. The maximal singular subspace N through p_1 intersecting M_2 in a submaximal singular subspace (that is, a singular subspace of dimension $r-2$) intersects M_3 in a point p_3, since N is necessarily of minus-type. Hence, the maximal singular subspace of minus-type through p_3, intersecting M_1 in a submaximal singular subspace, contains p_1. This shows that the projectivity $M_1 \barwedge M_2 \barwedge M_3 \barwedge M_1$ is a duality, each point of which is absolute. Lemma 3.2 of [**TTV11**] implies that it is a symplectic polarity. By conjugation, we can obtain every symplectic polarity of M_1 in this way. Applying Theorem J, together with Lemma 9.3.0.2, Case (A*) follows from the fact that the matrix corresponding to a symplectic polarity necessarily has square determinant (and every square can occur).

Case (A3) By Theorem 2.10.0.3, every self-projectivity preserves the residual form, hence

$$\Pi(F) \leq \mathsf{PGO}_6(\mathbb{K}).$$

Case (A*), for $r = 3$, together with Lemma 10.5.0.1 and the fact that $\mathsf{PGO}^\circ_6(\mathbb{K})$ is isomorphic to $\mathsf{PSL}_4(\mathbb{K}, 2)$, concludes this case.

Case (A**) By Theorem 2.10.0.3, $\Pi^+(F)$ contains $\mathsf{PSL}_r(\mathbb{K})$. Hence, it suffices to show that $\Pi^+(F)$ contains an element of $\mathsf{PGL}_r(\mathbb{K})$ whose corresponding matrix has an arbitrary determinant.

Let M_1 and M_3 be two maximal singular subspaces of plus-type, intersecting in a subspace U_{13} of dimension $r-3$. Let M_2 be a maximal singular subspace opposite both M_1 and M_3 (then M_2 has plus-type). Let U_{24} be a subspace of M_2 of dimension $r-3$ opposite U_{13}. Let L_1 be the unique line of M_1 collinear to U_{24}. Let L be an arbitrary line in M_1, joining a point $p_{13} \in U_{13}$ with some point $p_1 \in L_1$. Pick $p, p' \in L \setminus \{p_{13}, p_1\}$ and suppose $p \neq p'$ (this is possible as we assume $|\mathbb{K}| \geq 3$).

Let M be the maximal singular subspace of plus-type, containing p and intersecting M_2 in a hyperplane. Put $W = U_{24} \cap M$. Then W has dimension $r-4$ and is collinear to L. The intersection of M and M_3 is a point q, as both have the same type. Both p and p_{13} are collinear to q and p' is collinear to q. Hence, p' is collinear to $\langle q, W\rangle$, and $\langle p', q, W\rangle$ is a singular subspace of dimension $r-2$. Therefore, there is a unique maximal singular subspace M' of plus-type containing p', q and W. It intersects M_1 in p' and M_3 in q. We also have a unique maximal singular subspace M_4, containing U_{24} and intersecting M' in a hyperplane (and therefore it is of minus-type). Now, with this set-up, one verifies that the projectivity $M_1 \barwedge M_2 \barwedge M_3 \barwedge M_4 \barwedge M_1$ pointwise fixes both U_{13} and L_1, and maps p to p'. Choosing a basis in $U_{13} \cup L_1$, the matrix of such a homology in M_1 is diagonal of the form $\mathsf{diag}(k, k, \ell, \ell, \dots, \ell)$, and the arbitrariness of p' implies that k and ℓ are also arbitrary. Set $r = 2s+1$. Putting $k = \ell^{-s+1}$, we obtain the determinant $\ell^{-2s+2+2s-1} = \ell$. Since ℓ is arbitrary, the assertion follows.

Case (D′) First, set $\ell = 1$, that is, $r - 2\ell + 1 = r - 1$, and F is just a point of the polar space or hyperbolic quadric Q. Let p_1, p_2, p_3 be three mutually opposite points. Since $p_1^\perp \cap p_2^\perp$ is a hyperbolic quadric of rank $r-1$, we have that $p_1^\perp \cap p_2^\perp \cap p_3^\perp$ is either a parabolic subquadric or a degenerate quadric. In the latter case, $\{p_1, p_2, p_3\}^{\perp\perp}$ is a degenerate plane conic containing p_1, p_2, p_3, and hence p_3 is collinear to either p_1 or p_2, a contradiction. Consequently, $p_1^\perp \cap p_2^\perp \cap p_3^\perp$ is a parabolic quadric, and the projectivity $p_1 \barwedge p_2 \barwedge p_3 \barwedge p_1$ is a parabolic polarity. Every parabolic polarity of $\mathsf{Res}_\Delta(p_1)$ can be obtained this way. Thus, Theorem J, Lemma 9.3.0.1, and Lemma 5.1.2.9 yield

$$\Pi(p_1) = \mathsf{PGO}_{2r-2}(\mathbb{K}) \quad \text{and} \quad \Pi^+(p_1) = \mathsf{PGO}^\circ_{2r-2}(\mathbb{K}).$$

Let ℓ be such that $4 \leq r-2\ell+1 \leq r-1$, but otherwise arbitrary. Since the stabilizer of F in $\mathsf{PGO}_{2r}(\mathbb{K})$ preserves the residual form (in $\mathsf{Res}_\Delta(F)$), we see that $\Pi^+(F)$ is a subgroup of $\mathsf{PGO}^\circ_{2r-4\ell+2}(\mathbb{K})$, and therefore it coincides with it by Lemma 10.5.0.1 in the case $\ell = 1$. In order to show $\Pi(F) = \mathsf{PGO}_{2r-4\ell+2}(\mathbb{K})$, we only need to exhibit a parabolic polarity as a self-projectivity in $\mathsf{Res}_\Delta(F)$. This is done similarly as in the previous paragraph for the case $\ell = 1$: Choose three mutually opposite singular subspaces U_1, U_2, U_3 of dimension $2\ell-1$, contained in a parabolic subquadric obtained from Q by intersecting Q in its ambient projective space with a subspace of dimension 4ℓ. Suppose $U_1 \in F$. Then, as before, the projectivity

$$U_1 \barwedge U_2 \barwedge U_3 \barwedge U_1$$

is a parabolic polarity of $\mathsf{Res}_\Delta(F)$.

Case (D) This is completely similar to the case $\ell > 1$ of Case (D′), noting that $\Pi^+(F)$ coincides with $\Pi(F)$ by Theorem F.

□

10.6.3. Exceptional Cases

Proof of Theorem M – for exceptional types

Theorem 10.6.3.4. *Let Δ be a building of type E_6, E_7, or E_8 over the field $\mathbb{K}$. Let F be a simplex of Δ, such that $\mathsf{Res}_\Delta(F)$ is irreducible. Then $\Pi(F)$ and $\Pi^+(F)$ are given as in* Table 8*, where the last column contains a checkmark if $\mathsf{typ}(F)$ is polar closed. Again, a long hyphen in the table in the column of $\Pi(F)$ means that $\mathsf{typ}(F)$ is not self-opposite and so $\Pi(F)$ is trivially isomorphic to $\Pi^+(F)$ — it must again be read as a "bysame" symbol. Grey rows correspond to projectivity groups which are not necessarily full linear groups.*

Cf.	$\mathsf{typ}(\Delta)$	$\mathsf{Res}_\Delta(F)$	$\mathsf{cotyp}(F)$	$\Pi^+(F)$	$\Pi(F)$	
(A1)		A_1		$\mathsf{PGL}_2(\mathbb{K})$	$\mathsf{PGL}_2(\mathbb{K})$	
(A2)	E_6	A_2	$\{2,4\}$	$\mathsf{PGL}_3(\mathbb{K})$	$\mathsf{PGL}_3(\mathbb{K}).2$	
	E_6	A_2	$\neq\{2,4\}$	$\mathsf{PGL}_3(\mathbb{K})$	——	
	$\mathsf{E}_7,\mathsf{E}_8$	A_2		$\mathsf{PGL}_3(\mathbb{K})$	$\mathsf{PGL}_3(\mathbb{K}).2$	
(A3)	E_6	A_3	$\{3,4,5\}$	$\mathsf{PGL}_4(\mathbb{K})$	$\mathsf{PGL}_4(\mathbb{K})$	✓
	E_6	A_3	$\neq\{3,4,5\}$	$\mathsf{PGL}_4(\mathbb{K})$	——	
	$\mathsf{E}_7,\mathsf{E}_8$	A_3		$\mathsf{PGL}_4(\mathbb{K})$	$\mathsf{PGL}_4(\mathbb{K}).2$	
(A4)	E_6	A_4		$\mathsf{PGL}_5(\mathbb{K})$	——	
	$\mathsf{E}_7,\mathsf{E}_8$	A_4		$\mathsf{PGL}_5(\mathbb{K})$	$\mathsf{PGL}_5(\mathbb{K}).2$	
(A5)	E_6	A_5		$\mathsf{PSL}_6(\mathbb{K},3)$	$\mathsf{PSL}_6(\mathbb{K},3)$	✓
	E_7	A_5	$\{2,4,5,6,7\}$	$\mathsf{PSL}_6(\mathbb{K},2)$	$\mathsf{PSL}_6(\mathbb{K},2).2$	
	E_7	A_5	$2\notin\mathsf{cotyp}(F)$	$\mathsf{PGL}_6(\mathbb{K})$	$\mathsf{PGL}_6(\mathbb{K}).2$	
	E_8	A_5		$\mathsf{PGL}_6(\mathbb{K})$	$\mathsf{PGL}_6(\mathbb{K}).2$	
(A6)	$\mathsf{E}_7,\mathsf{E}_8$	A_6		$\mathsf{PGL}_7(\mathbb{K})$	$\mathsf{PGL}_7(\mathbb{K}).2$	
(A7)	E_8	A_7		$\mathsf{PGL}_8(\mathbb{K})$	$\mathsf{PGL}_8(\mathbb{K}).2$	
(D4)	E_6	D_4		$\overline{\mathsf{PGO}}^\circ_8(\mathbb{K})$	$\overline{\mathsf{PGO}}_8(\mathbb{K})$	
	$\mathsf{E}_7,\mathsf{E}_8$	D_4		$\overline{\mathsf{PGO}}^\circ_8(\mathbb{K})$	$\overline{\mathsf{PGO}}^\circ_8(\mathbb{K})$	✓
(D5)	E_6	D_5		$\overline{\mathsf{PGO}}^\circ_{10}(\mathbb{K})$	——	
	$\mathsf{E}_7,\mathsf{E}_8$	D_5		$\overline{\mathsf{PGO}}^\circ_{10}(\mathbb{K})$	$\overline{\mathsf{PGO}}_{10}(\mathbb{K})$	
(D6)	E_7	D_6		$\overline{\mathsf{P\Omega}}_{12}(\mathbb{K})$	$\overline{\mathsf{P\Omega}}_{12}(\mathbb{K})$	✓
	E_8	D_6		$\overline{\mathsf{PGO}}^\circ_{12}(\mathbb{K})$	$\overline{\mathsf{PGO}}^\circ_{12}(\mathbb{K})$	✓
(D7)	E_8	D_7		$\overline{\mathsf{PGO}}^\circ_{14}(\mathbb{K})$	$\overline{\mathsf{PGO}}_{14}(\mathbb{K})$	
(E6)	$\mathsf{E}_7,\mathsf{E}_8$	E_6		$\mathsf{PGE}_6(\mathbb{K})$	$\mathsf{PGE}_6(\mathbb{K}).2$	
(E7)	E_8	E_7		$\mathsf{PGE}_7(\mathbb{K})$	$\mathsf{PGE}_7(\mathbb{K})$	✓

Table 8. Projectivity groups in the exceptional cases $\mathsf{E}_6,\mathsf{E}_7,\mathsf{E}_8$.

PROOF. The case (A1) was handled in Theorem L. We now handle the other cases. Note that we may, again, assume that $|\mathbb{K}| \geq 3$, as otherwise the linear groups are unique.

Cases (A2) and (A3) Every subdiagram of type A_2 or A_3 of E_r, $r = 6, 7, 8$, is contained in a subdiagram of type A_3 or A_4, respectively. Then the assertions all follow from Corollary 10.5.0.3 and Theorem 10.6.1.2.

Case (A4) If $2 \notin \mathsf{cotyp}(F)$ for E_6, or if $\mathsf{cotyp}(F) \neq \{1, 2, 3, 4\}$ for $\mathsf{E}_7, \mathsf{E}_8$, then we can, again, embed the diagram of $\mathsf{Res}_\Delta(F)$ in the diagram of type A_5 and use Corollary 10.5.0.3 and Theorem 10.6.1.2.

Now suppose $2 \in \mathsf{cotyp}(F)$ for E_6 and $\mathsf{cotyp}(F) = \{1, 2, 3, 4\}$ for E_7 and E_8. Then the assertion follows from Corollary 10.5.0.3 and Case (A**), for $r = 5$ of Theorem 10.6.2.3.

Case (A5) In the Coxeter diagram of type E_8, every subdiagram of type A_5 is contained in one of type A_6, and hence, the assertion for E_8 follows from Corollary 10.5.0.3. The same thing holds in the Coxeter diagram of type E_7 if the subdiagram of type A_5 does not contain the vertex of type 2.

Now suppose Δ is the building of type E_6 over the field $\mathbb{K}$, and F is a vertex of type 2.

We argue in the corresponding Lie incidence geometry of type $\mathsf{E}_{6,1}$. There, F is a 5-space. Let F_1, F_2, F_3 be three 5-spaces, with both F_1 and F_3 opposite both F and F_2, and with F adjacent to F_2, and F_1 adjacent to F_3, that is, $\pi_0 := F \cap F_2$ and $\pi_1 := F_1 \cap F_3$ are planes. We also initially assume that π_0 and π_1 are opposite. Consider the projectivity $\rho\colon F \barwedge F_1 \barwedge F_2 \barwedge F_3 \barwedge F$. We claim that ρ fixes each point of π_0. Indeed, let $p_0 \in \pi_0$ be such a point. Then — since $F \cap F_2$ contains p_0 — the projectivities $F \barwedge F_1 \barwedge F_2$ and $F_2 \barwedge F_3 \barwedge F$ fix p_0, hence $p_0^\rho = p_0$, and the claim is proved. Likewise, ρ fixes each point of F collinear to a point of π_1. The set of such points forms a plane π_0' of F, disjoint from π_0. Choosing a basis of F in $\pi_0 \cup \pi_0'$, a matrix of ρ is a diagonal matrix with diagonal elements three times 1 and three times some scalar $k \in \mathbb{K}$. We now show that k can be arbitrary. This is equivalent to showing that,

(*) given F_1, F_2 and F_3 as above, given a line L_0 in F containing points $x_0 \in \pi_0$ and $x_0' \in \pi_0'$, and given two points $p, q \in L_0 \setminus \{x_0, x_0'\}$, we can re-choose F_3 through π_1, such that ρ maps p to q.

We now prove (*). Let p_1 be the projection of p onto F_1, and let p_2 be the projection of p_1 onto F_2. If p and p_2 were not collinear, then the symp $\xi(p, p_2)$ would contain p_1 and π_0, leading to additional points in π_0 collinear to p_1 and inside $\xi(p, p_2)$, contradicting the fact that F and F_1 are opposite. Therefore, p_1 is far from F, and there is some singular 4-space U, containing π_0, L, and p_2. (Note that, since U intersects F in a 3-space, Facts 4.5.0.11(ii) implies that U is really a 4-space and not a 4′-space.) Set $\xi := \xi(x_0, p_1)$. Then ξ contains p, q, p_1, p_2 and the unique point $x_1 \in \pi_1$, collinear to x_0'. We find that π_1 intersects ξ in only x_1, as otherwise there would be a point of π_1 collinear to x_0, contradicting the fact that π_0 and π_1 are opposite. So, Lemma 4.5.0.19 yields a plane $\alpha \subseteq \xi$, collinear to π_1. Facts 4.5.0.11

implies that α and π_1 are contained in a unique $4'$-space U_2, which is itself contained in a unique 5-space F_3. Both q and p_2 are (inside ξ) collinear to all points of respective lines of α, implying that they are collinear to a common point $p_3 \in F_3$. Now (*) follows.

It now also follows that the set of such projectivities ρ (as above with π_0 and π_1 opposite) is geometric (they are the homologies with two disjoint planes as centers). Next, we drop the assumption of π_0 being opposite π_1. We claim that in this more general case, the projectivity ρ, as defined above, is the product of homologies with disjoint planes as centers. Indeed, set $\pi_0' := \mathsf{proj}_F^{F_1}(\pi_1)$ as above. If π_0' is disjoint from π_0, then by Proposition 2.5.0.4, π_0 and π_1 are opposite. Now we treat the other cases. Set $d = \dim(\pi_0 \cap \pi_0')$ and note that $d = -1$ is precisely the case we already proved.

- $d = 0$ Let π_2 be a plane in F, sharing a line with π_0, but disjoint from π_0'. Then one can check that the unique 4-space U, containing the 3-space generated by π_2 and π_0, is disjoint from $\mathsf{proj}_{F_2}^{F_1}(\pi_1)$. Hence, there exists a 5-space $F_2' \neq F$, containing π_2 and opposite both F_1 and F_3, and we have that π_1 is opposite both $F \cap F_2'$ and $F_2 \cap F_2'$. We can now write ρ as the product of $F \barwedge F_1 \barwedge F_2' \barwedge F_3 \barwedge F$ and the conjugate of $F_3 \barwedge F_2' \barwedge F_1 \barwedge F_2 \barwedge F_3$ by $F \barwedge F_3$, reducing this case to the case $d = -1$, which we already proved.
- $d = 1$ Let π_2 be a plane in F, sharing a line with π_0 and exactly one point (necessarily in π_0) with π_0'. Then, similarly as in the case $d = 0$, we can choose a 5-space $F_2' \neq F$ through π_2, opposite both F_1 and F_3, and such that π_1 has a unique point collinear to some point of $F \cap F_2'$, and that point is also the unique point of π_1 collinear to some point of $F_2 \cap F_2'$. We have hence reduced this case to two times the case $d = 0$, which we proved above.
- $d = 2$ This case is similarly reduced to the case $d = 1$. We leave the (straightforward) details to the reader.

The claim is proved. Hence, thanks to Lemma 10.4.2.5, we can apply Theorem K and obtain that $\Pi^+(F)$ is generated by all homologies with disjoint planes as centers. This group contains $\mathsf{PSL}_6(\mathbb{K})$ and then corresponds to all 6×6 matrices with a determinant equal to some non-zero 3rd power. Also, $\Pi(F) = \Pi^+(F)$ by virtue of Theorem F.

Now suppose $\mathsf{cotyp}(F) = \{2, 4, 5, 6, 7\}$ in case of E_7. Here, we can take for F a pair consisting of a 5-space W and a symp ξ, containing W in the Lie incidence geometry of type $\mathsf{E}_{7,7}$ over the field $\mathbb{K}$. We employ the same method as in the previous case (Case A_5 in E_6), noting that a projectivity $\{W, \xi\} \barwedge \{W', \xi'\} \barwedge \{W'', \xi''\}$, where the simplices $\{W, \xi\}$ and $\{W'', \xi''\}$ are adjacent, is trivial as soon as $W = W''$, and so we may always assume that in such a (sub)sequence $W \neq W''$ and $\xi = \xi''$. However, since the action of the projectivity is independent of the symps ξ and ξ', we may only consider projections from 5-spaces onto 5-spaces.

Let W_1, W_2, W_3 be three 5-spaces, such that both W_1 and W_3 are opposite both W and W_2, and $\Sigma_1 := W_1 \cap W_3$ and $\Sigma_0 := W \cap W_2$ are 3-spaces, such that the symps ξ_0 and ξ_1, containing W, W_2, and W_1, W_3, respectively, are also opposite. Similarly as in the previous case (type A_5 inside E_6), we may assume from the beginning that Σ_0 and Σ_1 are opposite 3-spaces. Set $L_0 := \mathsf{proj}_W^{W_1}(\Sigma_1)$. Then, by Proposition 2.5.0.4, L_0 and Σ_0 are disjoint. Set $L_2 := \mathsf{proj}_{W_2}^{W_1}(\Sigma_1)$, then likewise L_2 and Σ_0 are disjoint.

Let x_0 be an arbitrary point on L_0. Then inside ξ_0, one sees that there is a unique point x_2 on L_2, collinear to x_0. We claim that the projectivity $\rho_1 \colon W \barwedge W_1 \barwedge W_2$ maps x_0 to x_2. Indeed, set $W_1' := \mathsf{proj}_{\xi_0}^{\xi_1}(W_1)$. Then, again by Proposition 2.5.0.4, W_1' is disjoint from both W and W_2. Set $U_1 := \mathsf{proj}_{W_1}^{W}(x_0)$ and $U_1' := \mathsf{proj}_{\xi_0}^{\xi_1}(U_1)$ and note that $\Sigma_1 \subseteq U_1$. Then $U_1' \subseteq W_1'$. Since x_0 is at distance 2 from each point of U_1, it follows by Fact 4.6.2.12 that x_0 is collinear to all points of U_1'. Hence, x_0 is contained in the unique $5'$-space V_0 of ξ_0, containing U_1'. Likewise, if $x_2' = \mathsf{proj}_{W_2}^{W_1}(U_1)$, then $x_2' \in V_0$. Therefore, x_0 and x_2', which is contained in L_2 as U_1 contains Σ_1, are collinear. Consequently, $x_2' = x_2$, and the claim is proved.

It now also follows that $\rho_3 \colon W_2 \barwedge W_3 \barwedge W$ maps x_2 back to x_0, since x_0 is the unique point on L_0, collinear to x_2. Consequently, the projectivity $\rho \colon W \barwedge W_1 \barwedge W_2 \barwedge W_3 \barwedge W$ fixes each point of L_0. One can see that it also fixes every point of Σ_0. Hence, it is a homology corresponding to a diagonal matrix with the diagonal consisting of four times a 1 and two times a scalar $k \in \mathbb{K}^\times$. If we can now show that every non-zero scalar k can occur, then, similarly to the case A_5 in E_6, using Theorem K and Lemma 10.4.2.5, we are done.

But it follows from the arguments in the previous paragraphs that the projectivity ρ_1 coincides with the projectivity $W \barwedge W_1' \barwedge W_2$ inside ξ_0. Likewise, the projectivity ρ_3 coincides with the projectivity $W_2 \barwedge W_3' \barwedge W$ inside ξ_0, with $W_3' := \mathsf{proj}_{\xi_0}^{\xi_1}(W_3)$. Now the assertion follows with exactly the same arguments as Case (A**) in the proof of Theorem 10.6.2.3.

This concludes Case (A5).

Case (A6) In a Coxeter diagram of type E_8, a subdiagram of type A_6 is either contained in a subdiagram of type A_7, in which case we are done by Corollary 10.5.0.3 and Theorem 10.6.1.2, or it has type $\{2, 4, 5, 6, 7, 8\}$, in which case it is contained in a subdiagram of type D_7, and we are done by Corollary 10.5.0.3 and Theorem 10.6.2.3(A**).

In a Coxeter diagram of type E_7, a subdiagram of type A_6 necessarily has type $\{1, 3, 4, 5, 6, 7\}$. Although a proof using the Lie incidence geometry of type $\mathsf{E}_{7,7}$, where F is a 6-space, is feasible, we prefer to consider the Lie incidence geometry of type $\mathsf{E}_{7,1}$, where the proof is entirely similar to the case (A7) of A_7 in E_8 below. So we refer to that case for the details.

Case (A7) In a Coxeter diagram of type E_8, a subdiagram of type A_7 has cotype 2, which corresponds to a 7-space. Let Δ be the building of type E_8 over the field $\mathbb{K}$.

Let W_0 and W_2 be two 7-spaces in Δ, intersecting in a 4-space that we denote by U. Let W_1 be a 7-space opposite to both W_0 and W_2. Then U projects to a plane α in W_1. Projection here means that U and α are special; for every pair of points (u, a) with $u \in U$ and $a \in \alpha$, there exists a point $p_{u,a}$ that is collinear to u and a. Let U' be a 4-space in W_1 that has no intersection with α. Let W_3 be a 7-space that intersects W_1 in U' and is opposite both W_0 and W_2. Set

$$\rho \colon W_0 \barwedge W_1 \barwedge W_2 \barwedge W_3 \barwedge W_0.$$

We claim that points of U are fixed under ρ. Indeed, a point of $W_0 \cap W_2$ first maps to a hyperplane of W_1, then back to itself, then to a hyperplane of W_3, and again back to itself. That means U is fixed pointwise under ρ. The claim is proved.

The projection of U' onto W_0 is a plane, which we will denote by β. A point p of β maps to a hyperplane of W_1 that contains U' and intersects α in a line. We will call this hyperplane H. The projection of H onto W_2 is a point, which we will denote by p'. We claim that the point p is collinear to p'.

Indeed, suppose for a contradiction that p were not collinear to p'. The points p and p' cannot have distance 3 or be a special pair, since they are both collinear to each point of U. That means p and p' are symplectic. The symp $\xi(p, p')$ intersects W_0 and W_2 each in a 6′-space that contains U. The projection of the 6′-space $W_0 \cap \xi(p, p')$ onto W_1 is a point w. The point w is in α, since α was the projection of W_0 onto W_1. Let F be a 6′-space in W_1 not through w. Then F is opposite $W_0 \cap \xi(p, p')$. Let ξ be the unique symp through F. Then ξ is opposite $\xi(p, p')$.

The projections of p and p' onto W_1 define the two 6′-spaces of W_1, for which it is not the case that all points are opposite p and p', and the intersection of them is a 5-space, which we will denote by V_1. Let V_0 be the 5-space that is the projection of V_1 onto $\xi(p, p')$. The 5-space spanned by U and p in $\xi(p, p')$ is disjoint from V_0, but p and p' are both collinear to V_0. Let x be an arbitrary point in V_1 and x' be a point at distance 2 from x in V_0. The point x' is collinear to both p and p'. There exists a unique symp ξ' that intersects $\xi(p, p')$ exactly in x' and ξ exactly in x.

The 6′-spaces $\langle V_0, p\rangle$ and $\langle V_0, p'\rangle$ in $\xi(p, p')$ are of the same type as the 6′-spaces $W_0 \cap \xi(p, p')$ and $W_2 \cap \xi(p, p')$. But the intersections $\langle V_0, p\rangle \cap (W_0 \cap \xi(p, p'))$ and $\langle V_0, p'\rangle \cap (W_2 \cap \xi(p, p'))$ only contain a unique point. This contradicts the fact that subspaces of the same type in polar spaces of type D_7 have to intersect in even codimension. We conclude that p and p' have to be collinear, and the claim is proved.

Now we claim that all points of β are fixed under ρ. Indeed, the point p projects to the hyperplane H in W_1. This hyperplane projects to p' in W_2. This already implies that p and p' are collinear. If we project a point of W_2 to W_3 and to W_0, we also get that this point and the point in W_0 have to be collinear. The point p is the only point of β that p' is collinear to. Hence, p' projects to a hyperplane in W_3 and then back to p.

Let xy be a line in W_0 between a point $x \in U$ and $y \in \beta$. Let a and b be two distinct points on xy, not equal to either x or y. We claim that we can re-define W_3, such that ρ maps a to b. Let $a' := W_0 \overline{\wedge} W_1 \overline{\wedge} W_2(a)$, $b' := W_0 \overline{\wedge} W_1 \overline{\wedge} W_2(b)$. Since a, b, a', y, y', and U form a 6-space, $\langle a, b, a'\rangle$ forms a plane, ba' and yy' intersect in a point s.

Let W' be a 7-space through U and s. The projection of W' onto U' is a 7-space that is not opposite W', and that we will denote by W_3. Since W_3 and W' are not opposite, there exists, by Lemma 5.5.1.18, a plane γ_3 in W_3 such that no point of γ_3 is opposite any point of W', and there exists a plane γ' in W' such that no point of γ' is opposite any point of W_3.

Let y' be the point in γ' on a line with the point y of β. The line yy' is the unique line between γ' and β. The points y and y' are not opposite all points of U'. Hence,

the point s is not opposite all points of U'. The point s is also not opposite any of the points of γ_3 since $s \in W'$. That means s is not opposite any point of W_3, and hence has to be contained in γ'.

The line $a'b$ contains s. That means the points a' and b project to the same hyperplane of W_3. Then the point a maps to a hyperplane of W_1, then to a' in W_2, and the point a' projects to a hyperplane of W_3 and then to b. Our claim is proved.

By Theorem 2.10.0.3, $\Pi^+(W_0)$ contains $\mathsf{PSL}_8(\mathbb{K})$. By the above, it also contains all diagonal matrices with diagonal $(1,1,1,1,1,k,k,k)$, $k \in \mathbb{K}^\times$ arbitrarily, and the entries k can be anywhere. This readily implies that $\Pi^+(W_0)$ contains the diagonal matrices with diagonal $(1,1,1,1,1,1,1,k)$, hence all homologies, and hence $\Pi^+(W_0) = \mathsf{PGL}_8(\mathbb{K})$ and $\Pi(W_0) = \mathsf{PGL}_8(\mathbb{K}).2$.

Case (D5) We first consider the case of a Coxeter diagram of type E_6. Without loss of generality, we may assume that F has type 6. Thus, we consider F as a symp in a geometry of type $\mathsf{E}_{6,1}$ over the field $\mathbb{K}$.

Let p_1 be a point in Δ and ξ_0 a symp opposite p_1 in Δ. Let U be a maximal singular subspace in ξ_0. Then U is a 4-space. Let ξ_2 be another symp through U opposite p_1. Opposite a 4-space are lines. Let L be a line through p_1 opposite U and $V := \mathsf{proj}_{\xi_0}(L)$. Let p_3 be any point on L opposite both ξ_0 and ξ_2, so that we have a projectivity $\rho\colon \xi_0 \barwedge p_1 \barwedge \xi_2 \barwedge p_3 \barwedge \xi_0$. We will show that ρ fixes U and V pointwise.

First, let x be a point in U. Then x projects to a symp $\xi(x,p_1)$, then back to x, since $x \in \xi_0 \cap \xi_2$, then to a symp $\xi(x,p_3)$ and then again back to x.

Now, let y be a point in V. The point $y = y_0$ projects to a symp $\xi(y_0,p_1) = \xi_y$ and then to a point $y_2 \in \xi_2$. Suppose y_0 and y_2 were not collinear. The symp ξ_y has to contain the closure of y_0 and y_2. Both y_0 and y_2 are collinear to a 3-space of U. The intersection of these 3-spaces contains a plane. This means that the closure has to contain a plane of U, which then has to be contained in ξ_y. But this contradicts the fact that U and p_1 are opposite, because p_1 would have to be collinear to elements of that plane. It follows that $y_0 \perp y_2$. Now, since $V = \mathsf{proj}_{\xi_0}(L)$, we see that $L \subseteq \xi_y$. So y_2 continues mapping to ξ_y and then back to y_0. Hence, points of V are fixed.

Next, we want to show that we can always define p_3 on L in a way such that the projectivity ρ defined above maps an arbitrary point p on a line xy, with $x \in U$ and $y \in V$, to another arbitrary point q on xy, for $p \notin U, V$ and $q \notin U, V$. Given U, V, L and p_1 as before and a line xy as described above, let p be an arbitrary point on xy not in U or V. Then projecting p to p_1 yields a symp $\xi(p,p_1)$, which projects to a point p_2 onto ξ_2. Let $y_2 := \mathsf{proj}_{\xi_2}(\mathsf{proj}_{p_1}(y))$. By the previous paragraph, the points x, y, y_2 generate a singular plane, which contains p, q and p_2. Let $a := p_2q \cap yy_2$. Suppose a were collinear to p_1. Then a would be in $\xi(p,p_1)$, and $\xi(p,p_1)$ would contain the plane $\langle x, y, y_2\rangle$, and in particular the line xy. But this contradicts the fact that $\xi(p,p_1)$ intersects ξ_0 only in p. It follows that a is not collinear to p_1. This means a is collinear to a different point of L that we will define as p_3. This point p_3 is not collinear to p_2, as otherwise $\xi(p,p_1)$ would contain L, forcing $p \in V$, a contradiction. Since a and p_2 are in $\xi(p_3,p_2)$, $\xi(p_3,p_2)$ contains the whole line ap_2 and hence the point q. With that, it follows that p maps to $\xi(p,p_1)$ to p_2 to $\xi(p_3,p_2) = \xi(p_3,q)$, and finally to q.

Now Lemma 5.7.1.3(i) proves the assertion in the E_6 case.

In a Coxeter diagram of type E_7 or E_8, a subdiagram of type D_5 is always contained in a subdiagram of type E_6, and so we can apply Corollary 10.5.0.3, the previous paragraphs, and Theorem F. This completes the proof for case of D_5.

Case (D4) Each subdiagram of type D_4 in a diagram of type E_n, $n = 6, 7, 8$, is contained in a subdiagram of type D_5. It follows that, if F is a simplex of cotype D_4 in a building Δ of type E_n, $n = 6, 7, 8$, then there is a subsimplex $F' \subseteq F$ of cotype D_5. By the previous case and Theorem 2.10.0.3, the stabilizer of F' in the little projective group $\mathsf{Aut}^\dagger(\Delta)$ of Δ acts on $\mathsf{Res}_\Delta(F')$ as the complete linear type-preserving group of automorphisms. Hence, the stabilizer of F in $\mathsf{Aut}^\dagger(\Delta)$, acting on $\mathsf{Res}_\Delta(F)$, contains the stabilizer of the vertex $F \setminus F'$ in the full linear type-preserving group of $\mathsf{Res}_\Delta(F')$. This is also the full linear type-preserving group of $\mathsf{Res}_\Delta(F)$.

Now, in case of $\mathsf{typ}(\Delta) = \mathsf{E}_6$, it follows from Theorem F that $\Pi^+(F)$ has index 2 in $\Pi(F)$, and so $\Pi(F)$ is the full linear group of the corresponding polar space of $\mathsf{Res}_\Delta(F)$. In case of E_7 or E_8, Theorem F implies that $\Pi(F) = \Pi^+(F)$.

Case (D7) In a Coxeter diagram of type E_8, a subdiagram of type D_7 is missing vertex 1. So F is a symp in the geometry Γ of type $\mathsf{E}_{8,8}$ over the field $\mathbb{K}$.

Let ξ_0 and ξ_2 be two symps in Γ, intersecting in a 6-space, that we will denote by U. Let ξ_1 be a symp opposite ξ_0 and ξ_2, and U' the 6′-space that is the projection of U onto ξ_1. Opposite U' in ξ_1 is a 6-space V. Let V' be the 6′-space that is the projection of V onto ξ_0. Suppose there is a symp $\xi_3 \supseteq V$ opposite both ξ_0 and ξ_2, such that we have a projectivity $\rho\colon \xi_0 \barwedge \xi_1 \barwedge \xi_2 \barwedge \xi_3 \barwedge \xi_0$. We first show that ρ fixes every point of U and V'. For points of U the assertion follows immediately. A point v_0 of V' first projects to a point v_1 in V, then to some point in ξ_2 and then back to v_1. The point v_1 maps back to v_0.

Next, we want to show that there always exists a ξ_3 through V opposite both ξ_0 and ξ_2 and that we can define it in a way such that we can map an arbitrary point p on a line xy, with $x \in U$ and $y \in V'$, to a point q on xy for $p \notin U, V'$ and $q \notin U, V'$.

We define:

$$y =: y_0,\ \mathsf{proj}_{\xi_1}(y_0) =: y_1,\ \mathsf{proj}_{\xi_2}(y_1) =: y_2,$$
$$p =: p_0,\ \mathsf{proj}_{\xi_1}(p_0) =: p_1,\ \mathsf{proj}_{\xi_2}(p_1) =: p_2.$$

First, we want to show that $p_2 \perp q$. Suppose p_2 and q were not collinear. Then there exists a symp $\xi(p_2, q) = \xi(p_2, p)$ and a point $u \in U \cap \xi(p_2, p)$, such that p_1 is opposite u. But by Lemma 4.7.0.10(iii) the point p_1 is supposed to be symplectic to only one point of $\xi_{p_2,p}$, and it is already symplectic to p and p_2. It follows that $p_2 \perp q$ and there exists a plane $\langle p, q, p_2\rangle$. Define $a := y_0y_2 \cap p_2q$.

Let $\xi(U, a)$ be the unique symp through U containing a, according to Lemma 5.5.1.16. Define $\xi_3 := \mathsf{proj}_V(\xi(U, a))$. Then $\xi(U, a)$ is the unique symp through U not opposite ξ_3.

In $\xi(U, a)$ let, according to Lemma 5.5.1.17, W be the maximal singular subspace, such that no point of W is opposite (at distance 3 from) any point of ξ_3. Let W_3 be the maximal singular subspace of ξ_3, such that no point of W_3 is opposite any point of $\xi(U, a)$. The points y and y_2 are not opposite any point of V. Hence, a is

not opposite any point of V. But the point a is also not opposite any point of W_3 by the definition of W_3. Hence, $a \in W$.

The same points of ξ_3 are not opposite q and p_2, since, if we take an arbitrary point s in ξ_3 which is not opposite q, then s is not opposite a and q, so also not opposite p_2. The same holds if we switch the roles of q and p_2. Let q_3 be the unique point of ξ_3 symplectic to q. Then q is opposite $\xi_3 \setminus q_3^\perp$. But then $\xi_3 \setminus q_3^\perp = \xi_3 \setminus p_3^\perp$. The perps of points s and t in a polar space are the same if $s = t$. Hence, it follows that $q_3 = p_3$. That means q is the unique point of ξ_0 that p_3 is symplectic to. With that, we get that $\rho(p) = q$.

Now, as before, Lemma 5.7.1.3(i) proves the assertion.

Case (D6) We first treat the case of type D_6 inside type E_7. Let ξ be a symp of the geometry of type $\mathsf{E}_{7,7}$ over the field $\mathbb{K}$. We first claim that $\Pi(\xi)$ (which is equal to $\Pi^+(\xi)$ by Theorem F) contains all homologies pointwise fixing two ξ-opposite maximal singular 5-spaces. Let M_{13} and M be two such subspaces of ξ. Let ξ_3 be an arbitrary symp distinct from ξ and containing M_{13}. Let ξ_2 be a symp opposite both ξ and ξ_3 (and note that this implies that each point of ξ_2 is opposite some point of ξ). There is a unique maximal singular subspace M_{24}, contained in ξ_2, each point of which is collinear to some point of M, that is, $M_{24} = \mathsf{proj}^{\xi}_{\xi_2}(M)$. Let L be any given line in ξ, joining a point $p_{13} \in M_{13}$ and $p \in M$. Choose two points $q, q' \in L \setminus \{p_{13}, p\}$. Set $q_2 = \mathsf{proj}_{\xi_2}(q)$ and $q_3 = \mathsf{proj}_{\xi_3}(q_2)$.

If q were not collinear to q_3, then the symp containing them would contain a 3-dimensional subspace of M_{13} and q_2; this would imply that q_2 is close to ξ, contradicting Fact 4.6.2.12, in view of our remark in the previous paragraph that says that q_2 is opposite some point of ξ. Hence, $\langle q, q_3, q' \rangle$ is a plane π, contained in the symp ζ containing p_{13} and q_2. Let ξ' be any symp containing M_{24}, but distinct from ξ_2. Let p_{24} be the unique point of ξ_2 collinear to p, and note $p_{24} \in M_{24}$, and that p_{24} and q_2 are collinear. Therefore, $p_{24} \in \zeta$. This implies that $\zeta \cap \xi'$ is either a line or a 5-space through p_{24}. In the latter case, p_{13}, being collinear to more than one point of that intersection, is close to ξ'. This contradicts Fact 4.6.2.12 and the fact that M_{24} is opposite M_{13}. Thus p_{13} is opposite some points of ξ'. Hence, $\zeta \cap \xi'$ is a line $K \ni p_{24}$. If q_2 were not collinear to K, then ζ would contain a 3-space of M_{24}; again a similar contradiction (since ζ contains p_{13}). The planes π and $\langle q_2, K \rangle$ are easily seen to be opposite in ζ, and hence there is a unique point $q_4 \in \langle q_2, K \rangle$, collinear to both q_3 and q'. Now let ξ_4 be the symp containing M_{24} and q_4, whose existence follows from Fact 4.6.2.19. Then one checks that ξ_4 is opposite both ξ and ξ_3, and the projectivity $\xi \barwedge \xi_2 \barwedge \xi_3 \barwedge \xi_4 \barwedge \xi$ pointwise fixes both M_{13} and M, and maps q to q'. This proves the claim.

Now, if we want to apply Theorem K, then we have to show that every projectivity

$$\rho\colon \xi_0 \barwedge \xi_1 \barwedge \xi_2 \barwedge \xi_3 \barwedge \xi_0,$$

with $M_0 := \xi_0 \cap \xi_2$ and $M_1 := \xi_1 \cap \xi_3$ singular 5-spaces, is the product of similar projectivities, but with M_0 opposite M_1. So suppose M_0 and M_1 are not opposite. As for the case of type A_5 in type E_6, there are 3 cases to consider, and they are again all quite similar to each other. So we consider, for example, the case where the set of points of M_0 collinear to a point of M_1 is a line L (the other possibilities are a 3-space and the whole space M_1). Then we consider an appropriate 5-space

M_2 in ξ_0, intersecting M_0 in a 3-space, contained in M_0, and disjoint from L. Then, we find a symp ξ_2', containing M_2, opposite both ξ_1 and ξ_3, and intersecting ξ_2 in a 5-space opposite M_1. As in the case of type A_5 in type E_6, we can now write ρ as the product of $\xi_0 \barwedge \xi_1 \barwedge \xi_2' \barwedge \xi_3 \barwedge \xi_0$ and the conjugate of $\xi_3 \barwedge \xi_2' \barwedge \xi_1 \barwedge \xi_2 \barwedge \xi_3$ by $\xi_0 \barwedge \xi_3$.

Now we can use Lemma 5.7.1.3 and, thanks to Lemma 10.4.2.5, also Theorem K, to conclude that $\Pi^+(F) = \overline{\mathsf{P\Omega}}_{12}(\mathbb{K})$.

Finally, consider the case of type D_6 inside type E_8 and let F be a simplex of cotype D_6. On the one hand, since the D_6 subdiagram is contained in a E_7 subdiagram, $\Pi^+(F)$ contains $\overline{\mathsf{P\Omega}}_{12}(\mathbb{K})$. On the other hand, since the D_6 subdiagram is contained in a D_7 subdiagram, $\Pi^+(F)$ contains $\mathsf{PGO}^\circ_{12}(\mathbb{K})$, as follows from Theorem 10.6.2.3(D′). Now, Theorem F and Lemma 5.7.1.3(*iii*) yield

$$\Pi(F) = \Pi^+(F) = \overline{\mathsf{PGO}}^\circ_{12}(\mathbb{K}).$$

Case (E6) Let Γ be the parapolar space of type $\mathsf{E}_{7,7}$ over the field $\mathbb{K}$. Let p_1, p_2, p_3 be three mutually opposite points of Γ. If we show that the self-projectivity

$$\rho\colon p_1 \barwedge p_2 \barwedge p_3 \barwedge p_1$$

is always a symplectic polarity, then Theorem J and Proposition 9.6.1.1 imply that $\Pi(p)$ is generated by all the symplectic polarities. By Proposition 6.8(*i*) of [**DSV22**], ρ pointwise fixes a subbuilding of type F_4. More exactly, if $\mathsf{Res}_\Delta(p_1)$ is viewed as a parapolar space Γ_{p_1} of type $\mathsf{E}_{6,1}$ with the lines through p_1 as points, then ρ pointwise fixes a geometric hyperplane inducing in Γ_{p_1} a geometry of type $\mathsf{F}_{4,4}$ over the field $\mathbb{K}$. It follows from [**DSV18**] that ρ is a symplectic polarity. Now Lemma 5.7.2.6 shows that $\Pi^+(p)$ is $\mathsf{PGE}_6(\mathbb{K})$ and $\Pi(p)$ is $\mathsf{PGE}_6(\mathbb{K}).2$. Since these groups are the respective full linear type-preserving and full linear (not necessarily type-preserving) groups, Corollary 10.5.0.3 implies the assertion for type E_6 in type E_8.

Case (E7) In a Coxeter diagram of type E_8, a subdiagram of type E_7 is missing vertex 8. Hence, given a building Δ of type E_8 over the field $\mathbb{K}$, a simplex $F = \{p\}$ with residue of type E_7 can be thought of as being a point in the Lie incidence geometry Γ of type $\mathsf{E}_{8,8}$ over the field $\mathbb{K}$. Let p_1 and p_3 be collinear points of Γ opposite p and let p_2 be a point of Γ collinear to p and opposite both p_1 and p_3. Note that the lines through p form the point set of a Lie incidence geometry of type $\mathsf{E}_{7,7}$, where a line consists of all lines of Γ through p contained in a given plane of Γ. Set $\rho\colon p \barwedge p_1 \barwedge p_2 \barwedge p_3 \barwedge p$. If π is a plane through the line pp_2, then $\mathsf{proj}_{p_2}^{p_1}\mathsf{proj}_{p_1}^{p}(\pi) = \pi$ and $\mathsf{proj}_{p}^{p_3}\mathsf{proj}_{p_3}^{p_2}(\pi) = \pi$. Hence, ρ fixes π. Likewise, if L is the projection onto p of the line $L_1 := p_1p_3$, all planes through L are fixed by ρ. By the basic properties of Lie incidence geometries of type $\mathsf{E}_{7,7}$, there is a unique line M through p, no point of which is special to any point of L (it is the projection of L on π in the residue $\mathsf{Res}_\Delta(p)$). Let K_1 and K_2 be two distinct lines in π containing p and distinct from both pp_2 and M. Let M_1 be the line of π containing all points of π not opposite p_1. Set $\{x_1\} = K_1 \cap M_1$. Then there exists a unique point x_1' with $x_1 \perp x_1' \perp p_1$. So the line p_1x_1' is the projection onto p_1 of the line K_1. Since $p_2 \perp x_1$, we see that the line p_2x_1 is the projection of the line p_1x_1' onto p_2.

Set $\{x_2\} = p_2x_1 \cap K_2$ and redefine p_3 as the projection of x_2 onto the line L_1. If $x_2 \perp x_2' \perp p_3$, then p_2x_1 is projected onto p_3x_2' from p_2 onto p_3, and p_3p_2' is projected

back onto $px_2 = K_2$ from p_3 onto p. Hence, ρ maps K_1 to K_2 and the result now follows from Lemma 5.7.3.10. □

This concludes the proofs of our main results regarding projectivity groups of buildings of simply laced type. We conclude this section with some remarks.

Remark 10.6.3.5. It follows from Theorem 10.6.3.4 that $\overline{\mathsf{P\Omega}}_{12}(\mathbb{K})$ does not always coincide with $\overline{\mathsf{PGO}}_{12}(\mathbb{K})$. Indeed, if it did, then the special projectivity groups in the buildings of type E_7 of all irreducible residues of types contained in D_6 would be the full linear groups. This contradicts the second grey row of Table 8 for fields containing non-square elements.

Remark 10.6.3.6. The argument for case E_7 in E_8 of the proof of Theorem 10.6.3.4 could also be used for the cases of D_6 in E_7 and A_5 in E_6, if we were to use the corresponding long root geometries. We chose to use the simpler and more accessible Lie incidence geometries of types $\mathsf{E}_{7,7}$ and $\mathsf{E}_{6,1}$, respectively, instead, also as a warm-up for the more complicated cases such as A_5 in E_7 and D_7 in E_8.

Remark 10.6.3.7. In the course of the proof of Theorem 10.6.3.4, we do not really need the full strength of Lemmata 5.7.1.3(i), 5.7.2.6, and 5.7.3.10, since we know by Theorem 2.10.0.3 that the little projective group is already contained in the group that we want to generate. This knowledge would simplify the proof, since we would only have to prove that the little projective group, together with said homologies, generates the full linear group.

10.7. Projectivity Groups for Upper Residues in Polar Spaces

We now move on to the determination of the projectivity groups for non-maximal simplices with irreducible residues in thick spherical buildings of non-simply laced type and first consider buildings of type B_n and C_n; i.e., thick polar spaces. Considering the reductively labeled version of the Coxeter diagrams, we see that they have a double bond on one end, but not on the other, and that that is the only factor that makes them asymmetrical. Hence, for each type, we consider the upper and the lower residues, which we have defined for polar spaces in Definition 3.6.2.17, see also Definition 3.6.2.22. We first tend to the upper residues of points, and then to the upper residues of subspaces of dimension at least 1.

10.7.1. Projectivity Groups of Points

10.7.1.1. The Generic Case

Proposition 10.7.1.1. *Let Δ be a thick embeddable polar space of rank $n \geq 3$. Let p_1 and p_2 be two opposite points of Δ, let Γ be the polar space $p_1^\perp \cap p_2^\perp$ of rank $n-1$ inside Δ, and let H be a hyperplane of Γ, obtained as the intersection of a projective*

hyperplane of some projective space with some embedding of Γ *in that projective space. Let* k *and* k' *be two distinct non-collinear points of* Γ*, both not contained in* H*. Suppose* $k^\perp \cap H = k'^\perp \cap H$*. Then there exists a point* p_3 *in* Δ *that is opposite both* p_1 *and* p_2 *with* $p_3^\perp \cap \Gamma = H$*, such that the odd projectivity* $p_1 \barwedge p_2 \barwedge p_3 \barwedge p_1$ *fixes all lines joining* p_1 *with a point of* H*, and moves* p_1k *to* p_1k'*.*

PROOF. The fact that $p_1 \barwedge p_2 \barwedge p_3 \barwedge p_1$ fixes all lines through p_1 having a point in H implies that p_3 is collinear to all points of H. Also, the fact that $p_1 \barwedge p_2 \barwedge p_3 \barwedge p_1$ maps p_1k to p_1k' implies that p_3 is on a line L that intersects both p_1k' and p_2k. We select an arbitrary point k'' on the line p_1k' distinct from both p_1 and k', and we let L be the unique line through k'' intersecting p_2k. Set $K = k^\perp \cap H$. If H contains lines, then K is a hyperplane of H, and so H is determined by K and some point $h \in H \setminus K$, meaning that, if a point is collinear to K and h, then it is collinear to H. If H only contains points, then the same follows from our assumption that H arises as the intersection of a projective hyperplane of some projective space with some embedding of Γ in that projective space.

We now define p_3 as the unique point of L collinear to h. Since all of p_1, p_2, k and k' are collinear to K, also k'' and all other points of L are collinear to K. In particular, p_3 is collinear to K, and since it is also collinear to h, it is collinear to H.

It follows that $p_1 \barwedge p_2 \barwedge p_3 \barwedge p_1$ fixes all lines joining p_1 with a point of H, and moves p_1k to p_1k'. □

Corollary 10.7.1.2. *Let* Δ *be a thick polar space of rank* $n \geq 3$*, and let* H *be a hyperplane of* Δ*. Let* k, k' *be two points not contained in* H*, such that*

$$k^\perp \cap H = k'^\perp \cap H.$$

Then there exists a unique collineation θ *of* Δ*, pointwise fixing* H*, and mapping* k *to* k'*.*

PROOF. First, we prove uniqueness.

Let K be the set $k^\perp \cap H = k'^\perp \cap H$. Let x be some point of K. Then the line kx maps to the line $k'x = k^\theta x$. Let a be some point on kx not equal to k or x. Then a^θ has to be on the line $k^\theta x$. Let y be some point in $(a^\perp \cap H) \setminus (k^\perp \cap H)$. Then a is the unique point of kx collinear to y, and with that, a^θ is uniquely determined as the unique point on $k^\theta x$ collinear to $y^\theta = y$. That means the images of all points in $k^\perp$ and H are uniquely determined. Playing the same game with all points of $k^\perp \setminus H$, and then again and again, shows that θ is uniquely determined, as the complement of a hyperplane in a polar space is always connected.

Next, we prove existence. If Δ is non-embeddable, then by [**CS90**], the only hyperplanes are of the form $p^\perp$, for some point p, and then θ is a central elation (see [**Van24**, Chapter 5]). If Δ is embeddable, then we can view it as the intersection of $p_1^\perp$ and $p_2^\perp$, for two opposite points p_1, p_2 of a polar space of rank $n+1$. Then existence follows from Proposition 10.7.1.1. □

Remark 10.7.1.3. It can happen that a (thick) polar space Δ of rank at least 3 possesses a hyperplane H, but that there is no non-trivial collineation of Δ pointwise fixing it. This is not in contradiction with Corollary 10.7.1.2, as for such hyperplanes H there do not exist two distinct points k, k' with $k^\perp \cap H = k'^\perp \cap H$. This situation

occurs, for instance, in symplectic polar spaces over fields of characteristic 2. Indeed, let Δ be a symplectic polar space of rank r at least 3 over a field $\mathbb{K}$ with $\mathsf{char}\,\mathbb{K} = 2$. The universal embedding of such a space corresponds with a (inseparable) quadric, and hence there are geometric hyperplanes of Δ which, as subsets of points of $\mathsf{PG}(2r-1,\mathbb{K})$ (in which Δ is naturally embedded), generate $\mathsf{PG}(2r-1,\mathbb{K})$. Let H be such a hyperplane. Select $2r-1$ points of H that generate a hyperplane J of $\mathsf{PG}(2r-1,\mathbb{K})$. Then $J = k^\perp$ for a unique point k of Δ. Hence, there is no second point k' with $k^\perp \cap H = k'^\perp \cap H$.

Proof of Theorem N(iii) – embeddable case for upper residues of points

If we combine our results of Theorem H, Theorem J', Proposition 10.4.1.1, and Proposition 10.7.1.1, we obtain the following theorem.

Theorem 10.7.1.4. *Let Δ be an embeddable polar space of rank at least 3 with at least 4 points per line, and let p be a point. Then $\Pi(x)$ is the subgroup of the automorphism group of $\mathsf{Res}(p)$ generated by all reflections of $\mathsf{Res}(p)$. If Δ is not a separable orthogonal polar space, then $\Pi^+(p) = \Pi(x)$. If Δ is a separable orthogonal polar space, then $\Pi^+(p)$ is the subgroup of $\mathsf{Aut}(\mathsf{Res}(p))$ consisting of products of an even number of reflections.*

We have the following immediate consequence (taking into account that a polar space which is not a separable quadric automatically has at least 5 points per line, except if it is a symplectic polar space over $\mathbb{F}_2$).

Corollary 10.7.1.5. *Let Δ be an embeddable, but not separable orthogonal or symplectic polar space of rank at least 2. Then the group of collineations generated by all reflections coincides with the group of collineations which are the product of an even number of reflections.*

10.7.1.2. Hermitian Case and Symplectic Polar Spaces

Noting that, in the case that Δ is embeddable, in the proof of Proposition 10.7.1.1, the hyperplanes H arise as intersections of subspaces with Γ, we gather some immediate consequences of Theorem 10.7.1.4 in the following statements.

Corollary 10.7.1.6.

(*i*) *If Δ is the polar space arising from a non-degenerate Hermitian form (including symmetric bilinear forms) in a finite-dimensional vector space over a (commutative) field, then, for each point p, the group $\Pi(p)$ is the full linear group preserving the Hermitian form that defines $\mathsf{Res}(p)$.*

(*ii*) *If Δ is a symplectic polar space, and p is a point of Δ, then $\Pi(p) = \Pi^+(p)$ is the simple symplectic group corresponding to $\mathsf{Res}(p)$, except if $\mathsf{Res}(p)$ has rank 2 and each line has exactly 3 points, in which case the group is isomorphic to the symmetric group on 6 letters.*

Proof. The bilinear case of (*i*) follows directly from Propositions 8 and 14 of [**Die43b**] (see also Appendix D). Now let the form that defines Δ be Hermitian with non-trivial field automorphism σ. Since $\Pi^+(p)$ contains the little projective group

of $\mathsf{Res}_\Delta(p)$, [**Die43b**, Théorème 5] (see also Appendix D) implies that it suffices to show that each field element x with $xx^\sigma = 1$ can be obtained as a determinant of a reflection. It suffices to consider the 2-dimensional case (vector space dimension), where this is immediate: the mapping $(x, y) \mapsto (x, ay)$, with $aa^\sigma = 1$, preserves the Hermitian form $xx^\sigma + kyy^\sigma$, has determinant a, and fixes the hyperplane $(0, 1)$ (and also the point $(1, 0)$, which is the perp of $(0, 1)$). Now (i) follows.

We now show (ii). By Remark 10.7.1.3, $\Pi^+(p)$ is generated by reflections for which the fixed hyperplanes are point perps. These are elations, and hence $\Pi^+(p)$ is the simple symplectic group corresponding to $\mathsf{Res}(p)$ if lines have at least 4 points. By Theorem H, $\Pi(p) = \Pi^+(p)$. The statement (ii) for polar spaces with exactly three points per line follows from Theorem 2.10.0.3, as the stabilizer of a point in the simple symplectic group is the symplectic group of the residue. (This argument can in fact also be used as an alternative for the larger symplectic polar spaces.) □

Three issues remain to be addressed in more detail:

(1) If Δ is a separable orthogonal polar space, can we be more specific about when $\Pi^+(p) = \Pi(p)$ (in other words, are there generic situations in which this equality *always* or perhaps *never* holds true)?
(2) If Δ is non-embeddable, then the description of $\Pi(p)$ above is not very transparent; can we provide a description using the bilinear form defining the dual (embeddable) generalized quadrangle?
(3) The case when lines of Δ have exactly three points.

We begin with (1).

10.7.1.3. Separable Orthogonal Polar Spaces

Proposition 10.7.1.7. *Let Δ be a separable, orthogonal polar space of rank n with at least 4 points per line, and anisotropic codimension r. Let p be a point. If r is even, then $\Pi^+(p)$ has index 2 in $\Pi(p)$. If r is odd, then $\Pi^+(p) = \Pi(p)$.*

Proof. If r is even, then each member of $\Pi^+(p)$ preserves the system of generators of the (imaginary) hyperbolic quadric over a splitting field of the quadric, whereas each single reflection interchanges these systems.

Now let r be odd. Since $\Pi(p)$ contains reflections with arbitrary spinor norm, in particular also non-trivial elements with trivial spinor norm, we find products of two reflections (which each are products of three perspectivities) with arbitrary spinor norm. The proposition follows. □

10.7.1.4. Polar Spaces with Three Points per Line

Now, we address (3). For symplectic polar spaces (which are isomorphic to parabolic polar spaces), this is already contained in Corollary 10.7.1.6(ii). Hermitian polar spaces never contain exactly three points per line. There remain the so-called *elliptic* ones, having anisotropic dimension 2. For these, Proposition 10.7.1.1 implies that $\Pi(p)$ contains the special (projective) orthogonal group (which is generated by reflections by [**Die43b**, Proposition 14]). Now, we see that Proposition 10.7.1.7 is

also valid in the present case, and therefore $\Pi^+(p)$ is the simple orthogonal group (which has index 2 in the special orthogonal group).

10.7.1.5. Non-embeddable Polar Spaces

In order to deal with the thick non-embeddable polar spaces, we first take a look at residues isomorphic to the pseudo-quadratic polar spaces of rank 2 that are dual to separable orthogonal polar spaces of rank 2.

Recall that a similitude of a form is a linear transformation preserving the form up to a non-zero constant, and in case of a quadratic form, it is called direct if the form is hyperbolic over a splitting field and the similitude preserves each natural system of maximal singular subspaces (cf. Definition 5.7.0.1).

Let $\mathbb{K}$ be a field with an involution σ, and let $\mathbb{F}$ be the subfield pointwise fixed by σ. Let $\Delta(\mathbb{K},\mathbb{F})$ be the pseudo-quadratic polar space of rank 3 given by the pseudo-quadratic form

$$X_{-3}^{\sigma}X_3 + X_{-2}^{\sigma}X_2 + X_{-1}^{\sigma}X_1 \in \mathbb{F}.$$

In what follows, it is also allowed that $\mathbb{K}$ has characteristic 2 and is an inseparable quadratic extension of $\mathbb{F}$ (and then σ is the identity).

Let p_i be the base point corresponding to the x_i-coordinate, $i \in \{-3,-2,-1,1,2,3\}$. Let p be the point with coordinates $(1, a_{-2}, 0, 0, a_2, k - a_{-2}^{\sigma}a_2)$, with $a_{-2}, a_2 \in \mathbb{K}$ and $k \in \mathbb{F}$. Also, let $p(\ell_1,\ell_2)$ be the point with coordinates $(0,\ell_1,0,0,\ell_2,0)$, with $\ell_1,\ell_2 \in \mathbb{F}$. We assume that p is not collinear to either p_{-3} or p_3, that is, $k - a_{-2}^{\sigma}a_2 \neq 0$.

An elementary calculation shows that the projectivity $p_{-3} \barwedge p_3 \barwedge p \barwedge p_{-3}$ maps the plane

$$\langle p_{-3}, p_{-1}, p(\ell_1,\ell_2)\rangle$$

to the plane

$$\langle p_{-3}, p_{-1}, p((a_{-2}^{\sigma}a_2 + a_2^{\sigma}a_{-2} - k)\ell_1 - a_{-2}^{\sigma}a_{-2}\ell_2, a_2a_2^{\sigma}\ell_1 - k\ell_2)\rangle.$$

It follows that, if Γ is the separable, orthogonal polar space of rank 2 dual to $p_{-3}^{\perp} \cap p_3^{\perp}$, and L is the line of Γ corresponding to the point p_{-1}, then the action on L by the above projectivity is given in binary coordinates by

$$\begin{pmatrix}\ell_1\\ \ell_2\end{pmatrix} \mapsto \begin{pmatrix} a_{-2}^{\sigma}a_2 + a_2^{\sigma}a_{-2} - k & -a_{-2}^{\sigma}a_{-2} \\ a_2^{\sigma}a_2 & -k\end{pmatrix} \cdot \begin{pmatrix}\ell_1\\ \ell_2\end{pmatrix} =: A \cdot \begin{pmatrix}\ell_1\\ \ell_2\end{pmatrix}.$$

The determinant of the 2×2-matrix A in the above expression is

$$(k - a_{-2}^{\sigma}a_2) \cdot (k - a_{-2}^{\sigma}a_2)^{\sigma}.$$

Let Γ be embedded in $\mathsf{PG}(5,\mathbb{F})$. Note that the fixed points of the projectivity above form an ovoid of $p_{-3}^{\perp} \cap p_3^{\perp}$, and hence a spread of Γ. Taking six base points on three spread lines, amongst which we can choose L, we claim that the determinant of the matrix of the corresponding collineation is

$$[(k - a_{-2}^{\sigma}a_2) \cdot (k - a_{-2}^{\sigma}a_2)^{\sigma}]^3.$$

Indeed, let M be a second line with two base points. We can choose the basis such that the equation of $\Gamma \cap \langle L, M\rangle$ is $x_0x_2 + x_1x_3 = 0$, where $L = \langle (1,0,0,0),(0,1,0,0)\rangle$

and $M = \langle (0,0,1,0), (0,0,0,1) \rangle$. The projectivity now fixes every line of $\Gamma \cap \langle L, M \rangle$ disjoint from L (and also L). One calculates that the matrix of the collineation is the block matrix

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Doing this for the third line with base points again, the claim follows.

Hence, we conclude that the special and general projectivity groups of the upper residue of p_{-3}, viewed as a collineation group of Γ, consist of all similitudes of the associated quadratic form, with factor a norm of the quadratic extension $\mathbb{K}/\mathbb{F}$ (note that each square of $\mathbb{F}$ is such a norm). Writing the equation (in some other basis) of Γ as $y_{-2}y_2 + y_{-1}y_1 = yy^\sigma$, where we view the underlying vector space as $\mathbb{F} \times \mathbb{F} \times \mathbb{K} \times \mathbb{F} \times \mathbb{F}$, with generic coordinates $(y_{-2}, y_{-1}, y, y_1, y_2)$, we see that the factor of each similitude is a norm.

Proof of Theorem N(iii) – non-embeddable case for upper residues of points

Proposition 10.7.1.8. *Let Δ be the thick non-embeddable polar space associated with the Cayley division algebra $\mathbb{O}$ over the field $\mathbb{F}$ with standard involution $x \mapsto \overline{x}$. Let p be any point of Δ. Then $\Pi(p) = \Pi^+(p)$ is the group of all direct similitudes of the quadratic form*

$$q : \mathbb{F} \times \mathbb{F} \times \mathbb{O} \times \mathbb{F} \times \mathbb{F} \to \mathbb{F} : (x_{-2}, x_{-1}, x, x_1, x_2) \mapsto x_{-1}x_1 + x_{-2}x_2 + x\overline{x},$$

and the factor of such a similitude is a norm of some element of $\mathbb{O}$.

Proof. Let p be a point of Δ. The rank 2 polar space $\mathsf{Res}_\Delta(p)$ is dual to a separable quadric of Witt index 2 in $\mathsf{PG}(11, \mathbb{F})$, given by the quadratic form mentioned in the statement. We have that $\Pi(p) = \Pi^+(p)$ is contained in the group of all direct similitudes of that quadratic form. So, it suffices to prove that each such similitude can occur. But this follows from the discussion above, taking into account that the pseudo-quadratic polar space $\Delta(\mathbb{K}, \mathbb{F})$, with $\mathbb{K}$ any 2-dimensional subfield of $\mathbb{O}$ containing $\mathbb{F}$, is a subpolar space of Δ, with the property that all planes of Δ through a line of $\Delta(\mathbb{K}, \mathbb{F})$ belong to $\Delta(\mathbb{K}, \mathbb{F})$. □

10.7.2. Subspaces of Dimension at least 1.

Lemma 10.7.2.9. *Let Δ be a polar space of rank $n \geq 3$ embedded into $\mathsf{PG}(V)$ that is not a separable quadric. Let A_1, A_2, A_3 and A_4 be singular subspaces of dimension $m \leq n-2$, such that $A_1 \cap A_3 =: B$ and $A_2 \cap A_4 =: C$, with B and C opposite and of dimension $m-1$, such that A_i is opposite A_{i+1} for $i \in \mathbb{Z}/4\mathbb{Z}$, and such that A_1 and A_3 are not contained in a common subspace. Then $A_1 \overline{\wedge} A_2 \overline{\wedge} A_3 \overline{\wedge} A_4 \overline{\wedge} A_1$ is a product of two collineations in $\mathsf{Res}_\Delta(A_1)$ that each fix a hyperplane.*

Proof. We may assume that $\mathsf{PG}(V)$ is minimal, that is, the relation $\perp$ defines a non-degenerate polarity of $\mathsf{PG}(V)$.

(I) We will first assume that $m \leq n-3$. Then $A_1^\perp \cap A_2^\perp$ has rank $n-m-1$. Since $n-m-1 \geq 2$, $A_1^\perp \cap A_2^\perp \cap A_3^\perp$ is a hyperplane of $A_1^\perp \cap A_2^\perp$ that we will denote by H.

If $A_4^\perp \cap A_1^\perp \cap A_2^\perp = H$, then H is fixed pointwise. So suppose $A_4^\perp \cap A_1^\perp \cap A_2^\perp \neq H$. Define:

$$x_1 := C^\perp \cap A_1, \quad x_2 := B^\perp \cap A_2,$$
$$x_3 := C^\perp \cap A_3, \quad x_4 := B^\perp \cap A_4.$$

Since B and C are opposite, $x_1 \neq x_3$ and $x_2 \neq x_4$. In $\mathsf{PG}(V)$, there are planes $\langle x_1, x_2, x_3\rangle$ and $\langle x_1, x_3, x_4\rangle$ which share the line $\langle x_1, x_3\rangle$ in $\mathsf{PG}(V)$. On the line $\langle x_1, x_3\rangle$ in $\mathsf{PG}(V)$, there exists some other point that is contained in Δ, since Δ is not a separable quadric, and we denote it by x_5. Since every point of C is collinear to x_1 and x_3, it follows that every point of C is collinear to x_5. The subspace $\langle C, x_5\rangle$ is a subspace in Δ and we denote it by A_5. Now $A_5^\perp \cap A_1^\perp \cap A_2^\perp = H$, and with that, we can write φ as the product of the projectivities

$$\varphi_1 \colon A_1 \barwedge A_2 \barwedge A_3 \barwedge A_5 \barwedge A_1 \text{ and } \varphi_2 \colon A_1 \barwedge A_5 \barwedge A_3 \barwedge A_4 \barwedge A_1.$$

Then φ_1 fixes H pointwise and, similarly, φ_2 fixes a hyperplane of $\mathsf{Res}_\Delta(A_1)$ pointwise. Therefore, $\varphi = \varphi_1\varphi_2$ is the product of two collineations in $\mathsf{Res}_\Delta(A_1)$ that each fix a hyperplane pointwise.

(II) Now suppose $m = n - 2$. Then the intersection $A_1^\perp \cap A_2^\perp \cap A_3^\perp$ could be empty. In that case, we consider the subspace of $\mathsf{PG}(V)$ spanned by $A_3^\perp$ and intersect it with the subspace spanned by $A_1^\perp \cap A_2^\perp$. This intersection will be a hyperplane of the subspace spanned by $A_1^\perp \cap A_2^\perp$, and we take this as H. If the hyperplane spanned by $A_4^\perp$ intersects the space spanned by $A_1^\perp \cap A_2^\perp$ in H as well, we are done. Otherwise, we do the same construction as before and obtain a subspace A_5, such that the hyperplane spanned by $A_5^\perp$ intersects the space spanned by $A_1^\perp \cap A_2^\perp$ in H, and $\varphi = \varphi_1\varphi_2$ as before with the same definitions for φ_1 and φ_2, and the same conclusion holds. □

Lemma 10.7.2.10. *Let Δ be a polar space of rank $n \geq 3$ embedded into $\mathsf{PG}(V)$ that is a separable quadric. Let A_1, A_2, A_3 and A_4 be singular subspaces of dimension $m \leq n - 2$, such that $A_1 \cap A_3 =: B$ and $A_2 \cap A_4 =: C$, with B and C opposite and of dimension $m - 1$, such that A_i is opposite A_{i+1} for $i \in \mathbb{Z}/4\mathbb{Z}$, and such that A_1 and A_3 are contained in a common subspace and A_2 and A_4 are contained in an opposite subspace. Then $\theta \colon A_1 \barwedge A_2 \barwedge A_3 \barwedge A_4 \barwedge A_1$ is a product of two collineations in $\mathsf{Res}_\Delta(A_1)$ that each fix a hyperplane in the ambient projective space.*

Proof. Since Δ is a separable quadric, we can view the perp relation as a non-degenerate polarity.

Let $\Gamma := A_1^\perp \cap A_2^\perp$. Then Γ has rank $n - m - 1$. Since $m \leq n - 2$, the rank of Γ is at least 1. Let H be the hyperplane $A_3^\perp \cap \langle\Gamma\rangle$ and G be the subhyperplane $A_4^\perp \cap H$. If $G = H$, we are done. So suppose G is a proper subset of H.

Define:

$$x_1 := C^\perp \cap A_1, \quad x_2 := B^\perp \cap A_2,$$
$$x_3 := C^\perp \cap A_3, \quad x_4 := B^\perp \cap A_4.$$

Let c_1 be the point $A_2^\perp \cap \langle A_1, A_3\rangle$ and c_2 be the point $A_1^\perp \cap \langle A_2, A_4\rangle$. Note that both c_1 and c_2 belong to Γ. We have $A_2^\perp \cap \langle A_1, A_3\rangle \subseteq C^\perp \cap \langle A_1, A_3\rangle = x_1x_3$. That means c_1 is on the line x_1x_3 and analogously c_2 is on the line x_2x_4. Every point of

G is collinear to x_1 and x_3 and with that also to c_1, and analogously to c_2. That means $G \subseteq c_1^\perp \cap c_2^\perp$, and since G is a subhyperplane of Γ, we have $G = \langle \Gamma \rangle \cap c_1^\perp \cap c_2^\perp$. We may view θ as a collineation in Γ; the fixed points correspond precisely with the points of G.

If $c_1 \perp c_2$, then the singular subspace generated by A_1 and A_3 is not opposite the one generated by A_2 and A_4, contradicting our hypothesis. So c_1 and c_2 are not collinear. Then we can apply Lemma 10.4.4.12 and the proof is complete. □

Lemma 10.7.2.11. *Let Δ be a polar space of rank $r \geq 3$, which is a separable quadric in $\mathsf{PG}(V)$. Let $1 \leq d \leq r-2$. Let U_1 and U_2 be two opposite singular subspaces of dimension d. Let W_1 and W_2 be two singular subspaces of dimension $d+1$, containing U_1 and U_2, respectively, and intersecting in a point p. Let U_3 be a singular subspace of dimension d intersecting $\langle U_1, U_2 \rangle$ in a $(d-1)$-dimensional subspace H. Suppose p is not collinear to all points of U_3. Then the image of W_2 under $U_2 \overline{\wedge} U_3 \overline{\wedge} U_1$ is W_1 if and only if d is odd.*

PROOF. Set $u_i = \mathsf{proj}_{U_i} H$, $i = 1, 2$. Since $U_1, U_2 \subseteq p^\perp$, we find that $H \subseteq p^\perp$. It follows that the line pu_i is the projection of H onto W_i, $i = 1, 2$. Consequently, $\mathsf{proj}_{W_i} U_3 =: x_i \in pu_i$, $i = 1, 2$. Note that by assumption, $p \notin \{x_1, x_2\}$. We conclude that the image of W_2 under $U_2 \overline{\wedge} U_3 \overline{\wedge} U_1$ is W_1 if and only if $x_1 \perp x_2$ (the "if"-part follows from the fact that the $(2d+1)$-dimensional subspace of $\mathsf{PG}(V)$ generated by U_1 and U_2 does not contain singular subspaces of dimension $d+1$ as it intersects Δ in a non-degenerate hyperbolic quadric). This is equivalent to $u_1 \perp u_2$, which, in turn, is equivalent to $u_1 \in \langle H, u_2 \rangle$. This happens if and only if U_1 and U_2 belong to the same natural system of maximal singular subspaces of the hyperbolic quadric obtained by restricting Δ to $\langle U_1, U_2 \rangle$. Since hyperbolic quadrics contain opposite maximal singular subspaces of the same natural system if and only if their rank is even (and equivalently, the dimension of the maximal singular subspaces is odd), the assertion follows. □

This has now the following consequences.

Proof of Theorem N(iii)&(iv) – embeddable case for upper residues of non-maximal singular subspaces

Theorem 10.7.2.12. *Let Δ be an embeddable polar space of rank r at least 3 with at least 4 points per line, and let U be a singular subspace of dimension at most $r-2$. Then $\Pi^+_{\geq}(U)$ is the subgroup of the automorphism group of $\mathsf{Res}(U)$ generated by products of two reflections of $\mathsf{Res}(U)$. If Δ is not a separable orthogonal polar space, then $\Pi^+_{\geq}(U) = \Pi_{\geq}(U)$ is also generated by all reflections. If Δ is a separable orthogonal polar space, then $\Pi_{\geq}(U) = \Pi^+_{\geq}(U)$, if $\dim U$ is odd, and $\Pi(U)$ is the subgroup of the automorphism group of $\mathsf{Res}(U)$ generated by all reflections, if $\dim U$ is even. In the latter case, $\Pi_{\geq}(U)$ is not equal to $\Pi^+_{\geq}(U)$, as soon as the anisotropic dimension of Δ (as a quadric) is even.*

PROOF. Let G_1 be the group of collineations of $\mathsf{Res}(U)$ generated by the reflections, and let $G_2 \leq G_1$ be its subgroup consisting of those members which are a product of an even number of reflections.

By Lemma 10.5.0.1 and Theorem 10.7.1.4, $G_2 \leq \Pi^+_{\geq}(U)$. By Lemma 10.7.2.9 and Lemma 10.7.2.10, we have $\Pi^+_{\geq}(U) \leq G_2$. This proves the first part of the theorem.

If Δ is not a separable orthogonal polar space, then $\Pi^+_{\geq}(U) = \Pi_{\geq}(U)$ by Theorem H. Then the second assertion follows directly from Corollary 10.7.1.5.

Now let Δ be a separable orthogonal polar space. If $\dim U$ is odd, then Theorem H implies $\Pi_{\geq}(U) = \Pi^+_{\geq}(U)$. Now let $\dim U$ be even. Then Lemma 10.7.2.11 implies that there exists a non-trivial self-projectivity of U of length 3 that fixes a hyperplane of $\mathsf{Res}(U)$. It follows now that $\Pi_{\geq}(U)$ is generated by reflections. Finally, if the anisotropic dimension of Δ (as a quadric) is even, then this is also the case for $\mathsf{Res}(U)$. Then each member of $\Pi^+_{\geq}(U)$ preserves the system of generators of the (imaginary) hyperbolic quadric over a splitting field of the quadric, whereas each single reflection interchanges these systems. □

The projectivity groups for upper residues in symplectic polar spaces follow directly from Theorem 2.10.0.3.

Proposition 10.7.2.13. *If Δ is a symplectic polar space of rank r, and U is a singular subspace of Δ of dimension $d \leq r-2$, then $\Pi_{\geq}(U) = \Pi^+_{\geq}(U)$ is the simple symplectic group corresponding to $\mathsf{Res}(U)$, except if $\mathsf{Res}(U)$ has rank 2 and each line has exactly 3 points, in which case the group is isomorphic to the symmetric group on 6 letters.*

Similarly to the case of points, we can also handle the case of an elliptic polar space with three points per line.

Proposition 10.7.2.14. *Let Δ be a polar space of rank r with three points per line and distinct from a symplectic polar space. Let U be a singular subspace of dimension $d \leq r-2$. Then $\Pi^+_{\geq}(U)$ is the simple orthogonal group associated to $\mathsf{Res}(U)$. If d is odd, then $\Pi_{\geq}(U) = \Pi^+_{\geq}(U)$, whereas if d is even, $\Pi_{\geq}(U)$ is the special orthogonal group associated to $\mathsf{Res}(U)$.*

A special case of Theorem 10.7.2.12 is worth mentioning.

Corollary 10.7.2.15. *Let Δ be a σ-Hermitian polar space over a commutative field $\mathbb{K}$ with anisotropic dimension 0, and let $\mathbb{F}$ be the fixed field of σ. Let U be a submaximal singular subspace. Then $\Pi^+_{\geq}(U) = \Pi_{\geq}(U)$ is permutation equivalent to $\mathsf{PGL}_2(\mathbb{F})$.*

Proof of Theorem N(iv) – non-embeddable case for upper residues of non-maximal singular subspaces

Since each non-embeddable polar space contains an ideal σ-Hermitian polar subspace (*ideal* means that every plane of the large space through a line of the subspace belongs entirely to the subspace), we immediately deduce from Corollary 10.7.2.15 the following result:

Corollary 10.7.2.16. *Let Δ be a non-embeddable polar space with planes isomorphic to projective planes over an octonion division algebra with center $\mathbb{K}$. Let L be a line of Δ. Then $\Pi^+_{\geq}(L) = \Pi_{\geq}(L)$ is permutation equivalent to $\mathsf{PGL}_2(\mathbb{K})$.*

10.8. Projectivity Groups for Lower Residues in Polar Spaces

We now turn to the lower residues, which are always projective, as we have seen in Definition 3.6.2.17. First, we handle the case of non-maximal projective residues, and after that the case of maximal ones.

10.8.1. Non-maximal Projective Residues

10.8.1.1. Embeddable Polar Spaces

Proof of Theorem N(i)&(v) – embeddable case for mixed residues and lower residues of non-maximal singular subspaces

This case is straightforward, using Lemma 10.5.0.1. Indeed, the special projectivity group of a subspace D of dimension d of a Desarguesian projective space over the skew field $\mathbb{L}$ is the full linear group $\mathsf{PGL}(d, \mathbb{L})$, so it also coincides with the special projectivity group inherited from the polar space. For the general projectivity group, we have to add a duality, which is linear if the polar space is orthogonal or symplectic, but which is semi-linear with companion skew field automorphism σ if the polar space is associated to a σ-quadratic form. If $\dim D = 1$, then this duality is just a collineation of $\mathsf{PG}(1, \mathbb{L})$ (with $\mathbb{L}$ the underlying skew field) induced by σ. In the orthogonal case, the duality belongs to $\mathsf{PGL}_2(\mathbb{L})$, and so the general and special groups coincide; in the other cases, the special group has index 2 in the general group. For the case of mixed residues the same conclusions hold.

10.8.1.2. Non-embeddable Polar Spaces

Here, the residues have type 1 or 2 in Bourbaki labeling. We look at the lower residue of a line, and at the planar line pencils. We first consider the lower residues of a line.

Proof of Theorem N(i) – non-embeddable case for lower residues of maximal singular subspaces

Proposition 10.8.1.1. *Let Δ be the non-embeddable polar space associated to the Cayley division algebra $\mathbb{O}$ over the field $\mathbb{K}$. Let L be a line of Δ. Then $\Pi^+_{\leq}(L)$ is permutation equivalent to the group of direct similitudes of the quadric of Witt index 1, defined by the anisotropic form given by the norm form of $\mathbb{O}$ over $\mathbb{K}$. Also, $\Pi_{\leq}(L)$ is isomorphic to the group generated by $\Pi^+_{\leq}(L)$ and a permutation induced by the standard involution of $\mathbb{O}$, when considering L as a projective line over $\mathbb{O}$.*

Proof. We begin by noting that the mentioned permutation group is equal to the projectivity group of a line in the Moufang plane $\mathsf{PG}(2, \mathbb{O})$, as proved by

Grundhöfer [**Gru83**, Satz]. So, in view of Lemma 10.5.0.1, it suffices to show that every (lower) even projectivity of L in Δ coincides with a projectivity inside some plane of Δ containing L. Thanks to Theorem K', Remark 10.3.2.4, Proposition 10.4.1.1, and Proposition 10.4.1.4, it suffices to show this for lower projectivities of the form $L \overline{\wedge} L_2 \overline{\wedge} L_3 \overline{\wedge} L_4 \overline{\wedge} L$, where L and L_3 intersect in a point p_1 and generate a plane π_1, where L_2 and L_4 intersect in a point p_2 and generate a plane π_2, and where p_1 and p_2 are opposite, and π_1 and π_2 are also opposite. Set $c_i = \mathsf{proj}_{\pi_1} L_i$, $i = 2, 4$. Then one observes that, for each $x \in L$, $x_3 := \mathsf{proj}_{L_3}\mathsf{proj}_{L_2} x$ is contained in the line xc_2, and $x' := \mathsf{proj}_L \mathsf{proj}_{L_4} x_3$ lies on the line $x_3 c_4$. Hence, $L \overline{\wedge} L_2 \overline{\wedge} L_3 \overline{\wedge} L_4 \overline{\wedge} L$ coincides with the projectivity $L \overline{\wedge} c_2 \overline{\wedge} L_3 \overline{\wedge} c_4 \overline{\wedge} L$ inside π_1. This completes the proof of the first part of the proposition.

The second part follows from Section 10.8.1.1 by noting that Δ contains (many) polar spaces over quaternion subfields. □

Proof of Theorem N(v) – non-embeddable case for mixed residues

Proposition 10.8.1.2. *Let Δ be the non-embeddable polar space associated to the Cayley division algebra $\mathbb{O}$ over the field $\mathbb{K}$. Let P be a planar line pencil of Δ. Then $\Pi^+(P)$ is permutation equivalent to the group of direct similitudes of the quadratic form of Witt index 1 defined by the norm form of $\mathbb{O}$ over $\mathbb{K}$. Also, $\Pi(P)$ is isomorphic to the group generated by $\Pi^+(P)$ and the permutation induced by the standard involution of $\mathbb{O}$, when considering P as a projective line over $\mathbb{O}$.*

Proof. We identify a line pencil with its point-plane pair. As in the previous proof, it suffices to show that, for line pencils P_2, P_3, P_4, with $P \sim P_3$, and $P_2 \sim P_4$ (see the next paragraph for more details on the adjacency $\sim$), such that both P and P_3 are opposite both P_2 and P_4, the projectivity $P \overline{\wedge} P_2 \overline{\wedge} P_3 \overline{\wedge} P_4 \overline{\wedge} P$ is a projectivity of P inside the plane it defines.

Assume that $P = \{p, \pi\}$, with p a point in the plane π, and likewise $P_i = \{p_i, \pi_i\}$, $i = 2, 3, 4$. Then $P \sim P_3$ means that either $\pi = \pi_3$ and $p \neq p_3$, or $p = p_3$ and $\pi \cap \pi_3$ is a line. Suppose first that $\pi = \pi_3$. Let L be the line pp_3 and set $M := \mathsf{proj}_\pi p_2$. Note $M \cap L \notin \{p, p_3\}$. Let K be an arbitrary line of P. Then its image under $P \overline{\wedge} P_2$ is the line K_2 through p_2 and $\mathsf{proj}_{\pi_2} K$. It follows that $\mathsf{proj}_\pi K_2 = K \cap M$. But then the image K_3 of K_2 under $P_2 \overline{\wedge} P_3$ is the line joining p_3 with $K \cap M$. It follows that $P \overline{\wedge} P_2 \overline{\wedge} P_3$ coincides with the projectivity $p \overline{\wedge} M \overline{\wedge} p_3$ in the projective plane π.

Now assume $p = p_3$ and $\pi \cap \pi_3$ is a line $L \in P \cap P_3$. Set $M = \mathsf{proj}_\pi p_2$, $M_3 = \mathsf{proj}_{\pi_3} p_2$, and $M_2 = \mathsf{proj}_{\pi_2} p$. Let K again be an arbitrary line of P and set $x = K \cap M$, $x_2 = \mathsf{proj}_{\pi_2} K = \mathsf{proj}_{M_2} x$, $K_2 = p_2 x_2$, $x_3 = \mathsf{proj}_{\pi_3} K_2 = \mathsf{proj}_{M_3} x_2$, and $K_3 = p x_3$. Then we find that K_3 is the image of K under $P \overline{\wedge} P_2 \overline{\wedge} P_3$, and x_3 is the image of x under $M \overline{\wedge} M_2 \overline{\wedge} M_3$ (lower perspectivities). Hence, $P \overline{\wedge} P_2 \overline{\wedge} P_3$ can be written as the product of $p \overline{\wedge} M$ in π, with $M \overline{\wedge} M_2 \overline{\wedge} M_3$ in Δ, and finally $M_3 \overline{\wedge} p$ in π_3. Since a similar decomposition holds for $P_3 \overline{\wedge} P_4 \overline{\wedge} P$, we conclude, using Proposition 10.8.1.1, that $P \overline{\wedge} P_2 \overline{\wedge} P_3 \overline{\wedge} P_4 \overline{\wedge} P$ is contained in the projectivity group of p inside π. The first part of the proposition is proved.

The second part follows from the second part of Proposition 10.8.1.1 by noting that, if $Q = \{q, \alpha\}$ is an arbitrary line pencil opposite P, the perspectivity $P \overline{\wedge} Q$ is the

product of the perspectivity $p \barwedge \mathsf{proj}_\pi q$ inside π, the lower perspectivity $\mathsf{proj}_\pi q \barwedge \mathsf{proj}_\alpha p$, and the perspectivity $\mathsf{proj}_\alpha p \barwedge q$ inside the plane α. □

Remark 10.8.1.3. The standard involution of a split octonion algebra interchanges the two systems of maximal singular subspaces, as follows immediately from the fact that it interchanges two of the eight standard basis vectors and maps the others to their additive inverse (in the representation of Zorn). Hence, the general projectivity groups in Proposition 10.8.1.1 and Proposition 10.8.1.2 are the groups of all similitudes of said quadratic forms. This also holds for the quadratic associative division algebras over the field $\mathbb{K}$ with non-trivial standard involution. For further use, given a quadratic alternative division algebra $\mathbb{A}$ over $\mathbb{K}$ with non-trivial standard involution, we denote the group of all permutations of the projective line $\mathsf{PG}(1, \mathbb{A})$ corresponding to direct similitudes by $\mathsf{PGL}_2^+(\mathbb{A})$, and the group of all permutations corresponding to all similitudes by $\mathsf{PGL}_2(\mathbb{A})$; except, if $\mathbb{A}$ is commutative (and this notation would be ambiguous), then we use $\mathsf{PGL}_2^+(\mathbb{A}/\mathbb{K})$ and $\mathsf{PGL}_2(\mathbb{A}/\mathbb{K})$, respectively.

10.8.2. Maximal Singular Subspaces

10.8.2.1. Reduction to the Composition of Four Perspectivities

Proposition 10.8.2.4. *Let Δ be a polar space of rank $r \geq 3$. Let M be a maximal singular subspace of Δ. Suppose the set of projectivities $M \barwedge M_2 \barwedge M_3 \barwedge M_4 \barwedge M$, with $M \equiv M_2 \equiv M_3 \equiv M_4 \equiv M$ and $U_0 := M \cap M_2$ and $U_1 := M_1 \cap M_3$ opposite submaximal singular subspaces, is geometric. Then $\Pi^+(M)$ is generated by all such projectivities.*

Proof. This follows from Remark 10.3.2.4 and Proposition 10.4.1.4. □

As we will see below, the projectivities $M \barwedge M_2 \barwedge M_3 \barwedge M_4 \barwedge M$, mentioned in the previous proposition, are homologies. So proving that said set is geometric is equivalent to showing that the set of factors of these homologies is closed under conjugation with any element of the underlying skew field.

10.8.2.2. Computation of the Special Projectivity Group in the General (Non-symplectic Embeddable) Case

We use the standard form, see [**Van24**, Chapter 4]. Let Δ be an embeddable polar space with pseudo-quadratic description in $\mathsf{PG}(V)$, for some right vector space V over some skew field $\mathbb{L}$, and let σ be an involution of $\mathbb{L}$. Let V_0 be a subspace of V of codimension $2n$, and let

$$\{e_i \mid i \in \{-n, -n+1, \ldots, -1, 1, \ldots, n\}\}$$

be a (suitably chosen) basis of a complementary subspace, such that

$$\{\langle e_i \rangle \mid i \in \{-n, -n+1, \ldots, -1, 1, \ldots, n\}\}$$

is a polar frame. For general vectors $v, w \in V$ we write $v = v_0 + \sum_{i \in I} e_i x_i$ and $w = w_0 + \sum_{i \in I} e_i y_i$ (where $v_0, w_0 \in V_0$ and $x_i, y_i \in \mathbb{L}$ for all $i \in I$). Then there

exists an anisotropic σ-quadratic form $\mathfrak{q}_0$ on V_0, where $\mathfrak{q}_0(v_0) = g_0(v_0, v_0) + \mathbb{L}_\sigma$, for the $(\sigma, 1)$-linear form $g_0 : V_0 \times V_0 \to \mathbb{L}$, with associated σ-Hermitian form f_0, such that

$$\begin{aligned} \mathfrak{q}(v) &= x_{-n}^\sigma x_n + \cdots + x_{-1}^\sigma x_1 + \mathfrak{q}_0(v_0), \\ f(v, w) &= x_{-n}^\sigma y_n + x_n^\sigma y_{-n} + \cdots + x_{-1}^\sigma y_1 + x_1^\sigma y_{-1} + f_0(v_0, w_0) \end{aligned}$$

define the points and collinearity, respectively, of the polar space Δ.

Let $v_0, w_0 \in V_0$, $t, u \in \mathbb{L}$ be such that

$$D := D(v_0, w_0, t, u) := (g_0(w_0, w_0) + u - u^\sigma)(g_0(v_0, v_0) + t - t^\sigma) + f_0(w_0, v_0) + 1 \neq 0.$$

We abbreviate $g_1 := g_0(v_0, v_0) + t - t^\sigma$ and $g_2 := g_0(w_0, w_0) + u - u^\sigma$ and define the following four maximal singular subspaces:

$$\begin{cases} M_1 = \langle e_{-1}, e_{-2}, \ldots, e_{-n} \rangle, \\ M_2 = \langle e_1, e_2, \ldots, e_n \rangle, \\ M_3 = \langle g_1 e_1 + v_0 + e_{-1}, e_{-2}, \ldots, e_{-n} \rangle, \\ M_4 = \langle g_2^\sigma e_{-1} + w_0 + e_1, e_2, \ldots, e_n \rangle. \end{cases}$$

Then, by our condition above, we have $M_1 \equiv M_2 \equiv M_3 \equiv M_4 \equiv M_1$ and $M_1 \sim M_3$ and $M_2 \sim M_4$ (cf. Definition 10.3.2.3). Set $\theta := M_1 \barwedge M_2 \barwedge M_3 \barwedge M_4 \barwedge M_1$. Then θ pointwise fixes the hyperplane $\langle e_{-2}, \ldots, e_{-n} \rangle$ of M_1 and also the point $\langle e_{-1} \rangle$. Hence, θ is a homology, and we determine its factor. We can work in the subspace W generated by e_{-2}, e_{-1}, e_1, e_2 and V_0. We write a generic element of that subspace as $(x_{-2}, x_{-1}, x_0, x_1, x_2)$, where all elements belong to $\mathbb{L}$, except $x_0 \in V_0$.

We consider a generic point $(1, x, 0, 0, 0)$ on $\langle e_{-2}, e_{-1} \rangle$ distinct from e_{-2}, so we may assume $x \neq 0$. Its image in $M_2 \cap W$ under the perspectivity $M_1 \barwedge M_2$ is the point $(0, 0, 0, 1, -x^\sigma)$. The image of that point in $M_3 \cap W$ under the perspectivity $M_2 \barwedge M_3$ is the point $(x^{-1}, 1, v_0, g_1, 0)$. Likewise, the image of the latter in $M_4 \cap W$, under the perspectivity $M_3 \barwedge M_4$, is the point $(0, g_2^\sigma, w_0, 1, -x^\sigma D^\sigma)$. Finally, the image of that point in $M_1 \cap W$, back again under the perspectivity $M_4 \barwedge M_1$, is the point $(1, Dx, 0, 0, 0)$.

We can now formulate the main result of this subsection. The set

$$\{D(v_0, w_0, t, u) \mid v_0, w_0 \in V_0, t, u \in \mathbb{L}\}$$

will be referred to as the *norm set*.

Proof of Theorem N(ii) – embeddable pseudo-quadratic case for lower residues of maximal singular subspaces

Proposition 10.8.2.5. *The special projectivity group of a maximal singular subspace of an embeddable polar space with pseudo-quadratic description in standard form as above is generated by homologies with factors in the norm set.*

Proof. In view of the preceding computations and Proposition 10.8.2.4, it only remains to show that the norm set is closed under conjugation with an arbitrary element $r \in \mathbb{L}^\times$. This follows straight from the identities

$$r^{-1} f_0(w_0, v_0) r = f_0(w_0 r^{-\sigma}, v_0 r), \quad r^\sigma (t - t^\sigma) r = (r^\sigma t r) - (r^\sigma t r)^\sigma.$$

□

It strongly depends on g_0 what exactly the special projectivity group of a maximal singular subspace is. We present a few special cases.

For a parabolic polar space of rank r, $V_0 = \mathbb{L}$, $\sigma = \mathsf{id}$, and $g_0(x_0, y_0) = x_0 y_0$. It follows that $f_0(x_0, y_0) = 2x_0 y_0$, and consequently,

$$D(x_0, y_0, t, u) = x_0^2 y_0^2 + 2x_0 y_0 + 1 = (x_0 y_0 + 1)^2.$$

Hence, the special projectivity group here is $\mathsf{PGL}_r(\mathbb{L})$ if r is odd, and is the linear subgroup of $\mathsf{PGL}_r(\mathbb{L})$ consisting of the matrices with a square determinant if r is even.

Now, suppose $V_0 = \mathbb{L} \oplus \mathbb{L}$, $\mathsf{char}\,\mathbb{L} \neq 2$, and $\sigma = \mathsf{id}$. Let g_0 be given by

$$g_0((x, y), (x', y')) = xx' + \ell yy'.$$

Then

$$D((x, y), (x', y'), t, u) = (1 + xx' + yy')^2 + \ell(xy' - x'y)^2,$$

and so the special projectivity group is generated by homologies with factor a norm of the quadratic extension of $\mathbb{L}$ corresponding to g_0. In the finite case, every element of $\mathbb{L}$ is a norm, and then we simply have $\mathsf{PGL}_r(q)$, $q = |\mathbb{L}|$.

For Hermitian polar spaces over commutative fields with non-trivial involution, see the next paragraph.

10.8.2.3. Special Projectivity Groups for Symplectic Polar Spaces and some Hermitian Ones

The previous subsection does not cover the symplectic polar spaces (it would if we considered skew-Hermitian forms rather than Hermitian forms). However, in this case, the simple group (generated by the root groups) is easy to describe, and hence we can directly use Theorem 2.10.0.3. The same method can be applied to Hermitian polar spaces over commutative fields with non-trivial field involution.

Proof of Theorem N(ii) – symplectic case for lower residues of maximal singular subspaces

Proposition 10.8.2.6. *Let Δ be a symplectic polar space of rank r, defined over the field $\mathbb{K}$. Let U be a maximal singular subspace. Then $\Pi^+(U) \cong \mathsf{PGL}_r(\mathbb{K})$ in its standard permutation representation. Also, $\Pi(U) \cong \mathsf{PGL}_r(\mathbb{K}) \rtimes \langle\rho\rangle$, with ρ any linear polarity.*

Proof. Let Δ be described by the alternating form

$$(x_{-r}y_r + x_{-r+1}y_{r-1} + \cdots + x_{-1}y_1) - (y_{-r}x_r + y_{-r+1}x_{r-1} + \cdots + y_{-1}x_1).$$

Using obvious notation, the subspace U, spanned by $e_{-1}, \ldots, e_{-r}$, is a maximal singular subspace, and the subspace W generated by $e_1, \ldots, e_r$ is an opposite maximal singular subspace. Set $X_- = (x_{-1}, \ldots, x_{-r})$ and $X_+ = (x_1, \ldots, x_r)$, which we also read as row matrices. Then, for every non-singular $r \times r$ matrix M, the linear transformation

$$X_- \mapsto X_- \cdot M^t, \qquad X_+ \mapsto X_+ \cdot M^{-1}$$

induces a linear collineation in $\mathsf{PG}(2r-1,\mathbb{K})$, preserving the above alternating form. Since such a collineation of Δ belongs to the automorphism group generated by the root elations, as follows from [**Die43b**, Théorème 1], we find, using Theorem 2.10.0.3, that $\Pi^+(U) \cong \mathsf{PGL}_r(\mathbb{K})$. Now a single perspectivity induces a linear duality (over $\mathbb{K}$) from one maximal singular subspace to another, concluding the proof of the proposition. □

A similar argument can be given to obtain a slightly more concrete version of Proposition 10.8.2.5 in the case of Hermitian polar spaces over a commutative field with non-trivial involution. For a given field $\mathbb{K}$ and subfield $\mathbb{F}$, let $\mathsf{SL}_r(\mathbb{K};\mathbb{F})$ be the multiplicative group of $r \times r$ matrices with a determinant in $\mathbb{F}^\times$, and let $\mathsf{PSL}_r(\mathbb{K};\mathbb{F})$ be the corresponding projective group, that is, the quotient group by its center.

Proof of Theorem N(ii) – hermitian case for lower residues of maximal singular subspaces

Proposition 10.8.2.7. *Let Δ be a Hermitian polar space of rank r, defined over the field $\mathbb{K}$, with non-trivial involution σ and fixed field $\mathbb{F}$. Let U be a maximal singular subspace.*

- (*i*) *If the anisotropic part is trivial, then $\Pi^+(U) \cong \mathsf{PSL}_r(\mathbb{K};\mathbb{F})$ in its standard permutation representation. Also, $\Pi(U) \cong \mathsf{PSL}_r(\mathbb{K};\mathbb{F}) \rtimes \langle\rho\rangle$, with ρ any polarity with companion involution σ.*
- (*ii*) *If the anisotropic part is non-trivial, then $\Pi^+(U) \cong \mathsf{PGL}_r(\mathbb{K})$ in its standard permutation representation. Also, $\Pi(U) \cong \mathsf{PGL}_r(\mathbb{K}) \rtimes \langle\rho\rangle$, with ρ any polarity with companion involution σ.*

Proof. First, we note that a single perspectivity induces a duality with companion field automorphism σ (as follows from the computations preceding Proposition 10.8.2.5). Hence, the special projectivity group will be a linear group. Let θ be any (linear) collineation of Δ in the little projective group $G^\dagger$ of Δ stabilizing U. Since the automorphism group of Δ acts transitively on opposite (ordered) pairs of maximal singular subspaces, we may assume that θ stabilizes an opposite maximal singular subspace. We consider the standard Hermitian form

$$(x_{-1}^\sigma x_1 + \cdots + x_{-r}^\sigma x_r) + (x_1^\sigma x_{-1} + \cdots + x_r^\sigma x_{-r}) + f_0(x_0, x_0),$$

where f_0 is an anisotropic Hermitian form on some vector space V_0 with shorthand coordinates x_0. We take for U the subspace generated by $e_{-r}, \ldots, e_{-1}$. By the above, we may assume that θ also stabilizes the maximal singular subspace W, generated by $e_1, \ldots, e_r$. Let θ be given by

$$\begin{pmatrix} x_{-1} & x_{-2} & \ldots & x_{-r} \end{pmatrix} \mapsto \begin{pmatrix} x_{-1} & x_{-2} & \ldots & x_{-r} \end{pmatrix} \cdot M^t,$$

with, for now, M any non-singular $r \times r$ matrix over $\mathbb{K}$ (the transpose is taken for notational reasons) with determinant $k \in \mathbb{K}^\times$. The action of θ on W is described by

$$\begin{pmatrix} x_1 & x_2 & \ldots & x_r \end{pmatrix} \mapsto \begin{pmatrix} x_1 & x_2 & \ldots & x_r \end{pmatrix} \cdot M^{-\theta}.$$

- (*i*) If V_0 is trivial, then the determinant of θ as a linear map in $\mathsf{PG}(2r-1,\mathbb{K})$ is equal to $\ell := kk^{-\theta}$. Then, according to [**Die43b**, Théorème 5], θ belongs to $G^\dagger$ if and only if $\ell = 1$, that is, $k = k^\theta$. Now the statement of (*i*) follows.

(*ii*) Suppose now that V_0 is not trivial. Since $\mathsf{PGL}_r(\mathbb{K})$ is the full group of linear transformations, it suffices to show the assertion in the case of $\dim V_0 = 1$. Then we may assume $f_0(x_0, x_0) = a_0 x_0^\sigma x_0$. We define the action of θ on the coordinate x_0 as $x_0 \mapsto \ell^{-1} x_0$. We find that

$$f_0(\ell x_0, \ell x_0) = a_0 \ell^\sigma \ell x_0^\sigma x_0 = a_0 x_0^\sigma x_0,$$

since $\ell^\sigma \ell = 1$, as one computes. With that, the determinant of θ as a linear map in $\mathsf{PG}(2r, \mathbb{K})$ is equal to 1, and hence, by [**Die43b**, Théorème 5] again, θ belongs to $G^\dagger$. This proves the assertions in (*ii*). □

10.8.2.4. Non-embeddable Polar Spaces

In this case, we could do a computation similar to the one performed in the previous subsection. However, using a result from [**PV24**], it is more efficient to use Theorem J'. Let $\mathbb{O}$ be a Cayley division algebra over the field $\mathbb{K}$ with standard involution $x \mapsto \overline{x}$, and let $\mathsf{PG}(2, \mathbb{O})$ be the associated projective plane, coordinatised as in Section 5.1.4. Then, for all $k, \ell \in \mathbb{K}$, the mapping with action on the points (x, y), with $y \neq 0$, given by

$$(x, y) \mapsto [k\overline{y}^{-1}\overline{x}, \ell\overline{y}^{-1}]$$

induces, by [**PV24**, §6.2], a polarity $\rho(k, \ell)$ of $\mathsf{PG}(2, \mathbb{O})$, which we call a *standard polarity*.

Remark 10.8.2.8. Replacing $\mathbb{O}$ with any quadratic associative division algebra $\mathbb{A}$, the above expression remains a polarity in the corresponding projective plane, and we also call such a polarity a *standard polarity* (with respect to the given quadratic structure, that is, relative to the field $\mathbb{K}$, which might not be determined by $\mathbb{A}$ if $\mathbb{A}$ is commutative). If the standard involution of $\mathbb{A}$ is non-trivial, then the polarity is Hermitian.

The following result is now an immediate consequence of the computations in [**PV24**, §6.2].

Lemma 10.8.2.9. *Let Δ be a thick non-embeddable polar space of rank 3. Then every perspectivity $\pi_1 \barwedge \pi_2 \barwedge \pi_3 \barwedge \pi_1$, with π_1, π_2, π_3 pairwise opposite planes, is a standard polarity.*

Noting that the class of standard polarities is geometric, we immediately conclude that $\Pi(\pi)$, for π a plane of a non-embeddable thick polar space, is the group generated by all standard polarities. We can be slightly more specific. Recall that a projective collineation is a collineation which induces a projectivity on at least one line (and hence on all lines). The set of such collineations forms a group, called the *full projective group*, which we denote by $\mathsf{PGL}_3(\mathbb{O})$, with slight abuse of notation (because there does not exist a 3-dimensional vector space over $\mathbb{O}$).

Proof of Theorem N(ii) – non-embeddable case for lower residues of maximal singular subspaces

Proposition 10.8.2.10. *Let Δ be the thick non-embeddable polar space associated with the Cayley division algebra $\mathbb{O}$ over the field $\mathbb{K}$. Let π be any plane of Δ. Then*

$\Pi^+(\pi)$ *is the full projective group of* $\mathsf{PG}(2,\mathbb{O})$*, whereas* $\Pi(\pi)$ *is the full projective group extended with a standard polarity.*

PROOF. By [**PV24**, Section 6.2], the action of $\rho(k,\ell)$ on lines $[m,k]$, with $k \neq 0$, is given by

$$[m,k] \mapsto (k^{-1}\ell\overline{m}\overline{k}^{-1}, \ell\overline{k}^{-1}).$$

We can calculate the action of $\rho(k,\ell)\rho(1,1)$ on the points (x,y), $y \neq 0$, and obtain

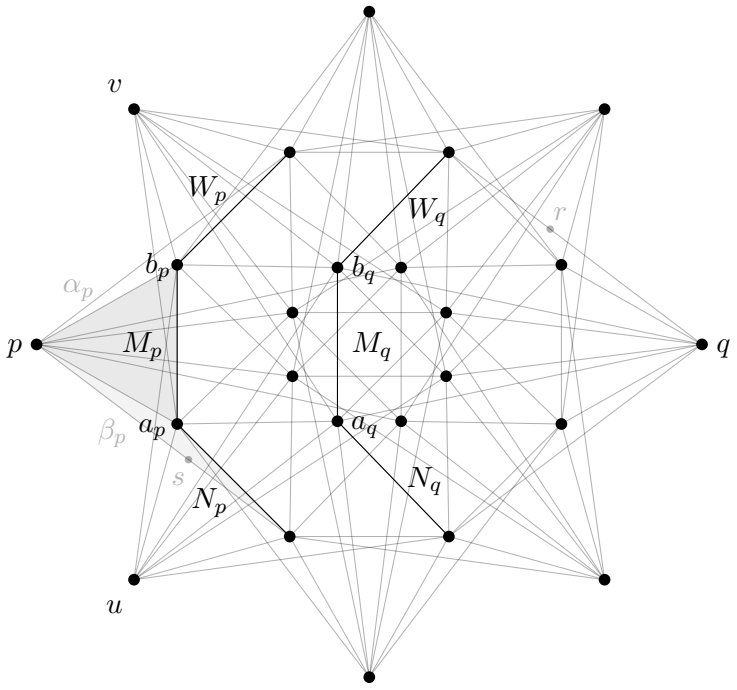
$$(x,y) \mapsto (k\ell^{-1}x, \ell^{-1}y).$$

Setting $\ell = 1$, we see that we obtain all homologies with axis $[0]$ and center (0). Then the assertion follows from [**Pic59**, Satz 3]. □

10.9. Projectivity Groups of Points of Metasymplectic Spaces

The only projectivity groups left to determine are the ones for non-maximal simplices with irreducible residues in buildings of type F_4, which correspond to Lie incidence geometries that are metasymplectic spaces. Again, we first take care of points, and after that, we consider all non-maximal residues.

For the first proposition of this section, we use the visualisation of an apartment of a building of type F_4 as provided in Section 5.6.4.



Again, we call a collineation of a polar space a *homology* if it pointwise fixes two opposite planes, which we call the *axes* of the homology.

Proposition 10.9.0.1. *Let p be a point of a metasymplectic space Γ. Let q be any opposite point, and let $s \perp p$ and $r \perp q$ be further points, such that $ps \equiv qr$, $p \equiv r$ and $q \equiv s$. Then the projectivity $\rho_{r,s}\colon p \barwedge q \barwedge s \barwedge r \barwedge p$ is a homology of the polar space Δ_p, corresponding to the residue at p, which pointwise fixes planes α_s and α_r, the planes corresponding to the lines sp and the projection of rq onto p, respectively. Moreover, if L is any line of Δ_p intersecting α_r and α_s non-trivially, then r and s can be chosen such that the projectivity $\rho_{r,s}$ maps a given plane of Δ through L, not adjacent to either α_r or α_s, to any other such plane.*

PROOF. The line L of Δ_p corresponds to a plane α through p in Γ. Let Σ be an apartment in Γ that contains the opposite lines ps and qr, the points p and q, as well as the plane β (which will only be relevant in the last part of the proof; thus, in the first part, we do not use this information). Let α_p and β_p be two planes in Σ through p that intersect in a line, and let $s \in \beta_p$. Also, for the last part of the proof (Claim 3), we choose these planes such that $\alpha_p = \alpha$. Let M_p and N_p be the lines in α_p and β_p, respectively, that are closest to q (meaning the points of these lines have distance two to q and are special to q). We denote by a_p the intersection point of M_p and N_p. Let M_q and N_q be the lines in $q^\perp \cap \Sigma$, such that each point of M_q (N_q, respectively) is collinear to a unique point of M_p (N_p). Let u be the point of Σ that is collinear to all points of M_p, N_p, M_q and N_q. Let W_p be the line of Σ that intersects α_p in exactly one point, whose points are all collinear to p and at distance 2 to q and which does not contain a point collinear to N_p. Let b_p be the intersection point of M_p and W_p. Again, let W_q be the line in $q^\perp \cap \Sigma$, such that each point of W_q is collinear to a unique point of W_p. Consider the plane $\langle M_p, u\rangle$ and let $S \subseteq \langle M_p, u\rangle$ be the line through a_p collinear to s (this line exists considering the symp containing p and u). We note that the plane $\langle S, s\rangle$ intersects α_p exactly in the point a_p, is not contained in $p^\perp$, and since s has distance 3 to q, we have $u \notin S$.

Let ξ be the octahedron of Σ containing M_p, M_q and u. Let v be the point in ξ that is opposite u and collinear to every point of M_p and M_q. Note that v is also collinear to W_p and W_q. Hence, v and q are contained in a common symp.

Let a_q be the point $M_q \cap N_q$; that is, the unique point of M_q and N_q that is collinear to a_p. Then $\langle a_p, a_q, v\rangle$ spans a plane of Σ. We denote the point $M_q \cap W_q$ by b_q.

Claim 1. *We claim that the projectivity $p \barwedge q \barwedge s \barwedge r \barwedge p$ stabilizes the planes $\langle p, N_p\rangle$ and $\langle p, W_p\rangle$ and fixes the lines pa_p and pb_p.*

Indeed, the plane $\langle p, N_p\rangle$ maps to $\langle q, N_q\rangle$ under the first projection and back to $\langle p, N_p\rangle$ under the second projection. It then maps to some plane through r, but since that plane is the image of $\langle p, N_p\rangle$, it maps back to $\langle p, N_p\rangle$ under the fourth projection. Similarly, $\langle p, W_p\rangle$ projects to $\langle q, W_q\rangle$, to some plane through s, and then back to $\langle q, W_q\rangle$ and $\langle p, W_p\rangle$.

The line pa_p maps to qa_q, sa_p, the line between r and $R \cap va_q$, and back to pa_p. The line pb_p maps to the line between q and $W_q \cap M_q$, maps to the line between s and $S \cap ub_p$, maps to the line between r and $W_q \cap M_q$, and back to pb_p.

Claim 2. *We claim that the projectivity $p \barwedge q \barwedge s \barwedge r \barwedge p$ stabilizes all planes through ps.*

This follows from Step 1 by varying Σ through $\{p, ps\}$ and $\{q, qr\}$.

Claim 3. *We claim that we can always define s and r in a way such that the projectivity $p \barwedge q \barwedge s \barwedge r \barwedge p$ moves a line px with x on M_p to a line px' with x' on M_p, $x \neq x'$ and $x, x' \notin \{a_p, b_p\}$.*

Let x and x' be two distinct points on M_p not equal to a_p or b_p. Let x_1 be the unique point of M_q collinear to x. Then px moves to qx_1 under the first projection.

Considering the symp spanned by ξ, we can see that x_1 must be collinear to a unique point x_2 of S. With that, qx_1 has to move to sx_2 under the second projection. The points x_2 and x' are each collinear to a line of $\langle v, a_q, b_q\rangle$. These lines are distinct as x_1 belongs to one of them but not the other. We define x_3 as the intersection point of these two lines. Let R be the connection line between b_q and x_3. Every point of $R \setminus \{b_q\}$ is collinear to the same line of $\langle q, W_q\rangle$ through b_q that we will denote by R'. In Σ, we see that W_q and qb_q form a triangle, and we denote the third point of that triangle in Σ by c_q. We define r to be the intersection point of R' and qc_q. Then sx_2 moves to rx_3 under the third projection and to px' under the last projection, as desired. □

We now first handle the case of type 1 vertices in $\mathsf{F}_4(\mathbb{K}, \mathbb{A})$, with $\mathbb{A}$ a quadratic alternative division algebra over $\mathbb{K}$. We may assume that $|\mathbb{A}| > 2$, as otherwise the groups of projectivity coincide with the full group of collineations $\mathsf{Sp}_6(2)$ by Theorem 2.10.0.3, since that group is also generated by the elations.

We denote by $\mathsf{PSp}_{2\ell}(\mathbb{K})$ the group of all collineations of $\mathsf{C}_{\ell,1}(\mathbb{K}, \mathbb{K})$ preserving the associated alternating form, $\ell \geq 2$. It is a simple group. We denote by $\overline{\mathsf{PS}}\mathsf{p}_{2\ell}(\mathbb{K})$ the group generated by $\mathsf{PSp}_{2\ell}(\mathbb{K})$ and the *diagonal* collineations, that is, the linear collineations of the underlying vector space mapping the associated alternating form to a non-zero scalar multiple and represented by a diagonal matrix.

If $\mathsf{char}\,\mathbb{K} = 2$, and $\mathbb{K}'$ is an field extension of $\mathbb{K}$ all of whose squares are contained in $\mathbb{K}$, then the polar space $\mathsf{C}_{\ell,1}(\mathbb{K}', \mathbb{K})$ is a polar subspace of $\mathsf{C}_{\ell,1}(\mathbb{K}', \mathbb{K}')$, and we denote the restriction of $\overline{\mathsf{PS}}\mathsf{p}_{2\ell}(\mathbb{K}')$ to $\mathsf{C}_{\ell,1}(\mathbb{K}', \mathbb{K})$ by $\overline{\mathsf{PS}}\mathsf{p}_{2\ell}(\mathbb{K}', \mathbb{K})$.

Likewise, for $\mathbb{A}$ a separable quadratic extension of $\mathbb{K}$, when $\mathsf{U}_6(\mathbb{A}/\mathbb{K})$ is the (simple) unitary group preserving a Hermitian form and whose elements correspond to matrices of determinant 1, then we denote by $\overline{\mathsf{U}}_6(\mathbb{A}/\mathbb{K})$ the group generated by $\mathsf{U}_6(\mathbb{A}/\mathbb{K})$ and all diagonal automorphisms with diagonal elements in $\mathbb{K}$ (with respect to the standard form).

Now let $\mathbb{A}$ be a quaternion division algebra over $\mathbb{K}$. Let $\mathsf{U}_6(\mathbb{A})$ denote the (simple) collineation group of $\mathsf{C}_{3,1}(\mathbb{A}, \mathbb{K})$ generated by the elations. Then the group generated by all elations and diagonal automorphisms (with factors in $\mathbb{K}$ and with respect to the standard form) is denoted by $\overline{\mathsf{U}}_6(\mathbb{A})$. Note that we do not need to mention $\mathbb{K}$ in the notation, since it is unique as the center of $\mathbb{A}$.

Finally, let $\mathbb{A}$ be a Cayley division algebra over $\mathbb{K}$. Here, there is no form of the corresponding polar space available (since it is non-embeddable) and hence we cannot consider diagonal automorphisms as in the previous paragraphs. However, we can either consider the group generated by all elations and homologies (which for the previous cases would have boiled down to the same groups), or all elations and the diagonal automorphisms of the universal embedding of the corresponding dual polar

space, see the proof of Lemma 10.9.0.2. We denote these automorphism groups by $\mathsf{E}_{7,3}^{(28)}(\mathbb{A})$ and $\overline{\mathsf{E}}_{7,3}^{(28)}(\mathbb{A})$, respectively.

Lemma 10.9.0.2. *Let Δ be a polar space of rank 3 isomorphic to $\mathsf{C}_{3,3}(\mathbb{A},\mathbb{K})$, with $\mathbb{A}$ a quadratic alternative division algebra over $\mathbb{K}$, and let α and β be two disjoint planes. Let L be a line intersecting both α and β non-trivially. Then there exists a unique homology with axes α and β mapping a given plane π through L, not adjacent to either α or β, to an arbitrary other given plane like that. In particular, there is a unique set of homologies with axes α and β, acting transitively on the set of planes through L, not adjacent with either α or β, and hence, such a set is geometric.*

Proof. This is best seen through the (universal) representation of the corresponding dual polar space in a projective space over $\mathbb{K}$, as established uniformly in [**DV16**]. We take the slightly more symmetric algebraic description from [**DSVV24**, Definition 10.1]. Let V be the vector space $\mathbb{K}^4\oplus\mathbb{A}^3\oplus\mathbb{K}^3\oplus\mathbb{A}^3\oplus\mathbb{K}$ (note that we view $\mathbb{A}$ as a vector space over $\mathbb{K}$ in the natural way, that is, coming from the algebra over $\mathbb{K}$). Then the projective (or Zariski) closure of the point set given by the following parameter form, with the induced line set, forms the dual polar space $\mathsf{C}_{3,3}(\mathbb{A},\mathbb{K})$. We denote by $x\mapsto\overline{x}$ the standard involution in $\mathbb{A}$.

$$\begin{gathered}(1,\ell_1,\ell_2,\ell_3,x_1,x_2,x_3,x_1\overline{x}_1-\ell_2\ell_3,x_1\overline{x}_1-\ell_2\ell_3,x_1\overline{x}_1-\ell_2\ell_3,\\ \ell_1\overline{x}_1-x_2x_3,\ell_1\overline{x}_1-x_2x_3,\ell_1\overline{x}_1-x_2x_3,\\ \ell_1x_1\overline{x}_1+\ell_2x_2\overline{x}_2+\ell_3x_3\overline{x}_3-x_1(x_2x_3)-\overline{x}_3(\overline{x}_2\overline{x}_1)-\ell_1\ell_2\ell_3),\end{gathered}$$

where $\ell_1,\ell_2,\ell_3\in\mathbb{K}$ and $x_1,x_2,x_3\in\mathbb{A}$. We let α and β correspond to the points $a=(1,0,0,\ldots,0)$ and $b=(0,0,\ldots,0,1)$, respectively. We write coordinates of points of $\mathsf{PG}(V)$ as 14-tuples with entries in $\mathbb{K}\cup\mathbb{A}$ in the natural way, according to the definition of V above. Then one sees that the set of points $a^\perp\cap b^{\not\perp}$ spans the subspace $(0,*,*,*,*,*,*,0,\ldots,0)$ and $a^{\not\perp}\cap b^\perp$ spans the subspace $(0,\ldots,0,*,*,*,*,*,*,0)$. Each homology with axes α and β has to fix all points of these subspaces, and one can see that such a homology stems from the following linear map, where $t\in\mathbb{K}^\times$:

$$\begin{aligned}&V\to V\\&(k,\ell_1,\ell_2,\ell_3,x_1,x_2,x_3,m_1,m_2,m_3,y_1,y_2,y_3,k')\\&\quad\mapsto(k,t\ell_1,t\ell_2,t\ell_3,tx_1,tx_2,tx_3,t^2m_1,t^2m_2,t^2m_3,t^2y_1,t^2y_2,t^2y_3,t^3k')\end{aligned}$$

According to [**DV16**], the line generated by

$$a'=(0,0,1,0,0,\ldots,0)\quad\text{and}\quad b'=(0,\ldots,0,1,0,0,0,0,0,0)$$

corresponds to a line joining a point of α with one of β (in the coordinates of [**DV16**] these points are $(0,0,0,0,0)$ and $(0,0)$, and the line has type **(I)**). The point a' is collinear to a and b' to b. The above linear mappings, for varying $t\in\mathbb{K}^\times$, act (sharply) transitively on the points of $a'b'\setminus\{a',b'\}$. This proves the lemma. □

Proof of Theorem O – for points

Corollary 10.9.0.3. *Let v be a vertex of type 1 in $\mathsf{F}_4(\mathbb{K},\mathbb{A})$, with $\mathbb{A}$ a quadratic alternative division algebra over $\mathbb{K}$, $|\mathbb{A}|>2$. Then $\Pi^+(\{p\})$ is generated by all homologies of the corresponding polar space.*

PROOF. This follows from Lemma 10.4.3.9, Lemma 10.9.0.2, and Proposition 10.9.0.1. □

In view of Theorem I, this proves the case (C3) of Table 9.

We now consider vertices of type 4 in $\mathsf{F}_4(\mathbb{K}, \mathbb{A})$. Since $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{K}') \cong \mathsf{F}_{4,4}(\mathbb{K}'^2, \mathbb{K})$, if $\mathsf{char}\,\mathbb{K} = 2$ and $\mathbb{K}'$ is an inseparable extension of $\mathbb{K}$, we may restrict ourselves to the case where $\mathbb{A}$ is separable; that is, either equal to $\mathbb{K}$ with $\mathsf{char}\,\mathbb{K} \neq 2$, or a separable quadratic extension of $\mathbb{K}$ (in any characteristic), or a quaternion or octonion division algebra over $\mathbb{K}$.

First, we consider the case $\mathbb{A} = \mathbb{K}$, with $\mathsf{char}\,\mathbb{K} \neq 2$. Let p be a point of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{K})$. We have the following analogue of Lemma 10.9.0.2.

Lemma 10.9.0.4. *Let Δ be a parabolic polar space of rank 3 over the field $\mathbb{K}$ with $\mathsf{char}\,\mathbb{K} \neq 2$, and let α and β be two disjoint planes. Let L be a line intersecting both α and β non-trivially. Then there exists a unique homology with axes α and β mapping a given plane π through L, not adjacent to either α or β, to an arbitrary other given plane like that. In particular, there is a unique set of homologies with axes α and β acting transitively on the set of planes through L, not adjacent with either α or β, and hence such a set is geometric. Also, there is a unique homology with axes α and β that is a reflection.*

PROOF. We consider the standard equation of a parabolic quadric in $\mathsf{PG}(6, \mathbb{K})$, that is,

$$X_{-3}X_3 + X_{-2}X_2 + X_{-1}X_1 = X_0^2.$$

We write the coordinates as $(X_{-3}, X_{-2}, X_{-1}, X_0, X_1, X_2, X_3)$. We can take, with self-explaining notation,

$$\begin{cases} \alpha = (*, *, *, 0, 0, 0, 0), \\ \beta = (0, 0, 0, 0, *, *, *), \\ L = \langle (1, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 1, 0) \rangle \end{cases}$$

Then $\pi_k = \langle L, (0, 0, k^{-1}, 1, k, 0, 0) \rangle$, $k \in \mathbb{K}^\times$, is a generic plane through L in Δ not adjacent to either α or β. With that, the linear map

$$(x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3) \mapsto (x_{-3}, x_{-2}, x_{-1}, kx_0, k^2x_1, k^2x_2, k^2x_3)$$

maps π_1 to π_k. This is a reflection if and only if it fixes each point with $x_0 = 0$, that is, if and only if $k^2 = 1$, or $k \in \{1, -1\}$.

Now suppose a homology φ with axes α and β also stabilizes the plane π_1. Then φ stabilizes each line of π_1 through $\pi_1 \cap \alpha$ (as such a line is the projection onto π_1 of some fixed point of α), and analogously, each line of π_1 through $\pi_1 \cap \beta$. Hence, π_1 is fixed pointwise, and this readily implies that $\mathsf{PG}(6, \mathbb{K})$ is fixed pointwise.

All assertions are proved. □

We denote by $\mathsf{O}_7(\mathbb{K})$ the group of linear collineations of a 7-dimensional vector space V over $\mathbb{K}$, preserving the bilinear form associated to the standard equation of a parabolic quadric in $\mathsf{PG}(6, \mathbb{K})$, and by $\mathsf{PO}_7(\mathbb{K})$ the quotient with its center. Note that the former coincides with $\mathsf{O}_7^+(\mathbb{K})$ with the notation of [**Die43b**], as $\dim V = 7$ is odd. We then have:

Lemma 10.9.0.5. *Let p be a point of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$. Then $\Pi^+(p) = \Pi(p) \cong \mathsf{PO}_7(\mathbb{K})$ in its standard action.*

PROOF. By Lemma 10.4.3.9, Proposition 10.9.0.1, and Lemma 10.9.0.4, $\Pi^+(p)$ is generated by all homologies; hence, it is contained in $\mathsf{PO}_7(\mathbb{K})$. Now let q be opposite p and consider the extended equator geometry $\widehat{E}(p,q)$. If we now restrict to perspectivities $x \barwedge y$ with x and y points of $\widehat{E}(p,q)$, then we see that the special projectivity group of p in $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{K}) \cong \widehat{E}(p,q)$ is a subgroup of the special projectivity group $\Pi^+(p)$ inside Δ. Hence, Corollary 10.7.1.6 yields $\mathsf{PO}_7(\mathbb{K}) \leq \Pi^+(p)$. Also, using Proposition 10.7.1.1, we find that there exists a self-projectivity of Δ of length 3, exhibiting a reflection of the residue in p. Hence, $\Pi^+(p) = \Pi(p) = \mathsf{PO}_7(\mathbb{K})$ and the proof is complete. □

This takes care of the first line of (B3) of Table 9 (and note that for $\mathsf{char}\,\mathbb{K} = 2$, the groups $\mathsf{PO}_7(\mathbb{K})$ and $\overline{\mathsf{PSp}}_6(\mathbb{K})$ coincide, so we do not have to require that $\mathsf{char}\,\mathbb{K} \neq 2$ on that line in the table).

Now let $\mathbb{A}$ be a separable quadratic alternative division algebra over the field $\mathbb{K}$, with $d := \dim_{\mathbb{K}} \mathbb{A} \in \{2,4,8\}$. Let Δ be the polar space of rank 3, obtained from the quadric in $\mathsf{PG}(5+d,\mathbb{K})$ with standard equation

$$X_{-3}X_3 + X_{-2}X_2 + X_{-1}X_1 = \mathsf{norm}(X_0),$$

where the coordinate X_0 belongs to $\mathbb{A}$, viewed as a vector space over $\mathbb{K}$ in the natural way, and where $\mathsf{norm}(X_0) = X_0\overline{X}_0$, with $X_0 \mapsto \overline{X}_0$ the standard involution in $\mathbb{A}$ with respect to the quadratic algebra structure. Let V be the vector space $\mathbb{K}^3 \oplus \mathbb{A} \oplus \mathbb{K}^3$. Then we denote by $\mathsf{O}_{d+6}(\mathbb{K},\mathbb{A})$ the group of all linear transformations of V preserving the quadratic form associated with the above equation, and by or $\overline{\mathsf{O}}_{d+6}(\mathbb{K},\mathbb{A})$ the group of all linear transformations of V mapping it to a scalar multiple (the similitudes; the corresponding scalar is called the *factor* of the similitude). We denote by $\mathsf{PO}_{d+6}(\mathbb{K},\mathbb{A})$ and $\overline{\mathsf{PO}}_{d+6}(\mathbb{K},\mathbb{A})$ the respective quotients with the center. Let $\mathbb{F}$ be a splitting field of $\mathbb{A}$, that is, a quadratic extension of $\mathbb{K}$ over which $\mathbb{A}$ splits as an algebra. Then there are two natural systems of maximal singular subspaces (corresponding to those of the associated hyperbolic quadric) of the corresponding quadratic form over $\mathbb{F}$. The subgroups of $\mathsf{O}_{d+6}(\mathbb{K},\mathbb{A})$ and $\overline{\mathsf{O}}_{d+6}(\mathbb{K},\mathbb{A})$, preserving each of these systems, will be denoted by $\mathsf{O}^+_{d+6}(\mathbb{K},\mathbb{A})$ and $\overline{\mathsf{O}}^+_{d+6}(\mathbb{K},\mathbb{A})$, respectively, partially following Dieudonné [**Die43b**] (instead of the bilinear form, we included the algebra in the notation). The corresponding projective groups are then $\mathsf{PO}^+_{d+6}(\mathbb{K},\mathbb{A})$ and $\overline{\mathsf{PO}}^+_{d+6}(\mathbb{K},\mathbb{A})$.

Furthermore, let α and β be two opposite planes of Δ. We say that the homology group with axes α and β *acts transitively*, or *is a transitive homology group*, if the following holds: for some line L intersecting both α and β non-trivially (and consequently for every such line), and each pair of planes π and π' through L but not intersecting either α or β in a line, there exists a homology with axes α and β mapping π to π'.

Note that the factors of the similitudes are precisely the non-zero norms of $\mathbb{A}$.

We now prepare for the determination of the projectivity groups of a point in $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$, with $\mathbb{A}$ separable and of dimension $2, 4$, or 8 over $\mathbb{K}$.

Lemma 10.9.0.6. *Let G be a group of collineations of $\mathsf{B}_{3,1}(\mathbb{A},\mathbb{K})$, with $\mathbb{A}$ a separable quadratic alternative division algebra over the field $\mathbb{K}$, with $d=\dim_{\mathbb{K}}\mathbb{A}\in\{2,4,8\}$, containing $\mathsf{PO}^+_{d+6}(\mathbb{K},\mathbb{A})$ and containing a transitive homology group. Then G contains $\overline{\mathsf{PO}}^+_{d+6}(\mathbb{K},\mathbb{A})$.*

Proof. It suffices to show that for each norm $r\in\mathbb{K}$, there exists an element of G that maps a given quadratic form describing $\mathsf{B}_{3,1}(\mathbb{A},\mathbb{K})$ to an r-multiple. We consider, with previous notation, the quadratic form

$$X_{-3}X_3+X_{-2}X_2+X_{-1}X_1-\mathsf{norm}(X_0).$$

As in the proof of Lemma 10.9.0.4, we set

$$\begin{cases}\alpha=(*,*,*,0,0,0,0),\\ \beta=(0,0,0,0,*,*,*),\\ L=\langle(1,0,0,0,0,0,0),(0,0,0,0,0,1,0)\rangle\end{cases}$$

Since r is a norm, there exists some $a\in\mathbb{A}$ with $\mathsf{norm}(a)=r$. Consider the plane π_a spanned by L and the point $p_a=(0,0,1,a,r,0,0)$. Let π_1 be the plane spanned by L and $p_1=(0,0,1,1,1,0,0)$. Then there exists some homology h with axes α and β mapping π_1 to π_a. Now p_1 and p_a are the unique points of π_1 and π_a, respectively, collinear to the fixed points $(0,1,0,0,0,0,0)$ and $(0,0,0,0,0,0,1)$. Hence, the point p_1 is mapped by h to p_a. Since all points of $\alpha\cup\beta$ are fixed, the matrix of h is diagonal. Hence, up to a scalar multiple, this diagonal equals $(1,1,1,M,r,r,r)$, where M is a $d\times d$ matrix whose first column is a (in coordinates over $\mathbb{K}$; the action is on the right). It follows that h is a similitude with factor r, and the lemma is proved. □

Lemma 10.9.0.7. *Let p be a point of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$, with $\mathbb{A}$ separable and of dimension $2,4$, or 8 over $\mathbb{K}$. Then $\Pi^+(p)\cong\overline{\mathsf{PO}}^+_{d+6}(\mathbb{K},\mathbb{A})$ and $\Pi(p)\cong\overline{\mathsf{PO}}_{d+6}(\mathbb{K},\mathbb{A})$.*

Proof. This time, the set of homologies obtained from Proposition 10.9.0.1 is not (necessarily) geometric. However, Lemma 10.4.3.9 still shows that $\Pi^+(p)$ is generated by a set of homologies, which contains a transitive group of homologies. It already shows that $\Pi^+(p)\leq\overline{\mathsf{PO}}_{d+6}(\mathbb{K},\mathbb{A})$, such that $\Pi^+(p)$ contains a similitude with an arbitrary norm as factor.

However, splitting the forms over $\mathbb{F}$ (see above) produces the exceptional buildings of types E_6, E_7, and E_8 for $d=2,4,8$, respectively. In these buildings, an odd projectivity *always* switches the natural classes of maximal singular subspaces of the hyperbolic quadratic associated with the splitting of the symps, lines labelled (D4), (D5), and (D7). Therefore, we deduce that $\Pi^+(p)\leq\overline{\mathsf{PO}}^+_{d+6}(\mathbb{K},\mathbb{A})$.

Now we select a point q opposite p and restrict the projectivity group $\Pi(p)$ to self-projectivities using only perspectivities between points of the extended equator geometry $\widehat{E}(p,q)$. Then Proposition 10.7.1.1 implies that every reflection is contained in $\Pi(p)$, seen as a polar space.

Hence, $\mathsf{PO}_{d+6}(\mathbb{K},\mathbb{A})\leq\Pi(p)$, and the previous paragraphs together with Lemma 10.9.0.6 now imply that $\Pi(p)$ contains $\overline{\mathsf{PO}}_{d+6}(\mathbb{K},\mathbb{A})$.

Since $\Pi^+(p) \leq \overline{\mathsf{PO}}^+_{d+6}(\mathbb{K},\mathbb{A})$, and $\Pi^+(p)$ has index at most two in $\Pi(p)$ while $\overline{\mathsf{PO}}^+_{d+6}(\mathbb{K},\mathbb{A})$ has exactly index two in $\overline{\mathsf{PO}}_{d+6}(\mathbb{K},\mathbb{A})$, all projectivity groups now follow. □

10.10. Projectivity Groups of Non-maximal Residues in Metasymplectic Spaces

Proof of Theorem O – for non-maximal residues

We start with the residues isomorphic to generalized quadrangles, that is, buildings of type B_2 (or C_2). The groups $\mathsf{PO}_5(\mathbb{K})$ and $\overline{\mathsf{PO}}^+_{d+4}(\mathbb{K},\mathbb{A})$, with $\mathbb{A}$ a separable quadratic alternative division algebra over $\mathbb{K}$ of dimension $2, 4$, or 8, are similarly defined as their higher-dimensional analogues above. We then have the following lemma.

Lemma 10.10.0.1. *Let F be a simplex of type $\{1,4\}$ of a building $\Delta := \mathsf{F}_4(\mathbb{K},\mathbb{A})$. Then $\Pi^+(F) = \Pi(F)$. If $\mathbb{A} = \mathbb{K}'$ is an inseparable extension of $\mathbb{K}$, when $\mathsf{char}\,\mathbb{K} = 2$, then $\Pi(F) \cong \overline{\mathsf{PS}}\mathsf{p}_4(\mathbb{K}',\mathbb{K})$; this includes $\mathbb{K} = \mathbb{K}'$ for $\mathsf{char}\,\mathbb{K} = 2$. If $\mathbb{K} = \mathbb{A}$, then $\Pi(F) \cong \mathsf{PO}_5(\mathbb{K}) \cong \overline{\mathsf{PS}}\mathsf{p}(\mathbb{K})$. If $\mathbb{A}$ is separable with $\dim_{\mathbb{K}} \mathbb{A} \in \{2,4,8\}$, then*

$$\Pi(F) \cong \overline{\mathsf{PO}}^+_{d+4}(\mathbb{K},\mathbb{A}).$$

Proof. The fact that $\Pi^+(F) = \Pi(F)$ follows from Lemma 10.2.2.3. More concretely, let $F_1 = \{x_1, \xi_1\}$ be an incident point-symp pair of $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$, and let $F_2 = \{x_2, \xi_2\}$ be an opposite point-symp pair. Let $x_3 \notin \{x_1, x_2\}$ be a point on the imaginary line defined by x_1 and x_2, that is, $x_3 \in \{x_1, x_2\}^{\amalg\amalg} \setminus \{x_1, x_2\}$. Finally, let ξ_3 be a symp through x_3, intersecting $E(x_1, x_2)$ in a point of the hyperbolic line of $E(x_1, x_2)$, defined by $E(x_1, x_2) \cap \xi_i$, $i = 1, 2$. Since $E(x_1, x_2)$ is either a symplectic, a mixed, a unitary, or a thick non-embeddable polar space, we can choose ξ_3 disjoint from $\xi_1 \cup \xi_2$, and it follows that ξ_3 is opposite both ξ_1 and ξ_2. Now one can check that $F_1 \mathbin{\overline{\wedge}} F_2 \mathbin{\overline{\wedge}} F_3 \mathbin{\overline{\wedge}} F_1$ is the identity.

Set $F = \{x, \xi\}$, with x of type 1 and ξ of type 4. By Theorem 2.10.0.3, $\Pi^*(F)$ is the group induced on $\mathsf{Res}_\Delta(F)$ by the little projective group $G^\dagger$ of Δ. Then also $\mathsf{Res}_\Delta(\xi)$ is stabilized, and it follows that $\Pi^+(F)$ is the restriction of the stabilizer of x in $\Pi^+(\xi)$ to $\mathsf{Res}_\Delta(F)$. All assertions now follow. □

We note the following isomorphisms (for appropriate definitions of the unitary groups, similarly to their higher-dimensional analogues):

$$\mathsf{PO}_5(\mathbb{K}) \cong \overline{\mathsf{PS}}\mathsf{p}_4(\mathbb{K}); \quad \overline{\mathsf{PO}}^+_6(\mathbb{K},\mathbb{L}) \cong \overline{\mathsf{U}}_4(\mathbb{L}/\mathbb{K}); \quad \overline{\mathsf{PO}}^+_8(\mathbb{K},\mathbb{H}) \cong \overline{\mathsf{U}}_4(\mathbb{H}),$$

where $\mathbb{L}$ is a separable quadratic extension of $\mathbb{K}$, and $\mathbb{H}$ is a quaternion division algebra over $\mathbb{K}$.

We now turn to the planes and the rank 1 residues. Recall the notion of a standard polarity in planes $\mathsf{PG}(2,\mathbb{A})$, with $\mathbb{A}$ a quadratic alternative division algebra over $\mathbb{K}$ (see Remark 10.8.2.8).

Lemma 10.10.0.2. *If F is a simplex of a building $\Delta := \mathsf{F}_4(\mathbb{K},\mathbb{A})$, whose residue is isomorphic to $\mathsf{PG}(2,\mathbb{B})$, with $\mathbb{B} \in \{\mathbb{K},\mathbb{A}\}$, then $\Pi^+(F) \cong \mathsf{PGL}_3(\mathbb{B})$ and $\Pi(F) \cong \mathsf{PGL}_3(\mathbb{B}) \rtimes 2$, where the extension is with a standard polarity if $\mathbb{B} = \mathbb{A}$, and with an ordinary orthogonal polarity if $\mathbb{B} = \mathbb{K}$. If $\mathsf{Res}_\Delta(F)$ is the rank 1 building over $\mathbb{B}$, then $\Pi^+(F) = \Pi(F) \cong \mathsf{PGL}_2(\mathbb{B})$, if $\mathbb{B} = \mathbb{K}$ or if the standard involution of $\mathbb{B}$ is trivial; otherwise $\Pi^+(F) = \mathsf{PGL}_2^+(\mathbb{B})$ and $\Pi(F) \cong \mathsf{PGL}_2(\mathbb{B})$.*

Proof. Let α be a plane of $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$, let ξ be a symplecton containing α, and set $F = \{\alpha,\xi\}$ (then F is a simplex of type $\{3,4\}$ of the corresponding building). By Theorem 2.10.0.3, $\Pi^+(F)$ is induced by the little projective group $G^\dagger$ of the building. Since also $\Pi^+(\{\xi\})$ is induced by $G^\dagger$, we find that $\Pi^+(F)$ contains the stabilizer of α in the group $\Pi^+(\{\xi\})$. The latter is determined in Corollary 10.9.0.3 (for the inseparable case), Lemma 10.9.0.5, and Lemma 10.9.0.7 (for the other cases). These stabilizers are the full linear groups; hence, $\Pi^+(F) = \mathsf{PGL}_3(\mathbb{K})$. Similarly, if F is a simplex of type $\{1,2\}$, then $\Pi^+(F) = \mathsf{PGL}_3(\mathbb{A})$. Now we determine the general groups. So we exhibit a self-projectivity of length 3. Let $F = \{x, L\}$, with x a point incident with some line of $\mathsf{F}_{4,i}(\mathbb{K},\mathbb{A})$, with $i \in \{1,4\}$, and let $F' = \{x', L'\}$ be an opposite simplex. The residue of F induces a plane π in $E(x,x')$, viewed as a polar space (the set of points symplectic with both x and x' via a symp through L). Similarly, the residue of F' also induces a plane π' in $E(x,x')$. Now, let ζ be a symp through L, intersecting $E(x,x')$ in a point $w \in \pi$. The plane β' through L' nearest to ζ is characterized by the property that every symp ζ' through β' is special to ζ. It follows that such symps ζ' intersect $E(x,x')$ in a point symplectic to $E(x,x') \cap \zeta$. This implies that these intersections form the projection of w to π'. Hence, we can consider a self-projectivity of length 3 in a symp of $\mathsf{F}_{4,5-i}(\mathbb{K},\mathbb{A})$. If $i = 4$, then it suffices to consider a self-projectivity in a parabolic polar space, and a direct computation shows that this is always a linear duality. If $i = 1$, then, according to Lemma 10.8.2.9 (which is valid for arbitrary $\mathbb{A}$ and not only for $\mathbb{A} = \mathbb{O}$), this is a standard polarity.

Now suppose that F has size 3. Then the residue of F is also a residue in a residue that is a projective plane, and hence, $\Pi^+(F)$ is the same as the special projectivity group in that projective plane. Completely similarly to the previous paragraph, a self-projectivity of length 3 can be equivalently seen as a self-projectivity of length 3 of either (the lower residue of) a line or a planar line pencil. This reduces the determination of $\Pi(F)$ to the polar spaces $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ and $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$. For the former, the results follow from Proposition 10.8.1.1, Proposition 10.8.1.2, and Remark 10.8.1.3; for the latter, it follows from Section 10.8.1.1. □

We note the following isomorphisms:

$$\mathsf{PO}_3(\mathbb{K}) \cong \mathsf{PGL}_2(\mathbb{K});\ \ \overline{\mathsf{PO}}_4^+(\mathbb{K},\mathbb{L}) \cong \mathsf{PGL}_2^+(\mathbb{L}/\mathbb{K});$$
$$\overline{\mathsf{PO}}_6^+(\mathbb{K},\mathbb{H}) \cong \mathsf{PGL}_2^+(\mathbb{H}),\quad \overline{\mathsf{PO}}_{10}^+(\mathbb{K},\mathbb{O}) \cong \mathsf{PGL}_2^+(\mathbb{O}),$$

where $\mathbb{L}$ is a separable quadratic extension of $\mathbb{K}$, $\mathbb{H}$ is a quaternion division algebra over $\mathbb{K}$, and $\mathbb{O}$ is an octonion division algebra over $\mathbb{K}$.

Conclusion for Buildings of Type F_4

We summarize the results obtained in the previous two sections for metasymplectic spaces in the following concluding theorem, including a tabular form.

Theorem O. *Let $\mathsf{F}_4(\mathbb{K}, \mathbb{A})$ be a building of type F_4, with $\mathbb{K}$ a field, and $\mathbb{A}$ a quadratic alternative division algebra over $\mathbb{K}$. Let F be a simplex with an irreducible residue. Then $\Pi^+(F)$ and $\Pi(F)$ are as given in* Table 9*, where*

- $\mathbb{L}$ *denotes a separable quadratic extension of* $\mathbb{K}$,
- $\mathbb{H}$ *denotes a quaternion division algebra over* $\mathbb{K}$,
- $\mathbb{O}$ *denotes a Cayley (octonion) division algebra over* $\mathbb{K}$,
- $\mathbb{K}'$ *denotes an inseparable extension of* $\mathbb{K}$, *with* $\mathsf{char}\,\mathbb{K} = 2$,
- $\mathbb{A}'$ *denotes a separable quadratic alternative division algebra over* $\mathbb{K}$ *with* $\dim_{\mathbb{K}} \mathbb{A}' = d \in \{2, 4, 8\}$.

Cf.	Δ	$\mathsf{Res}_\Delta(F)$	$\mathsf{cotyp}(F)$	$\Pi^+(F)$	$\Pi(F)$	
(A1)	$\mathsf{F}_4(\mathbb{K}, \mathbb{A})$	$\mathsf{A}_1(\mathbb{K})$	$\{1\}, \{2\}$	$\mathsf{PGL}_2(\mathbb{K})$	$\mathsf{PGL}_2(\mathbb{K})$	✓
	$\mathsf{F}_4(\mathbb{K}, \mathbb{K})$	$\mathsf{A}_1(\mathbb{K})$	$\{3\}, \{4\}$	$\mathsf{PGL}_2(\mathbb{K})$	$\mathsf{PGL}_2(\mathbb{K})$	
	$\mathsf{F}_4(\mathbb{K}, \mathbb{A}')$	$\mathsf{A}_1(\mathbb{A}')$	$\{3\}, \{4\}$	$\overline{\mathsf{PO}}^+_{d+2}(\mathbb{K}, \mathbb{A}')$	$\overline{\mathsf{PO}}_{d+2}(\mathbb{K}, \mathbb{A}')$	
(A2)	$\mathsf{F}_4(\mathbb{K}, \mathbb{A})$	$\mathsf{A}_2(\mathbb{K})$	$\{1, 2\}$	$\mathsf{PGL}_3(\mathbb{K})$	$\mathsf{PGL}_3(\mathbb{K}) \rtimes 2$	
	$\mathsf{F}_4(\mathbb{K}, \mathbb{A})$	$\mathsf{A}_2(\mathbb{A})$	$\{3, 4\}$	$\mathsf{PGL}_3(\mathbb{A})$	$\mathsf{PGL}_3(\mathbb{A}) \rtimes 2$	
	$\mathsf{F}_4(\mathbb{K}, \mathbb{K}')$	$\mathsf{A}_2(\mathbb{K})$	$\{1, 2\}$	$\mathsf{PGL}_3(\mathbb{K})$	$\mathsf{PGL}_3(\mathbb{K}) \rtimes 2$	
(B2)	$\mathsf{F}_4(\mathbb{K}, \mathbb{K})$	$\mathsf{B}_2(\mathbb{K}, \mathbb{K})$	$\{2, 3\}$	$\mathsf{PO}_5(\mathbb{K})$	$\mathsf{PO}_5(\mathbb{K})$	✓
	$\mathsf{F}_4(\mathbb{K}, \mathbb{A}')$	$\mathsf{B}_2(\mathbb{K}, \mathbb{A}')$		$\overline{\mathsf{PO}}^+_{d+4}(\mathbb{K}, \mathbb{A}')$	$\overline{\mathsf{PO}}^+_{d+4}(\mathbb{K}, \mathbb{A}')$	✓
(C2)	$\mathsf{F}_4(\mathbb{K}, \mathbb{K}')$	$\mathsf{B}_2(\mathbb{K}, \mathbb{K}')$	$\{2, 3\}$	$\overline{\mathsf{PSp}}_4(\mathbb{K}', \mathbb{K})$	$\overline{\mathsf{PSp}}_4(\mathbb{K}', \mathbb{K})$	✓
(B3)	$\mathsf{F}_4(\mathbb{K}, \mathbb{K})$	$\mathsf{B}_3(\mathbb{K}, \mathbb{K})$	$\{1, 2, 3\}$	$\mathsf{PO}_7(\mathbb{K})$	$\mathsf{PO}_7(\mathbb{K})$	
	$\mathsf{F}_4(\mathbb{K}, \mathbb{A}')$	$\mathsf{B}_3(\mathbb{K}, \mathbb{A}')$		$\overline{\mathsf{PO}}^+_{d+6}(\mathbb{K}, \mathbb{A}')$	$\overline{\mathsf{PO}}_{d+6}(\mathbb{K}, \mathbb{A}')$	
(C3)	$\mathsf{F}_4(\mathbb{K}, \mathbb{K})$	$\mathsf{C}_3(\mathbb{K}, \mathbb{K})$	$\{2, 3, 4\}$	$\overline{\mathsf{PSp}}_6(\mathbb{K})$	$\overline{\mathsf{PSp}}_6(\mathbb{K})$	✓
	$\mathsf{F}_4(\mathbb{K}, \mathbb{L})$	$\mathsf{C}_3(\mathbb{L}, \mathbb{K})$		$\overline{\mathsf{U}}_6(\mathbb{L}/\mathbb{K})$	$\overline{\mathsf{U}}_6(\mathbb{L}/\mathbb{K})$	✓
	$\mathsf{F}_4(\mathbb{K}, \mathbb{H})$	$\mathsf{C}_3(\mathbb{H}, \mathbb{K})$		$\overline{\mathsf{U}}_6(\mathbb{H})$	$\overline{\mathsf{U}}_6(\mathbb{H})$	✓
	$\mathsf{F}_4(\mathbb{K}, \mathbb{O})$	$\mathsf{C}_3(\mathbb{O}, \mathbb{K})$		$\overline{\mathsf{E}}^{28}_{7,3}(\mathbb{O})$	$\overline{\mathsf{E}}^{28}_{7,3}(\mathbb{O})$	✓
	$\mathsf{F}_4(\mathbb{K}, \mathbb{K}')$	$\mathsf{C}_3(\mathbb{K}', \mathbb{K})$		$\overline{\mathsf{PSp}}_6(\mathbb{K}', \mathbb{K})$	$\overline{\mathsf{PSp}}_6(\mathbb{K}', \mathbb{K})$	✓

Table 9. Projectivity groups in the exceptional case F_4

The last column of Table 9 contains a checkmark (✓) if Lemma 10.2.2.3 automatically yields $\Pi^+(F) = \Pi(F)$; see also Section 10.2.4. Note that there are other cases for which $\Pi^+(F) = \Pi(F)$ generically holds, in contrast to the simply laced case.

A. Algebra Basics

Although we expect the reader to be familiar with the concepts in this appendix, we briefly recall some basic notions, both for clarity and because definitions and notation vary across authors and epochs. We begin by defining groups, centers of groups, normal subgroups, quotient groups, rings, fields, skew fields, alternative fields, vector spaces, and modules. Next, we introduce the groups $\mathsf{GL}_n(\mathbb{K})$, $\mathsf{PGL}_n(\mathbb{K})$, $\mathsf{SL}_n(\mathbb{K})$, and $\mathsf{PSL}_n(\mathbb{K})$. Afterwards, we define forms, in particular bilinear, quadratic, and sesquilinear forms. Finally, we introduce $\mathbb{K}$-algebras, composition algebras, including alternative and associative composition algebras, split algebras, quadratic algebras, and division algebras, in particular quaternion, and octonion division algebras.

A.1. Definitions of Fundamental Concepts

Definition A.1.0.1 (**Monoid**, **group**, **abelian group**)**.** Let M be a set endowed with a binary operation

$$\circ\colon M \times M \to M,$$

such that the following conditions hold.

(M1) There exists an element $\mathsf{id} \in M$, such that for all elements $m \in M$, we have

$$\mathsf{id} \circ m = m = m \circ \mathsf{id};$$

(and we also call id the *identity element of M with respect to $\circ$*.)

(M2) For all $m, n, o \in M$, we have $(m \circ n) \circ o = m \circ (n \circ o)$, and we call the operation *associative*.

Suppose we are given a monoid G that additionally satisfies the following condition.

(G3) For every element $g \in G$, there exists an element $h \in G$, such that

$$g \circ h = \mathsf{id} = h \circ g.$$

(In this case, we will also call h the *inverse of g* and write $h = g^{-1}$.)

Then we call G a *group* with respect to $\circ$. A group is called *abelian* or *commutative*, if

(AbGr) for all $g, h \in G$, we have $g \circ h = h \circ g$.

Definition A.1.0.2 (**Center of a group**)**.** The *center of a group G* is defined as

$$Z(G) := \{z \in G \mid z \circ g = g \circ z, \text{ for all } g \in G\}.$$

Definition A.1.0.3 (**Normal subgroups**, **quotient group**)**.** Let G be a group. A subgroup N of G is called *normal*, if the sets

$$gN := \{gn \mid n \in N\} \quad \text{and} \quad Ng := \{ng \mid n \in N\}$$

are equal for all $g \in G$. If N is a normal subgroup of G, then we define the set $G/N := \{gN \mid g \in G\}$, and a binary operation on G/N, by putting

$$(gN)(hN) := (gh)N, \quad \text{for} \quad g, h \in G.$$

This is well-defined and yields a group structure on G/N. We call G/N the quotient group of G with respect to N. The identity in G/N is always given by N. Going from G to G/N, we also say that *we factor out the normal subgroup* N.

Example A.1.0.4. Let G be $\{0, 1, 2, 3, 4, 5\}$ with addition. Let $N := \{0, 3\}$. Then N is normal in G and

$$G/N = \{g + N \mid g \in G\} = \{\{0, 3\}, \{1, 4\}, \{2, 5\}\} = \{0 + N, 1 + N, 2 + N\}.$$

Definition A.1.0.5 (**Ring**, **commutative ring**)**.** Let R be a set endowed with two binary operations $+\colon R \times R \to R$ and $\cdot\colon R \times R \to R$, called *addition* and *multiplication*, such that the following conditions hold.

(R1) $(R, +)$ is an abelian group, where the identity is denoted by 0, the inverse of a is denoted by $-a$, and instead of writing $a + (-b)$, we also write $a - b$.
(R2) The set $R \setminus \{0\} =: R^\times$, endowed with the multiplication, defines a monoid.
(R3) We have $a(b + c) = ab + ac$, and $(a + b)c = ac + bc$, for all $a, b, c \in R$.

We call a ring *commutative*, if its multiplication is commutative; that is

(Comm) for all $a, b \in R$, we have $ab = ba$.

Note A.1.0.6. Note that, in a ring, $+$, as well as $\cdot$, are associative.

Definition A.1.0.7 (**Field**, **skew field**, **alternative field**)**.** Let $\mathbb{K}$ be a set endowed with two binary operations $+\colon \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ and $\cdot\colon \mathbb{K} \times \mathbb{K} \to \mathbb{K}$, called *addition* and *multiplication*, such that the following conditions hold.

(F1) $(\mathbb{K}, +)$ is an abelian group, where the identity is denoted by 0, the inverse of a is denoted by $-a$, and instead of writing $a + (-b)$, we also write $a - b$.
(F2) The multiplication defines a commutative group structure on $\mathbb{K} \setminus \{0\} =: \mathbb{K}^\times$ (and $0x = 0 = x0$, for all $x \in \mathbb{K}$).
(F3) We have $a(b + c) = ab + ac$, and $(a + b)c = ac + bc$, for all $a, b, c \in \mathbb{K}$, i.e., the distributive law holds.

Then we call $(\mathbb{K}, +, \cdot)$ a *field*. Sometimes we will only write $\mathbb{K}$ for $(\mathbb{K}, +, \cdot)$, if that seems clear enough.

Suppose we are given a set $\mathbb{L}$ with two binary operations like above and (F1), (F3) hold, but not (F2). We call $\mathbb{L}$ a *skew field* or *division ring*, if $\mathbb{L}$ satisfies the following condition.

(SF2) The multiplication defines a group structure on $L^\times$.

Suppose we are given a set $\mathbb{A}$ with two binary operations like above and (F1), (F3) hold, but not (F2). We call $\mathbb{A}$ an *alternative field*, if $\mathbb{A}$ satisfies the following condition.

(AF2) $\mathbb{A}^\times$, endowed with the multiplication, contains an identity element, for every element in $\mathbb{A}^\times$, there exists an inverse in $\mathbb{A}^\times$, and for all $a, b \in \mathbb{A}^\times$, we have $a(ab) = a^2b$ and $(ab)b = ab^2$, i.e., the multiplication is not associative, but *alternative*.

Note A.1.0.8.

- All fields are skew fields, and every skew field that is not a field, is non-commutative. An example for the latter are the *quaternions*, which we will define in Definition A.4.0.11.
- Every finite field has $q = p^n$ elements, where p is a prime number and $n \geq 1$. We will denote the finite field with q elements by $\mathbb{F}_q$ and also refer to q as the *order* of $\mathbb{F}_q$.

Definition A.1.0.9 (**Vector space**)**.** Let $\mathbb{L}$ be a skew field and 1 its multiplicative identity. A *right vector space over* $\mathbb{L}$ consists of an abelian group V with operation $+$, whose elements are called *vectors*, and an operation $\cdot\colon V \times \mathbb{L} \to V$, $(v, a) \mapsto va$, called *scalar multiplication*, such that for all $a, b \in \mathbb{L}$ and $u, v \in V$, the following conditions hold.

(i) $v(a + b) = va + vb$.
(ii) $(v + u)a = va + ua$.
(iii) $v(ab) = (va)b$.
(iv) $v1 = v$.

A *left vector space over* $\mathbb{L}$ is defined similarly by switching the order of the scalar multiplication, i.e., $\cdot$ is defined as $\mathbb{L} \times V \to V$, $(a, v) \mapsto av$.

Definition A.1.0.10 (**Module**)**.** Let R be a ring and 1 its multiplicative identity. A *right R-module* consists of an abelian group M with operation $+$, and an operation $\cdot\colon M \times R \to M$, $(m, r) \mapsto mr$, such that for all $r, s \in R$ and $m, n \in M$, the following conditions hold.

(i) $r(m + n) = rm + rn$.
(ii) $(m + n)r = mr + nr$.
(iii) $(rs)m = r(sm)$.
(iv) $m1 = m$.

Again, the operation $\cdot$ is called a *scalar multiplication*. A *left R-module* is defined similarly as before.

Note A.1.0.11. Modules can be seen as generalizations of vector spaces, where we replace the division ring by a ring; a vector space is a module over a skew field.

Definitions A.1.0.12. Let V be a vector space.

- A *square matrix* is a matrix with the same number of rows and columns.
- A *diagonal matrix* is a matrix, in which the entries outside the main diagonal are all zero.
- A square matrix M is called *diagonizable*, if there exists an invertible matrix I and a diagonal matrix D, such that $I^{-1}MI = D$.
- A *scalar matrix* is a square matrix, where all the diagonal elements are the same scalar value, and all other elements are zero.

A.2. The Groups $\mathsf{GL}_n(\mathbb{K})$, $\mathsf{PGL}_n(\mathbb{K})$, $\mathsf{SL}_n(\mathbb{K})$ & $\mathsf{PSL}_n(\mathbb{K})$

For a field $\mathbb{K}$ the group $\mathsf{GL}_n(\mathbb{K})$ is defined as the group of all invertible $n \times n$ matrices with entries in $\mathbb{K}$ and matrix multiplication as operation. The group $\mathsf{SL}_n(\mathbb{K})$ consists of all elements of $\mathsf{GL}_n(\mathbb{K})$ with determinant 1. One can see that $\mathsf{SL}_n(\mathbb{K})$ is a normal subgroup of $\mathsf{GL}_n(\mathbb{K})$, by considering it as the kernel of the map $\det\colon \mathsf{GL}_n(\mathbb{K}) \to \mathbb{K}^\times$, where $\mathbb{K}^\times = \mathbb{K} \setminus \{0\}$, as defined above[1].

Let $\mathsf{Sc}_n(\mathbb{K})$ be the group of all scalar matrices over $\mathbb{K}$ (see Definitions A.1.0.12). Then $\mathsf{Sc}_n(\mathbb{K})$ can be seen as the center of the group $\mathsf{GL}_n(\mathbb{K})$ and we can write

$$\mathsf{PGL}_n(\mathbb{K}) = \mathsf{GL}_n(\mathbb{K})/\mathsf{Sc}_n(\mathbb{K}).$$

That means $\mathsf{PGL}_n(\mathbb{K})$ consists of equivalence classes of invertible matrices under scalar multiplication (see Definition A.1.0.3). Similarly,

$$\mathsf{PSL}_n(\mathbb{K}) = \mathsf{SL}_n(\mathbb{K})/(\mathsf{Sc}_n(\mathbb{K}) \cap \mathsf{SL}_n(\mathbb{K})) = \{M \in \mathsf{GL}_n(\mathbb{K}) \mid \det M = 1\}/(\mathsf{Sc}_n(\mathbb{K}) \cap \mathsf{SL}_n(\mathbb{K})).$$

The set $\mathsf{Sc}_n(\mathbb{K}) \cap \mathsf{PSL}_n(\mathbb{K}))$ is the center of the group $\mathsf{SL}_n(\mathbb{K})$.

A.3. Forms

Definition A.3.0.1 (**Polynomials**, **homogeneous polynomials**, **polynomial functions**)**.**

A *polynomial* is a mathematical expression, consisting of variables and coefficients, that involves only the operations of addition, subtraction, multiplication and exponentiation to non-negative integer powers, and has a finite number of terms.

The exponent of a variable in a term is called the *degree of that variable in that term*. The *degree of the term* is the sum of the degrees of the variables in that term and the *degree of a polynomial* is the largest degree of any term with non-zero coefficient.

A *homogeneous polynomial* is a polynomial, whose non-zero terms all have the same degree. If the degree of all non-zero terms is n, we also say that we have a *homogeneous polynomial of degree* n. For example,

$$x^8 + 3x^5y^3 + 5xy^7$$

is a homogeneous polynomial of degree 8 in two variables, since the sum of the exponents in each term is always 5. An example of a polynomial, which is not homogeneos, is the following.

$$y^7 + 2xy^3 + 3x^7y^2.$$

A *polynomial map* or *polynomial function* is a map, that can be defined by evaluating a polynomial. For example:

$$f(x, y) = y^7 + 2xy^3 + 3x^7y^2 - 3.$$

[1]The kernel of a group homomorphism $G \to H$ is always a normal subgroup of G.

In the following, let V always be a vector space over a skew field $\mathbb{F}$.

Definition A.3.0.2 (**Forms**)**.**
In general, we say that a *form* on V is any map from V to $\mathbb{F}$. Polynomial maps defined by homogeneous polynomials are examples of forms.

Definition A.3.0.3 (**Bilinear**, **symmetric**, and **non-degenerate forms**)**.**
A *bilinear form* on V is a map

$$f\colon V \times V \longrightarrow \mathbb{F}$$

such that, for each $v \in V$, both maps

$$f(v, \cdot)\colon V \longrightarrow \mathbb{F}, \qquad f(\cdot, v)\colon V \longrightarrow \mathbb{F}$$

are linear. A bilinear form f is called *symmetric* if, for all $v, w \in V$, we have

$$f(v, w) = f(w, v).$$

The *radical* of f is the following subspace of V.

$$\mathsf{Rad}(f) := \{x \in V \mid f(x, y) = 0, \text{ for all } y \in V\}.$$

If $\mathsf{Rad}(f) = \{0\}$, then f is called *non-degenerate* (compare [**BC13**, §1.4]).

Definition A.3.0.4 (**Quadratic forms**, **associated bilinear forms**)**.**
Consider a homogeneous polynomial map κ of degree 2 on V. There exist linear forms on V, defined by polynomials f_i $(i \in I)$ of degree 1 such that

$$\kappa = \sum_{i,j \in I} f_i f_j.$$

This means that, for $v \in V$, we have

$$\kappa(v) = \sum_{i,j \in I} f_i(v)\, f_j(v).$$

Such a map is called a *quadratic form* on V. In particular,

$$\kappa(x\lambda) = \lambda^2 \kappa(x), \quad \text{for all } \lambda \in \mathbb{F},\ x \in V,$$

and

$$\kappa(x + y) = \kappa(x) + \kappa(y) + f(x, y), \quad \text{for all } x, y \in V,$$

for a uniquely determined symmetric bilinear form f on V. We call f the *bilinear form associated with* κ (cf. [**BC13**, Definition 4.4.2]).

Definition A.3.0.5 (**Associated sesquilinear forms**)**.**
Let V be a vector space over a field $\mathbb{F}$ and let σ be an automorphism of $\mathbb{F}$ with $\sigma^2 = \mathsf{id}$.
If we consider a map

$$f\colon V \times V \longrightarrow \mathbb{F},$$

such that, for all $a, b, c, d \in V$ and all $\lambda, \mu \in \mathbb{F}$, we have

$$\begin{aligned} f(a + b,\, c + d) &= f(a, c) + f(b, c) + f(a, d) + f(b, d), \\ f(a\lambda,\, b\mu) &= \sigma(\lambda)\, f(a, b)\, \mu, \end{aligned}$$

then we call f a *sesquilinear form on V with respect to σ* (compare [**BC13**, p. 45]).

A.4. Algebras

In this section, our main reference is [**DTV25**]. We adopt definitions that differ from those found in the standard literature on composition algebras, since their purpose for us will not be to do algebra; we require these concepts only for defining projective planes, not just over a field $\mathbb{K}$, but also over a quadratic alternative division algebra $\mathbb{A}$ over $\mathbb{K}$, as will be discussed in Section 4.8.

Definition A.4.0.1 ($\mathbb{K}$**-algebras**)**.** Let $\mathbb{K}$ be a field and let $\mathbb{A}$ be a vector space over $\mathbb{K}$ endowed with a multiplication $\cdot_{\mathbb{A}}$ such that

(1) $(ka)\cdot_{\mathbb{A}} b = k(a\cdot_{\mathbb{A}} b) = a\cdot_{\mathbb{A}}(kb)$, for all $k \in \mathbb{K}$, $a, b \in \mathbb{A}$.
(2) There is a unit vector denoted $1_{\mathbb{A}} \in \mathbb{A}$ such that $1_{\mathbb{A}} \cdot_{\mathbb{A}} a = a = a \cdot_{\mathbb{A}} 1_{\mathbb{A}}$, for all $a \in \mathbb{A}$.
(3) The distributive laws hold, that is,

$$a \cdot_{\mathbb{A}} (b + c) = a \cdot_{\mathbb{A}} b + a \cdot_{\mathbb{A}} c \quad \text{and} \quad (a + b) \cdot_{\mathbb{A}} c = a \cdot_{\mathbb{A}} c + b \cdot_{\mathbb{A}} c, \quad \text{for all } a, b, c \in \mathbb{A}.$$

Then we call $\mathbb{A}$ a *unital* $\mathbb{K}$*-algebra*, or just a $\mathbb{K}$*-algebra* for short.[2] The $\mathbb{K}$-multiples of $1_{\mathbb{A}} \in \mathbb{A}$ form a subalgebra isomorphic to $\mathbb{K}$, and we identify that subalgebra with $\mathbb{K}$. This enables us to write $a \cdot_{\mathbb{A}} b$ as ab, since this is consistent with the scalar multiplication and the multiplication in $\mathbb{K}$, and to write 1 instead of $1_{\mathbb{A}}$ (cf. [**DTV25**, Definition 1.4.1]).

Definitions A.4.0.2 (**Composition algebras**, **alternative**, **associative**)**.** Let $\mathbb{A}$ be a unital $\mathbb{K}$-algebra.

(i) A member $a \in \mathbb{A}$ is called *quadratic*, if there exist elements $t, n \in \mathbb{K}$ such that
$$a^2 - ta + n = 0.$$
In that case, we also say that *a is a solution for the quadratic equation*
$$x^2 - tx + n = 0, \tag{$\star$}$$
or that *a satisfies the quadratic equation* ($\star$).
(ii) If there exists a map $\mathfrak{n}\colon \mathbb{A} \to \mathbb{K}$ such that
$$\mathfrak{n}(xy) = \mathfrak{n}(x)\mathfrak{n}(y),$$
for all $x, y \in \mathbb{A}$, then we say that $\mathbb{A}$ is a *composition algebra* (over $\mathbb{K}$).
(iii) If
$$(a^2)b = a(ab) \quad \text{and} \quad a(b^2) = (ab)b,$$
for all $a, b \in \mathbb{A}$, then we say that $\mathbb{A}$ is *alternative*.
(iv) If
$$(ab)c = a(bc),$$
for all $a, b, c \in \mathbb{A}$, then we say that $\mathbb{A}$ is *associative*.

Fact A.4.0.3. An alternative algebra $\mathbb{A}$ over an arbitrary field $\mathbb{K}$, in which every element is quadratic, is a composition algebra.

[2] In some areas of mathematics, the assumption (ii) is not made, and then $\mathbb{K}$-algebras can be *non-unital*. These will not play a role for us.

REFERENCE. This is [**DTV25**, Theorem 1.4.2]. □

Fact A.4.0.4. An element $a \in \mathbb{A}$ is *invertible* if and only if $\mathfrak{n}(x) \neq 0$, and, if so, then

$$x^{-1} = \overline{x}\mathfrak{n}(x)^{-1}.$$

REFERENCE. This is [**Van24**, Lemma A.2.1]. □

Definition A.4.0.5 (**Division algebra**, **split algebra**)**.** If every $a \in \mathbb{A} \setminus \{0\}$ is invertible, then we call $\mathbb{A}$ a *division algebra*; if not, we say that $\mathbb{A}$ is *split* (cf. [**Van24**, Definition A.2.2]).

Fact A.4.0.6. An alternative division algebra $\mathbb{A}$ over an arbitrary field $\mathbb{K}$, in which every element is quadratic, admits a unique $\mathbb{K}$-linear involution

$$\mathbb{A} \to \mathbb{A},\ x \mapsto \overline{x},$$

such that $x + \overline{x} \in \mathbb{K}$, $x\overline{x} \in \mathbb{K}$ and $x\overline{x} = \overline{x}x$.

REFERENCE. This is [**DTV25**, Theorem 1.4.3]. □

Definition & Fact A.4.0.7. The *linearization* f of $\mathfrak{n}$ is defined as

$$f\colon \mathbb{A} \times \mathbb{A} \to \mathbb{K},\ (x, y) \mapsto \mathfrak{n}(x + y) - \mathfrak{n}(x) - \mathfrak{n}(y).$$

The map $\mathfrak{n}$ is a quadratic form,[3] whose linearization is the bilinear form

$$(x, y) \mapsto x\overline{y} + y\overline{x} = \overline{x}y + \overline{y}x.$$

The map $\mathfrak{n}$ is also called a *norm (form)*.

REFERENCE. This is [**Van24**, Definition A.1.3] and [**DTV25**, Theorem 1.4.3]. □

Definition A.4.0.8 (**Quadratric algebras**)**.** A unital $\mathbb{K}$-algebra $\mathbb{A}$, for which

- every element is quadratic,
- the mapping $\varphi\colon \mathbb{A} \to \mathbb{K}$, that maps
  - every $x \in \mathbb{A} \setminus \mathbb{K}$ to the unique element $n \in \mathbb{K}$ such that
    $$x^2 - tx + n = 0,\ \text{for } t \in \mathbb{K},$$
  - and every $x \in \mathbb{A} \cap \mathbb{K}$ to x^2,

  is a quadratic form,
- and similarly, the mapping $\psi\colon x \mapsto t$, that maps
  - every $x \in \mathbb{A} \setminus \mathbb{K}$ to the unique element $t \in \mathbb{K}$ such that
    $$x^2 - tx + n = 0,\ \text{for } n \in \mathbb{K},$$
  - and every $x \in \mathbb{A} \cap \mathbb{K}$ to $2x$,

  is a linear form,

is called a *quadratic algebra* (cf. [**DTV25**, Definition 1.4.5]).

[3] A *quadratic form* is a polynomial map defined by a homogeneous polynomial with terms all of degree two (cf. Appendix A).

Definition A.4.0.9 (**Anisotropic norm forms**)**.** If $\mathbb{A}$ is a quadratic alternative division algebra, then the unique $\mathbb{K}$-linear involution $x \mapsto \overline{x}$ is called the *standard involution*. In this case the norm form is only 0 for the zero-vector and we say the norm form is *anisotropic* (cf. [**DTV25**, Definition 1.4.5]).

Fact A.4.0.10. Quadratic alternative division algebras have been classified.

Reference. This is [**DTV25**, §.1.4.2]. □

We will mention more about the classification in Section 4.8 (when we need it). For now we will limit ourselves to defining quaternion and octonion division algebras.

Definition A.4.0.11 (**Quaternion division algebras**)**.** We call quadratic alternative division algebras of dimension 4, which are not commutative, *quaternion division algebras* and denote them by $\mathbb{H}$. (The '$\mathbb{H}$' stands for 'Hamilton', since the quaternions were first introduced by the mathematician William Rowan Hamilton in 1843.)

Definition A.4.0.12 (**Octonion division algebras**)**.** We call quadratic alternative division algebras of dimension 8, which are not commutative, *octonion division algebras* and denote them by $\mathbb{O}$.

For more information about quaternion and octonion division algebras, we refer to [**Van24**, Appendix A].

D. Some Results of Dieudonné

This appendix collects several results of Dieudonné that are cited in Chapter 10. The results are presented in their original French formulation alongside an English translation, providing the reader with immediate access to these statements and clarifying the foundations on which various arguments rely, without the need to consult the original references.

Proposition 8. *Toute transformation orthogonale à n variables sur un corps de caractéristique $\neq 2$ est un produit de n symétries au plus.*

Every orthogonal transformation (with respect to a non-degenerate bilinear form in n variables) is, in characteristic $\neq 2$, the product of at most n reflections.

Proposition 14. *Toute transformation orthogonale est un produit de transvections orthogonales, sauf pour $\mathbb{K} = \mathbb{F}_2$, $n = 4$ et $\nu = 2$.*

This appears in the section on characteristic 2, not mentioned in the proposition.

Every orthogonal transformation (with respect to a non-degenerate bilinear form) is, in characteristic 2, the product of orthogonal transvections, except if the underlying field is $\mathbb{F}_2$, the dimension of the vector space is 4, and the Witt index of the form is 2.

A transvection fixes a tangent hyperplane pointwise and is hence a special case of a reflection.

Théorème 1. *À l'exception des groupes $\mathsf{Sp}_2(\mathbb{F}_2)$, $\mathsf{Sp}_2(\mathbb{F}_3)$ et $\mathsf{Sp}_4(\mathbb{F}_2)$, tout groupe symplectique $\mathsf{Sp}_n(\mathbb{K})$ ne contient aucun sous-groupe distingué distinct de lui-même et non contenu dans son centre.*

The symplectic group $\mathsf{PSp}_n(\mathbb{K})$ is simple, except for $n = 2$ and $\mathbb{K} \in \mathbb{F}_2, \mathbb{F}_3$, and $n = 4$ and $\mathbb{K} = \mathbb{F}_2$.[1]

Théorème 5. *Pour $n \geq 2$ et $\nu \geq 1$, le groupe $\mathsf{U}_n^+(\mathbb{K}, f)$ ne contient aucun sous-groupe distingué distinct de lui-même et non contenu dans son centre, sauf pour $\mathbb{K}_0 = \mathbb{F}_3$, $n = 2$, et pour $\mathbb{K}_0 = \mathbb{F}_2$, $n = 2$, et $n = 3$.*

The group of unitary transformations (with respect to a Hermitian form in n variables, $n \geq 2$, of Witt index ≥ 1) has no normal subgroup distinct from itself or contained in the center, except for $n = 2$ and $\mathbb{K} \in \mathbb{F}_2, \mathbb{F}_3$, and $n = 3$ and $\mathbb{K} = \mathbb{F}_2$, where $\mathbb{K}$ is the field fixed by the companion field automorphism.

[1] In the French text we write $\mathsf{Sp}_n(\mathbb{K})$, and in the English text we write $\mathsf{PSp}_n(\mathbb{K})$. The formulation in French follows a common convention, stating simplicity up to the center. For the English translation we rephrased a bit and divided out the center.

C. Chevalley Groups

In the following, we would like to give a brief insight into Chevalley groups and how to obtain BN-pairs for them. What we will regard as a Chevalley group, is also sometimes called the *adjoint Chevalley group* (cf. [**Car72**, p. 198]). In the case of thick irreducible spherical buildings Δ of simply laced type and rank at least 3, the attached Chevalley group is just the little projective group of Δ (which is always simple in this case (compare with [**Tit64**, Main Theorem])). Our treatment will remain rudimentary, omitting substantial parts of the general theory and their proofs. Everything contained in this Appendix — apart from Appendix A.4 — closely follows [**Car72**].

C.1. Lie Algebras, Cartan Subalgebras & Root Systems

In the following, we will denote the neutral element with regards to vector addition by 0.

Definition C.1.0.1 (**Lie algebras**)**.** A *Lie algebra* is a vector space $\mathfrak{L}$ over a field $\mathbb{K}$, together with an operation $[\cdot,\cdot]\colon \mathfrak{L}\times\mathfrak{L}\to\mathfrak{L}$, $(x,y)\mapsto[x,y]$, for $x,y,z\in\mathfrak{L}$, satisfying the following axioms:

(i) $[\cdot,\cdot]$ is bilinear: For all $x,y,z\in\mathfrak{L}$, $a,b\in\mathbb{K}$, we have
$$[ax+by,z]=a[x,z]+b[y,z],$$
$$[z,ax+by]=a[z,x]+b[z,y].$$

(ii) $[\cdot,\cdot]$ is alternating: For all $x\in\mathfrak{L}$, we have $[x,x]=0$.

(iii) $[\cdot,\cdot]$ satisfies the *Jacobi identity*: For all $x,y,z\in\mathfrak{L}$, we have
$$[[x,y],z]+[[y,z],x]+[[z,x],y]=0.$$

We call $[\cdot,\cdot]$ the *Lie multiplication* in $\mathfrak{L}$, or the *Lie bracket* on $\mathfrak{L}$ (cf. [**Car72**, p. 33] & [**Wie24**, Definition 3.1.1]).

Note C.1.0.2. Any associative algebra can be made into a Lie algebra by defining the Lie multiplication by $[x,y]=xy-yx$ (cf. [**Car72**, §3.1]).

Then $[x,y]$ is bilinear, $[x,x]=0$ and

$$\begin{aligned}&[[x,y],z]+[[y,z],x]+[[z,x],y]\\ =&(xy-yx)z-z(xy-yx)+(yz-zy)x-x(yz-zy)+(zx-xz)y-y(zx-xz)\\ =&0\end{aligned}$$

However, generally, Lie multiplication is not associative.

Definition C.1.0.3 (**Ideals**, **Lie subalgebras**)**.** Let $\mathbb{K}$ be a field, $\mathfrak{L}$ a Lie algebra over $\mathbb{K}$, and let U be a $\mathbb{K}$-subspace of $\mathfrak{L}$. Then U is called an *ideal* of $\mathfrak{L}$, if $[x, y] \in U$, for all $x \in \mathfrak{L}$ and $y \in U$, and it is called a *Lie subalgebra* of $\mathfrak{L}$, if $[x, y] \in U$, for all $x, y \in U$ (cf. [**Wie24**, Definition 3.1.3]).

Definition C.1.0.4 (**Simple Lie algebras**)**.** A Lie algebra $\mathfrak{L}$ is called *simple*, if the only ideals in $\mathfrak{L}$ are the trivial ones; i.e., $\{0\}$ and $\mathfrak{L}$, and if $[\mathfrak{L}, \mathfrak{L}] \neq \{0\}$.

Definition C.1.0.5. For any $\mathbb{K}$-vector space V, we denote by $\mathfrak{gl}(V)$ the Lie algebra consisting of the vector space $\mathsf{End}_{\mathbb{K}}(V)$ with the commutator bracket, where the set $\mathsf{End}_{\mathbb{K}}(V)$ consists of all linear maps $V \to V$. Recall that, for two elements $a, b \in V$, the *commutator bracket* is defined as $[a, b] = [a, b]_{\text{comm}} = ab - ba$. The elements of $\mathsf{End}_{\mathbb{K}}(V)$ are called *endormorphisms*.

Note C.1.0.6. To motivate the notation and for some context, we quickly recall that, usually, for a $\mathbb{K}$-vector space V, the group of all automorphisms of V is denoted by $\mathsf{GL}(V)$. Automorphisms are invertible endomorphisms. For finite-dimensional vector spaces, there always exists a basis $\mathscr{B}$ such that we can write every linear map $V \to V$ as a matrix. In that case, if the dimension of V is n, the group $\mathsf{GL}(V)$ can be seen as the group of all invertible $n \times n$ matrices and then one also writes $\mathsf{GL}_n(\mathbb{K})$ or $\mathsf{GL}(n, \mathbb{K})$ instead of $\mathsf{GL}(V)$. The group $\mathsf{GL}_n(\mathbb{R})$ is an example for a *Lie group*. We will not explain what a Lie group is here, but we mention that every Lie group has an associated Lie algebra. For $\mathsf{GL}_n(\mathbb{R})$, the Lie algebra is given by all $n \times n$ matrices with entries in $\mathbb{R}$, together with the commutator bracket $[\cdot, \cdot]_{\text{comm}}$, and is denoted by $\mathfrak{gl}_n$.

Definition C.1.0.7 (**Dimension of a Lie algebra**)**.** Let $\mathfrak{L}$ be a Lie algebra over $\mathbb{K}$. Then we call the vector space dimension $\dim_{\mathbb{K}} \mathfrak{L}$ the *dimension* of $\mathfrak{L}$. If $\dim_{\mathbb{K}} \mathfrak{L} < \infty$, then we say that $\mathfrak{L}$ is of *finite dimension*.

Definition C.1.0.8 (**Representations**)**.** A *representation of* $\mathfrak{L}$ is a homomorphism $\delta \colon \mathfrak{L} \to \mathfrak{gl}(V)$, for some $\mathbb{K}$-vector space V. It is called *finite-dimensional*, if V is finite-dimensional as a $\mathbb{K}$-vector space, and it is called *faithful*, if it is injective. In this context, the vector space V is also called an $\mathfrak{L}$-*module*. An $\mathfrak{L}$-module V is called *irreducible*, if $V \neq \{0\}$ and if there exists no non-trivial submodule of V.

Definition & Fact C.1.0.9. For each element x of a Lie algebra $\mathfrak{L}$, we define a map

$$\mathsf{ad}_x \colon \mathfrak{L} \to \mathfrak{L},$$

such that

$$\mathsf{ad}_x(y) = [x, y],$$

for $y \in \mathfrak{L}$. The map ad_x is a linear map and

$$\mathsf{ad}_x([y, z]) = [x, [y, z]] = [[x, y], z] + [y, [x, z]] = [\mathsf{ad}_x(y), z] + [y, \mathsf{ad}_x(z)].$$

This is defined on page 34 in [**Car72**].

Definition C.1.0.10 (**Adjoint Representation**)**.** We define $\mathsf{ad} \colon \mathfrak{L} \to \mathfrak{gl}(V)$, $\mathsf{ad}(x) = \mathsf{ad}_x$. Then ad is a representation and we call it the *adjoint representation of* $\mathfrak{L}$ (cf. [**Wie24**, Example 3.2.4]).

Note C.1.0.11. Since the adjoint representation always maps to $\mathfrak{gl}(V)$ and we defined $\mathfrak{gl}(V)$ as a Lie algebra with commutator bracket, we have for the images ad_x under the adjoint representation, that $\mathsf{ad}_x(y) = [x, y]_{\text{comm}}$, but we also write $[x, y]$.

Reminder C.1.0.12. Recall that the *trace* of an $n \times n$ matrix A — denoted $\mathsf{tr}(A)$ — is the sum of the elements on its main diagonal, $a_{11} + a_{22} + \cdots + a_{nn}$. It is only defined for square matrices. The linear maps ad_x, for $x \in \mathfrak{L}$, correspond to square matrices, since they are endomorphisms.

Definition C.1.0.13 (**Killing form**)**.** Let $\mathfrak{L}$ be a Lie algebra over $\mathbb{K}$ of finite dimension. For each $x, y \in \mathfrak{L}$, we define a bilinear form $\kappa(\cdot, \cdot)$ by

$$\kappa(x, y) = \mathsf{tr}(\mathsf{ad}_x \circ \mathsf{ad}_y),$$

and we call it the *Killing form* on $\mathfrak{L}$.

Note C.1.0.14. As the trace of the linear map $\mathsf{ad}_x \circ \mathsf{ad}_y$, $\kappa(x, y)$ is an element of the field $\mathbb{K}$. The Killing form is bilinear and symmetric. For simple (and *semi-simple*) Lie algebras, the Killing form is non-degenerate (cf. [**Hum72**, §5.1]).

Definition C.1.0.15 (**Abelian Lie algebras**)**.** Let $\mathfrak{L}$ be a Lie algebra over $\mathbb{K}$ of finite dimension. Then we call $\mathfrak{L}$ *abelian* or *commutative* if

$$[x, y] = 0, \quad \text{for all } x, y \in \mathfrak{L}\,.$$

Compare [**Hum72**, §1.4].

Definition C.1.0.16 (**Torus, maximal torus**)**.** A *torus* of a finite-dimensional Lie algebra $\mathfrak{L}$ is a subalgebra $\mathfrak{T}$ of $\mathfrak{L}$ such that

(i) $\mathsf{ad}_t \colon \mathfrak{L} \to \mathfrak{L}$ is diagonalizable for all $t \in \mathfrak{T}$.
(ii) $\mathfrak{T}$ is commutative.

A torus in $\mathfrak{L}$ is called *maximal*, if it is not properly contained in any other torus. If $\mathfrak{L}$ has finite dimension, such a torus always exists.

Definition C.1.0.17 (**Cartan subalgebra**)**.** A subalgebra $\mathfrak{H}$ of a finite-dimensional Lie algebra $\mathfrak{L}$ is called a *Cartan subalgebra*, if it satisfies the following two conditions:

(i) For some $n \in \mathbb{N}$, repeating Lie multiplication of $\mathfrak{H}$ with $\mathfrak{H}$ n times, leads to
$$[[[\mathfrak{H}, \mathfrak{H}], \mathfrak{H}], \dots] = 0.$$
Subalgebras satisfying this condition are also called *nilpotent*.
(ii) If $[x, h] \in \mathfrak{H}$, for all $h \in \mathfrak{H}$, then $x \in \mathfrak{H}$.

Condition (ii) means that $\mathfrak{H}$ is not contained as an ideal in any larger subalgebra of $\mathfrak{L}$. With that, Cartan subalgebras, if they exist, are maximal tori.

Fact C.1.0.18. Let $\mathfrak{L}$ be a simple Lie algebra over $\mathbb{C}$ of finite dimension. Then $\mathfrak{L}$ has Cartan subalgebras and any two Cartan subalgebras are isomorphic; given any two Cartan subalgebras of $\mathfrak{L}$, there is an automorphism of $\mathfrak{L}$, which transforms one into the other (cf. [**Car72**, §3.2]).

In the following, when we talk about Cartan subalgebras $\mathfrak{H}$, we always assume that the underlying Lie algebra $\mathfrak{L}$ is of finite dimension, simple and over the field $\mathbb{K} = \mathbb{C}$. Most results can be shown more generally for *semi-simple* Lie algebras over algebraically closed fields, but that will not be important for us here.

Definition C.1.0.19 (**Root subspaces**, **roots**, **root systems**)**.** Let $\mathfrak{L}$ be a simple Lie algebra over $\mathbb{C}$ of finite dimension and let $\mathfrak{H}$ be a Cartan subalgebra. For any $\mathbb{C}$-linear map $\alpha\colon \mathfrak{H} \to \mathbb{C}$, we call

$$\mathfrak{L}_\alpha := \{x \in \mathfrak{L} \mid [h, x] = \alpha(h)x, \text{ for all } h \in \mathfrak{H}\}$$

the *root subspace* of $\mathfrak{L}$ corresponding to α. In other words, a root subspace with respect to $\mathfrak{H}$, consists of elements of $\mathfrak{L}$, for which Lie multiplication with an element $h \in \mathfrak{H}$, is the same as scalar multiplication with $\alpha(h)$. We call α a *root* of $\mathfrak{L}$ with respect to $\mathfrak{H}$ if $\mathfrak{L}_\alpha \neq \{0\}$ and $\alpha \neq 0$. We denote the set of all roots of $\mathfrak{L}$, with respect to $\mathfrak{H}$, by $\phi := \phi(\mathfrak{L}, \mathfrak{H})$ and call it the *root system* of $\mathfrak{L}$ with respect to $\mathfrak{H}$ (cf. [**Wie24**, Definition 3.1.6]).

Note C.1.0.20. We introduced root systems for Euclidean vector spaces in Definition 1.5.0.4, and here we introduce them for Lie algebras. We will see how this is connected in Note C.1.0.27 and Theorem C.1.0.28.

Definition & Fact C.1.0.21 (**Cartan decomposition**)**.** Let $\mathfrak{L}$ be a simple Lie algebra over $\mathbb{C}$ of finite dimension with a Cartan subalgebra $\mathfrak{H}$. Then $[\mathfrak{H}, \mathfrak{H}] = 0$ and $\mathfrak{L}$ can be decomposed into a direct sum of $\mathfrak{H}$ with a number of 1-dimensional subspaces, which are all invariant[1] under multiplication by $\mathfrak{H}$, i.e.,

$$\mathfrak{L} = \mathfrak{H} \oplus \mathfrak{L}_{\alpha_1} \oplus \mathfrak{L}_{\alpha_2} \oplus \cdots \oplus \mathfrak{L}_{\alpha_n},$$

where $\dim \mathfrak{L}_{\alpha_i} = 1$ and $[\mathfrak{H}, \mathfrak{L}_{\alpha_i}] = \mathfrak{L}_{\alpha_i}$, for each $i \in \{1, \ldots, n\}$. This is called a *Cartan decomposition* of $\mathfrak{L}$ (cf. [**Car72**, §3.2]) or a *root space decomposition* of $\mathfrak{L}$ (cf. [**Hum72**, §8.1] or [**Wie24**, Definition 3.1.13]).

Example C.1.0.22. Let $\mathfrak{L}$ be the Lie algebra over $\mathbb{C}$ of all $n \times n$ matrices of trace 0. Then the diagonal matrices of trace 0 form a Cartan subalgebra $\mathfrak{H}$ and we have

$$\mathfrak{L} = \mathfrak{H} \oplus \sum_{i \neq j} \mathbb{C}e_{ij},$$

where e_{ij} is the elementary matrix with 1 in the (i, j)-position and 0 elsewhere.

$$\begin{array}{c} \\ \\ \\ \\ i \\ \\ \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{cccccccc} & & & & j & & & \end{array} \\ \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & 1 & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \end{array} = e_{ij}$$

This direct decomposition is a Cartan decomposition and the 1-dimensional subspace $\mathbb{C}e_{ij}$ is invariant under $\mathfrak{H}$.

Note & Notation C.1.0.23 ($\mathfrak{H}^*$, t_α)**.** For any Cartan subalgebra $\mathfrak{H}$, we will denote by $\mathfrak{H}^*$ the vector space of $\mathbb{C}$-linear maps from $\mathfrak{H}$ to $\mathbb{C}$. In other words, $\mathfrak{H}^*$ is the dual vector space of $\mathfrak{H}$.

[1]To be very precise: By *invariant* we mean that for each $x \in \mathfrak{L}_{\alpha_i}$, for $i \in \{1, 2, \ldots, n\}$, $xh \in \mathfrak{L}_{\alpha_i}$, for all $h \in \mathfrak{H}$.

For a finite-dimensional vector space V over $\mathbb{K}$, its dual V^*, and a non-degenerate bilinear form χ of V, one can define maps

$$f_v \colon V \to \mathbb{K},\ x \mapsto \chi(v,x) \text{ for all } v \in V, \text{ and } \theta_\chi \colon V \to V^*,\ v \mapsto f_v.$$

Then θ_χ is an isomorphism of vector spaces over $\mathbb{K}$. With that, for all $f \in V^*$, there exists a unique element $t_f \in V$ such that for all $v \in V$, we have $f(v) = \chi(t_f, v)$ (put $t_f := \theta_\chi^{-1}(f)$). Note as well, that the dual of an Euclidean vector space is an Euclidean vector space.

Since every root is a $\mathbb{C}$-linear map from $\mathfrak{H}$ into $\mathbb{C}$, ϕ is contained in $\mathfrak{H}^*$. For a simple Lie algebra with Cartan subalgebra $\mathfrak{H}$, the Killing form is non-degenerate and bilinear and the restriction of the Killing form to $\mathfrak{H}$ as well, as described in [**Hum72**, §8.2]. Thus, with the above, we know that for each $\alpha \in \mathfrak{H}^*$, there exists a unique element $t_\alpha \in \mathfrak{H}$ such that

$$\kappa(t_\alpha, h) = \alpha(h), \text{ for all } h \in \mathfrak{H}.$$

Note C.1.0.24 (**Co-roots**)**.** Let $\mathfrak{L}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$, with Cartan subalgebra $\mathfrak{H}$. Let $\phi \subseteq \mathfrak{H}^*$ be the root system of $\mathfrak{L}$. We consider the real span of the roots, $\operatorname{span}_{\mathbb{R}}(\phi) \subseteq \mathfrak{H}^*$, and use the Killing form to identify $\mathfrak{H}^* \cong \mathfrak{H}$. This allows us to view the root system ϕ as a subset of a real Euclidean vector space — namely, the real span $\mathfrak{H}_{\mathbb{R}} \subseteq \mathfrak{H}$, generated by the elements $t_\alpha \in \mathfrak{H}$ corresponding to $\alpha \in \phi$. The Killing form κ, restricted to this subspace $\mathfrak{H}_{\mathbb{R}}$, gives a positive definite inner product, and in this way, the root system ϕ becomes a (reduced, crystallographic) root system in the classical Euclidean sense. For any $\alpha \in \mathfrak{H}^*$, we have:

$$\begin{aligned}
\kappa(t_\alpha, t_\alpha^\vee) = \alpha(t_\alpha^\vee) =& \alpha\left(\frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}\right) \\
=& 2\frac{\alpha(t_\alpha)}{\kappa(t_\alpha,t_\alpha)} && \text{(For all } x,y \in \mathfrak{L},\ \kappa(x,y) \in \mathbb{C},\ 2 \in \mathbb{C},\ \alpha \text{ } \mathbb{C}\text{-linear.)} \\
=& 2\frac{\kappa(t_\alpha,t_\alpha)}{\kappa(t_\alpha,t_\alpha)} && (\kappa(t_\alpha,h) = \alpha(h), \text{ for all } h \in \mathfrak{H}, \text{ and } t_\alpha \in \mathfrak{H}.) \\
=& 2
\end{aligned}$$

Viewing $t_\alpha \in \mathfrak{H}_{\mathbb{R}}$ as the representative of the root $\alpha \in \mathfrak{H}^*$ via the Killing form, we can identify the co-root associated to α with $t_\alpha^\vee$.

$$\alpha^\vee = t_\alpha^\vee = \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}.$$

Definition & Facts C.1.0.25 (**Cartan integers**)**.** Let $\mathfrak{L}$ be a simple Lie algebra over $\mathbb{C}$ of finite dimension with Cartan subalgebra $\mathfrak{H}$. Let ϕ be the root system for $\mathfrak{L}$ with respect to $\mathfrak{H}$.

(i) Then ϕ spans $\mathfrak{H}^*$.

Let $\mathfrak{L} = \mathfrak{H} \oplus \sum_{i=1}^n \mathfrak{L}_{\alpha_i}$ be a Cartan decomposition of $\mathfrak{L}$.

(ii) For every $\alpha \in \phi$, we have $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$.

(iii) If $\alpha \in \phi$, the only scalar multiples of α, which are roots, are α and $-\alpha$.

(iv) If $\alpha, \beta \in \phi$, then $\beta(\alpha^\vee) \in \mathbb{Z}$, and $\beta - \beta(\alpha^\vee)\alpha \in \phi$.
The numbers $\beta(\alpha^\vee)$ are called *Cartan integers*.

(v) Let $\alpha, \beta \in \phi$, $\beta \neq \pm\alpha$. Let r and q be, respectively, the largest integers for which $\beta - r\alpha$ and $\beta + q\alpha$ are roots. Then all $\beta + i\alpha \in \phi$, for $-r \leq i \leq q$, and $\beta(\alpha^\vee) = r - q$.

(vi) As a Lie algebra, $\mathfrak{L}$ is generated by the root spaces $\mathfrak{L}_\alpha$.

REFERENCE. This is [**Hum72**, §8.3 and §8.4]. □

The following lemma is a known result for which we will provide an explicit proof for clarity and for our own understanding similar to the one found in [**Hum72**, §8.5].

Lemma C.1.0.26. *Let $\mathfrak{L}$ be a simple Lie algebra over $\mathbb{C}$ of finite dimension with Cartan subalgebra $\mathfrak{H}$, let ϕ be the root system of $\mathfrak{L}$ with respect to $\mathfrak{H}$, and let*

$$\mathfrak{L} = \mathfrak{H} \oplus \sum_{i=1}^{n} \mathfrak{L}_{\alpha_i}$$

be a Cartan decomposition of $\mathfrak{L}$. Then we have $\kappa(t_\alpha, t_\beta) \in \mathbb{Q}$, for all $\alpha, \beta \in \phi$.

PROOF. For all $\gamma, \delta \in \mathfrak{H}^*$, we have

$$\kappa(t_\gamma, t_\delta) = \mathsf{tr}(\mathsf{ad}_{t_\gamma} \circ \mathsf{ad}_{t_\delta}) = \sum_{i=1}^{n} \alpha_i(t_\gamma)\alpha(t_\delta) = \sum_{i=1}^{n} \kappa(t_{\alpha_i}, t_\gamma)\kappa(t_{\alpha_i}, t_\delta),$$

and for $\beta \in \phi$, we get

$$0 \neq \kappa(t_\beta, t_\beta) = \sum_{i=1}^{n} \kappa(t_{\alpha_i}, t_\beta)^2.$$

Dividing by $\kappa(t_\beta, t_\beta)^2$, we get

$$\frac{1}{\kappa(t_\beta, t_\beta)} = \sum_{i=1}^{n} \frac{\kappa(t_{\alpha_i}, t_\beta)^2}{\kappa(t_\beta, t_\beta)^2}.$$

Furthermore,

$$\begin{aligned}\frac{\kappa(t_{\alpha_i}, t_\beta)}{\kappa(t_\beta, t_\beta)} = \frac{1}{\kappa(t_\beta, t_\beta)}\kappa(t_{\alpha_i}, t_\beta) = \kappa(t_{\alpha_i}, \frac{t_\beta}{\kappa(t_\beta, t_\beta)}) = \frac{1}{2}\kappa\left(t_{\alpha_i}, \frac{2t_\beta}{\kappa(t_\beta, t_\beta)}\right) \\ = \frac{1}{2}\kappa(t_{\alpha_i}, t_\beta^\vee) = \frac{1}{2}\alpha_i(t_\beta^\vee) \in \mathbb{Q}.\end{aligned}$$

That means, $\sum_{i=1}^{n} \frac{\kappa(t_{\alpha_i}, t_\beta)^2}{\kappa(t_\beta, t_\beta)^2} \in \mathbb{Q}$, and thus $\frac{1}{\kappa(t_\beta, t_\beta)} \in \mathbb{Q}$, and $\kappa(t_\beta, t_\beta) \in \mathbb{Q}$.

Now we can write

$$\begin{aligned}\kappa(t_\alpha, t_\beta) = \kappa\left(t_\alpha, \frac{2\kappa(t_\beta, t_\beta)}{2\kappa(t_\beta, t_\beta)}t_\beta\right) = \frac{\kappa(t_\beta, t_\beta)}{2}\kappa\left(t_\alpha, \frac{2}{\kappa(t_\beta, t_\beta)}t_\beta\right) \\ = \frac{\kappa(t_\beta, t_\beta)}{2}\kappa(t_\alpha, t_\beta^\vee) = \frac{\kappa(t_\beta, t_\beta)}{2}\alpha(t_\beta^\vee)\end{aligned}$$

and $\frac{\kappa(t_\beta, t_\beta)}{2}\alpha(t_\beta^\vee) \in \mathbb{Q}$. □

Note C.1.0.27. Let $\mathfrak{L}$ be a simple Lie algebra over $\mathbb{C}$ of finite dimension with Cartan subalgebra $\mathfrak{H}$ and ϕ a root system of $\mathfrak{L}$ with respect to $\mathfrak{H}$.

We define $\kappa^* \colon \mathfrak{H}^* \times \mathfrak{H}^* \to \mathbb{Q}$ as $(\alpha, \beta) \mapsto \kappa(t_\alpha, t_\beta)$, for all $\alpha, \beta \in \mathfrak{H}^*$.

We know that ϕ spans $\mathfrak{H}^*$. Let $\alpha_1, \ldots, \alpha_n$ be a basis of $\mathfrak{H}^*$, consisting of roots. Now, for every $\beta \in \phi$, we can write β uniquely as $\beta = \sum_{i=1}^{n} c_i \alpha_i$, and one can show that all c_i are elements of $\mathbb{Q}$ (cf. [**Hum72**, §8.5]). Let $E_{\mathbb{Q}}$ be the $\mathbb{Q}$-subspace of $\mathfrak{H}^*$ spanned by all the roots. Then $E_{\mathbb{Q}}$ has $\mathbb{Q}$-dimension $n = \dim_{\mathbb{C}}(\mathfrak{H}^*)$.

With a similar calculation as in Lemma C.1.0.26, one can show that $\kappa^*(\alpha, \beta) \in \mathbb{Q}$, for all $\alpha, \beta \in \phi$ (cf. [**Hum72**, §8.5]). It follows that all inner products of vectors in $E_{\mathbb{Q}}$ are rational and thus we obtain a non-degenerate form on $E_{\mathbb{Q}}$.

If we now extend the base field from $\mathbb{Q}$ to $\mathbb{R}$ and denote by E the real vector space obtained from that, the form extends canonically to E. With that, E is an Euclidean vector space, ϕ contains a basis of E and $\dim_{\mathbb{R}}(E) = n$ (cf. [**Hum72**, §8.5]).

Theorem C.1.0.28. *With the notation as in Note C.1.0.27, ϕ is a root system, as defined in Definition 1.5.0.4, for the Euclidean vector space E. There is a one-to-one correspondence between pairs $(\mathfrak{L}, \mathfrak{H})$ and (ϕ, E), up to isomorphism.*

Reference. This is [**Hum72**, §8.5, Ch. IV and V]. □

C.2. Chevalley Bases & Chevalley Groups

Construction, Theorem & Definitions C.2.0.1 (Chevalley basis, structure constants). Let $\mathfrak{L}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$, with Cartan subalgebra $\mathfrak{H}$, root system ϕ with respect to $\mathfrak{H}$, and fundamental system

$$\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \phi^+.$$

Let

$$\mathfrak{L} = \mathfrak{H} \oplus \sum_{\beta \in \phi} \mathfrak{L}_\beta \,.$$

be a Cartan decomposition of $\mathfrak{L}$. For each root $\gamma \in \phi$, denote by $\gamma^\vee \in \mathfrak{H}$ the corresponding co-root.

Choose, for every positive root $\beta_i \in \phi^+$, a nonzero element $e_{\beta_i} \in \mathfrak{L}_{\beta_i}$. Then for each negative root $-\beta_i \in \phi^-$, there exists a unique element $e_{-\beta_i} \in \mathfrak{L}_{-\beta_i}$ such that

$$[e_{\beta_i}, e_{-\beta_i}] = \beta_i^\vee \qquad \text{(cf. [**Car72**, §4.1], [**Hum72**, §25.2]).}$$

Moreover, the elements $e_{\beta_i} \in \mathfrak{L}_{\beta_i}$ may be chosen such that for all $\beta_i, \beta_k \in \phi$,

$$\begin{aligned} [e_{\beta_i}, e_{-\beta_i}] &= \beta_i^\vee, \\ [e_{\beta_i}, e_{\beta_k}] &= \pm(p_{\beta_i,\beta_k} + 1) e_{\beta_i + \beta_k}, \end{aligned}$$

where p_{β_i,β_k} is the largest integer p such that $\beta_k - p\beta_i \in \phi$ (cf. C.1.0.25).
The scalars $c_{\beta_i,\beta_k} \in \{\pm(p_{\beta_i,\beta_k} + 1)\}$ satisfying

$$[e_{\beta_i}, e_{\beta_k}] = c_{\beta_i,\beta_k} e_{\beta_i + \beta_k}$$

are called the *(Chevalley) structure constants* of $\mathfrak{L}$ with respect to $\mathscr{C}$ ([**Car72**, p. 52]). Furthermore, one defines

$$a_{\beta_i,\beta_k} = \frac{2[\beta_i, \beta_k]}{[\beta_i, \beta_i]},$$

which are known to be integers ([**Car72**, p. 38], [**Hum72**, §25.2]).

A *Chevalley basis* of $\mathfrak{L}$ with respect to $\mathfrak{H}$ is a set

$$\mathscr{C} := \{\alpha_j^\vee \mid \alpha_j \in \Pi\} \cup \{e_{\beta_i} \mid \beta_i \in \phi\}$$

satisfying the following relations:

(i) $[e_{\beta_i}, e_{-\beta_i}] = \beta_i^\vee$,
(ii) $[e_{\beta_i}, e_{\beta_k}] = c_{\beta_i,\beta_k} e_{\beta_i+\beta_k}$,
(iii) $[e_{-\beta_i}, e_{-\beta_k}] = c_{-\beta_i,-\beta_k} e_{-\beta_i+-\beta_k}$, and $c_{-\beta_i,-\beta_k} = -c_{\beta_i,\beta_k}$.

It can be shown that any Chevalley basis of $\mathfrak{L}$ indeed forms a basis of $\mathfrak{L}$ (cf. [**Car72**, Theorem 4.2.1], [**Hum72**, §25], [**Wie24**, §3.1]).

Facts C.2.0.2. Let $\mathfrak{L}$ be a simple Lie algebra of finite dimension over $\mathbb{C}$.

- A Chevalley basis of $\mathfrak{L}$ exists.
- A Chevalley basis $\mathscr{C}$ is a basis of the $\mathbb{C}$-vector space $\mathfrak{L}$, and the $\mathbb{Z}$-span of $\mathscr{C}$ is a Lie subalgebra of $\mathfrak{L}$ over $\mathbb{Z}$.
- The Chevalley structure constants of $\mathfrak{L}$ with respect to $\mathscr{C}$ have the following properties:
  (i) For all roots α, β, we have $c_{\alpha,\beta} = -c_{\beta,\alpha}$ and $c_{-\alpha,-\beta} = -c_{\alpha,\beta}$.
  (ii) All structure constants of $\mathfrak{L}$, with respect to $\mathscr{C}$ are integers, and their absolute values are uniquely determined by the root system ϕ.
  (iii) If $\mathscr{C}'$ is another Chevalley basis of $\mathfrak{L}$, with structure constants $(c'_{\alpha,\beta})_{\alpha,\beta\in\phi}$, then $c'_{\alpha,\beta} \in \{\pm c_{\alpha,\beta}\}$, for all roots α, β.
  (iv) If $\mathscr{C}'$ is another Chevalley basis of $\mathfrak{L}$, with structure constants $(c'_{\alpha,\beta})_{\alpha,\beta\in\phi}$ such that $c_{\alpha,\beta} = c'_{\alpha,\beta}$, for all roots α, β lying in some positive system $\Pi \subseteq \phi$, then $c_{\alpha,\beta} = c'_{\alpha,\beta}$, for all roots α, β.
  (v) In a simply laced root system, we have $c_{\alpha,\beta} \in \{\pm 1\}$, for all roots α, β, for which $\alpha + \beta$ is a root.

Compare [**Car72**, Theorem 4.2.1], [**Hum72**, §25.2] and [**Wie24**, Lemma 3.1.21, 22, 23 & Remark 3.1.24].

Construction C.2.0.3.

i) Let $\mathfrak{L}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$ with Chevalley basis

$$\mathscr{C} := \{\alpha_j^\vee \mid \alpha_j \in \Pi\} \cup \{e_{\beta_i} \mid \beta_i \in \phi\}.$$

Let $\xi \in \mathbb{C}$. We put

$$z_{\beta_i,\xi} := \exp(\xi \mathsf{ad}_{e_{\beta_i}}) = \sum_{k=0}^{\infty} \frac{(\xi \mathsf{ad}_{e_{\beta_i}})^k}{k!}.$$

Note that this sum is finite, because $\mathsf{ad}_{e_{\beta_i}}$ is nilpotent (cf. [**Wie24**, Lemma 3.2.5]). Every $z_{\beta_i,\xi}$ is an automorphism of $\mathfrak{L}$ and

$$\begin{aligned} z_{\beta_i,\xi}(e_{\beta_i}) &= e_{\beta_i}, \\ z_{\beta_i,\xi}(e_{-\beta_i}) &= e_{-\beta_i} + \xi\beta_i^\vee - \xi^2 e_{\beta_i}, \\ z_{\beta_i,\xi}(\beta_i^\vee) &= \beta_i^\vee - 2\xi e_{\beta_i}. \end{aligned}$$

As described in [**Car72**, §4.3], the automorphism $z_{\beta_i,\xi}$ transforms each element of the Chevalley basis into a linear combination of basis elements with coefficients, which are non-negative integral powers of ξ with integer coefficients.

ii) Next, we denote by $\mathfrak{L}_\mathbb{Z}$ the subset of $\mathfrak{L}$ of all linear combinations of the basis elements with coefficients in $\mathbb{Z}$. Then $\mathfrak{L}_\mathbb{Z}$ is an additive abelian group, closed under Lie multiplication and thus a Lie algebra over $\mathbb{Z}$. Now, let $\mathbb{K}$ be any field and let $1_\mathbb{K}$ be the identity element of $\mathbb{K}$ with regards to multiplication. We consider elements of the form

$$\sum_{\alpha_j\in\Pi} \lambda_{\alpha_j}(1_\mathbb{K}\otimes\alpha_j^\vee) + \sum_{\beta_i\in\phi} \mu_{\beta_i}(1_\mathbb{K}\otimes e_{\beta_i}),$$

where $\lambda_{\alpha_j}, \mu_{\beta_i} \in \mathbb{K}$. Then the set of all such elements forms an additive abelian group that we will denote by $\mathfrak{L}_\mathbb{K}$. Let

$$\overline{\alpha_j}^\vee := 1_\mathbb{K}\otimes\alpha_j^\vee, \text{ and } \overline{e_{\beta_i}} := 1_\mathbb{K}\otimes e_{\beta_i}$$

Then $\mathfrak{L}_\mathbb{K}$ is a vector space over $\mathbb{K}$ with basis $\overline{\mathscr{C}} = \{\overline{\alpha_j}^\vee \mid \alpha_j\in\Pi\}\cup\{\overline{e_{\beta_i}} \mid \beta_i\in\phi\}$ (cf. [**Car72**, §4.4]). We will define Lie multiplication $[\cdot,\cdot]_{\mathfrak{L}_\mathbb{K}}$ on $\mathfrak{L}_\mathbb{K}$ as follows.

$$\begin{aligned}[\cdot,\cdot]_{\mathfrak{L}_\mathbb{K}}\colon\ &\mathfrak{L}\times\mathfrak{L}\to\mathfrak{L}\\ &(x,y)\mapsto [1_\mathbb{K}\otimes x, 1_\mathbb{K}\otimes y]_{\mathfrak{L}_\mathbb{K}} := 1_\mathbb{K}\otimes[x,y]\end{aligned}$$

With that $(\mathfrak{L}_\mathbb{K}, [\cdot,\cdot]_{\mathfrak{L}_\mathbb{K}})$ is a Lie algebra over $\mathbb{K}$ (cf. [**Car72**, §4.4]).

iii) We now define automorphisms of $\mathfrak{L}_\mathbb{K}$ analogous to the automorphisms $z_{\beta_i,\xi}$ of $\mathfrak{L}$, following [**Car72**, §4.4, p. 63]. For that, let $A_{\beta_i,\xi}$ be the matrix representing $z_{\beta_i,\xi}$ with respect to $\mathscr{C}$. The coefficients of $A_{\beta_i,\xi}$ have the form $a\xi^k$, with $a\in\mathbb{Z}$ and $k\geq 0$. Let ℓ be an element of $\mathbb{K}$. We define the matrix $\overline{A}_{\beta_i,\ell}$ to be the matrix that we obtain from $A_{\beta_i,\xi}$, by replacing each coefficient $a\xi^k$ by $\overline{a}\ell^k$, where $\overline{a}$ is the element of the prime field of $\mathbb{K}$, corresponding to $a\in\mathbb{Z}$. The $\overline{a}\ell^k$, for $\ell\in\mathbb{K}$, are again elements of $\mathbb{K}$. We define $\overline{z}_{\beta_i,\ell}$ to be the linear map of $\mathfrak{L}_\mathbb{K}$ into itself, represented by the matrix $\overline{A}_{\beta_i,\ell}$ with respect to $\overline{\mathscr{C}}$.

iv) Now one can show that, for all $\beta_i\in\phi$ and $\ell\in\mathbb{K}$, $\overline{z}_{\beta_i,\ell}$ is an automorphism of $\mathfrak{L}_\mathbb{K}$. That is exactly [**Car72**, Proposition 4.4.2].

Definition C.2.0.4 (**Chevalley groups**)**.** In the context of Construction C.2.0.3, the *Chevalley group* $\mathfrak{L}(\mathbb{K})$ of $\mathfrak{L}$ over $\mathbb{K}$ is the group of automorphisms of the Lie algebra $\mathfrak{L}_\mathbb{K}$, generated by the elements $\overline{z}_{\beta_i,\ell}$.

Notation C.2.0.5. In the following, for a Chevalley group $\mathfrak{L}(\mathbb{K})$, we will write $\alpha_j^\vee$, e_{β_i}, $z_{\beta_i,\ell}$ and $A_{\beta_i,\ell}$ instead of $\overline{\alpha_j}^\vee$, $\overline{e_{\beta_i}}$, $\overline{z}_{\beta_i,\ell}$ and $\overline{A}_{\beta_i,\ell}$, respectively.

There is no danger of confusion since the objects originally called $\alpha_j^\vee$, e_{β_i}, $z_{\beta_i,\ell}$ and $A_{\beta_i,\ell}$ are special cases of $\overline{\alpha_j}^\vee$, $\overline{e_{\beta_i}}$, $\overline{z}_{\beta_i,\ell}$ and $\overline{A}_{\beta_i,\ell}$ when $\mathbb{K}=\mathbb{C}$ ([**Car72**, p. 64]).

Proposition C.2.0.6. *The group $\mathfrak{L}(\mathbb{K})$ is determined up to isomorphism by the simple Lie algebra $\mathfrak{L}$ over $\mathbb{C}$ and the field $\mathbb{K}$.*

Reference. This is [**Car72**, Proposition 4.4.3]. □

C.3. BN-Pairs for Chevalley Groups, Root & Weight Lattices

For this whole section, let $\mathfrak{L}$ be a simple Lie algebra of finite dimension over $\mathbb{C}$ with root system ϕ (with respect to some Cartan subalgebra $\mathfrak{H}$) and Chevalley basis $\mathscr{C}$. Let $\mathbb{K}$ be a field and let

$$G = \mathfrak{L}(\mathbb{K}) = \langle \{ z_{\beta,\lambda} \mid \beta \in \phi,\ \lambda \in \mathbb{K} \} \rangle$$

be the Chevalley group of $\mathfrak{L}$ over $\mathbb{K}$.

Definition C.3.0.1. Let X_α be the subgroup of $\mathfrak{L}(\mathbb{K})$ generated by all $z_{\alpha,k}$, for $k \in \mathbb{K}$. Then the subgroups X_α, for $\alpha \in \phi$, are called the *root subgroups* of $\mathfrak{L}(\mathbb{K})$ (cf. [**Car72**, p. 68]).

Notation C.3.0.2. In the following, let U^+ be the subgroup of $\mathfrak{L}(\mathbb{K})$ generated by all $z_{\beta_i,\ell}$, for $\ell \in \mathbb{K}$ and $\beta_i \in \phi^+$. Similarly, let U^- be the subgroups of $\mathfrak{L}(\mathbb{K})$ generated by all $z_{\beta_i,\ell}$, for $\ell \in \mathbb{K}$ and $\beta_i \in \phi^-$.

Definition C.3.0.3. For $i \in \{1, 2, \ldots, m\}$, let U_i be groups with centers Z_i such that

$$U_{j+1} = U_j / Z_j, \text{ for } j \in \{1, 2, \ldots, m-1\},$$

and such that U_m/Z_m is the trivial group. Then we write

$$U_1 \supseteq U_2 \supseteq \cdots \supseteq U_m \supseteq 1$$

and call this a *central series* for U_1.

Definition C.3.0.4. A linear transformation u of a vector space V into itself is called *nilpotent*, if there exists some positive integer n such that $u^n = 0$. It is called *unipotent*, if $u - 1$ is nilpotent, i.e., $(u-1)^n = 0$ (cf. [**Car72**, p. 68]). A group of linear transformations of a vector space that purely consists of unipotent elements is called a *unipotent group*.

Facts C.3.0.5.

- Each root subgroup is isomorphic to the additive group of $\mathbb{K}$.
- The subgroups U^+ and U^- generate $\mathfrak{L}(\mathbb{K})$.
- The elements of U^+ and U^- act on $\mathfrak{L}_\mathbb{K}$ as unipotent linear transformations.

Facts C.3.0.6. Let $\mathfrak{L}(\mathbb{K})$ be a Chevalley group, let U be the subgroup of $\mathfrak{L}(\mathbb{K})$ generated by the root subgroups X_α ($\alpha \in \phi^+$), let V be the real vector space spanned by ϕ, let $h\colon V \to \mathbb{R}$ be the linear map, which coincides with the height function on ϕ, and let U_l be the subgroup generated by the X_α with $\alpha \in \phi^+$ and $h(\alpha) \geq l$. Then the following holds.

(i) Each U_l, for $l \geq 1$, is normal in U.

(ii) Let $[U_l, U_j]$ be the subgroup generated by all commutators $[a, b]$, for $a \in U_l$ and $b \in U_j$. Let $\alpha, \beta \in \phi^+$ such that $h(\alpha) \geq j$ and $h(\beta) \geq l$. Then $[z_{\beta,e}, z_{\alpha,f}] \in U_{l+j}$ $(e, f \in \mathbb{K})$. Furthermore, elements of the form $z_{\beta,e}$ generate U_l, elements of the form $z_{\alpha,f}$ generate U_j, and $[U_l, U_j] \subseteq U_{l+j}$. In particular, $[U, U_j] \subseteq U_{j+1}$, for each j, and thus, each element of U_j/U_{j+1} commutes with each element of U/U_{j+1} and U_j/U_{j+1} is in the center of U/U_{j+1}.

(iii) U is nilpotent and, if we denote the greatest height of a root in $\mathfrak{L}$ by m, we have that
$$U = U_1 \supseteq U_2 \supseteq \cdots \supseteq U_m \supseteq 1$$
is a central series for U.

(iv) Each element of U is uniquely expressible in the form
$$\prod_{\alpha_i \in \phi^+} z_{\alpha_i, k_i},$$
where the product is taken over all positive roots in increasing order.

REFERENCE. This is [**Car72**, Theorem 5.3.3]. □

Reminder C.3.0.7 (**Special linear group**)**.** Let $\mathbb{K}$ be an arbitrary field. Recall that the *special linear group* over $\mathbb{K}$ is defined as
$$\mathsf{SL}_2(\mathbb{K}) := \left\langle \left\{ \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} \mid f \in \mathbb{K} \right\} \right\rangle.$$
It is the group of 2×2 matrices with determinant 1 (cf. [**Car72**, §6.1]). It is the kernel of the determinant
$$\det\colon \mathsf{GL}_2(\mathbb{K}) \to \mathbb{K}^\times.$$
Thus, it is a normal subgroup of the general linear group $\mathsf{GL}_2(\mathbb{K})$.

Theorem C.3.0.8. *In the following, let α be a root, $-\alpha$ its opposite root and X_α and $X_{-\alpha}$ the corresponding root subgroups. There is a homomorphism from $\mathsf{SL}_2(\mathbb{C})$ into $\langle X_\alpha, X_{-\alpha}\rangle \subseteq \mathfrak{L}(\mathbb{C})$ such that*
$$\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \mapsto z_{\alpha,f}, \quad \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} \mapsto z_{-\alpha,f}, \quad \textit{where } f \in \mathbb{K}.$$

REFERENCE. This is [**Car72**, Theorem 6.3.1]. □

We will denote the homomorphism from $\mathsf{SL}_2(\mathbb{K})$ into $\langle X_\alpha, X_{-\alpha}\rangle$ by φ_α. We put
$$y_{\alpha,\lambda} := \varphi_\alpha \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \; n_\alpha := \varphi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
In words, the elements $y_{\alpha,\lambda}$ and n_α of the Chevalley group $\mathfrak{L}(\mathbb{K})$ are the images of the diagonal and the monomial matrices, respectively, under the homomorphism from $\mathsf{SL}_2(\mathbb{K})$ into $\langle X_\alpha, X_{-\alpha}\rangle$.

Definition & Fact C.3.0.9 (**Root lattices**)**.** Let $\mathfrak{L}$ be a simple Lie algebra of finite dimension over $\mathbb{C}$ with root system ϕ and Chevalley basis $\mathscr{C}$. Let $\mathbb{K}$ be a field and let $\mathfrak{L}(\mathbb{K})$ be the Chevalley group of $\mathfrak{L}$ over $\mathbb{K}$. We denote the set of all linear combinations of elements of ϕ with integer coefficients by $\mathbb{Z}\phi$. Then $\mathbb{Z}\phi$ is an additive group generated by the roots of $\mathfrak{L}$. It is a free abelain group and has a basis consisting of the set of fundamental roots. We call $\mathbb{Z}\phi$ the *root lattice* of $\mathfrak{L}(\mathbb{K})$. A homomorphism from the additive group $\mathbb{Z}\phi$ into the multiplicative group $\mathbb{K}^\times$ of non-zero elements of $\mathbb{K}$ is called a $\mathbb{K}$-*character* of $\mathbb{Z}\phi$ (cf. [**AB08**, Proposition B.2] and [**Car72**, p. 97] or [**Ste68**, pp. 42–47]).

Note C.3.0.10. The extension of a map $\phi \to \mathbb{K}^\times$ to a character of $\mathbb{Z}\phi$, if it exists, is uniquely determined by its values on the fundamental roots. Let $\alpha \in \phi$, $\lambda \in \mathbb{K}^\times$

be arbitrary. Let $\chi\colon \phi \to \mathbb{K}^\times$ be a map such that

$$\beta \mapsto \chi(\beta) = \lambda^{2(\alpha,\beta)/(\alpha,\alpha)}.$$

We define

$$\chi_{\alpha,\lambda}\colon \mathbb{Z}\phi \to \mathbb{K}^\times$$
$$\chi_{\alpha,\lambda}(\gamma) = \lambda^{2(\alpha,\gamma)/(\alpha,\alpha)}.$$

Then $\chi_{\alpha,\lambda}$ is a character of $\mathbb{Z}\phi$, which takes the value $\lambda^{2(\alpha,\beta)/(\alpha,\alpha)}$ at the root β (cf. [**Car72**, pp. 97–98]).

Theorem C.3.0.11. *We define H to be the subgroup of $\mathfrak{L}(\mathbb{K})$ generated by the elements $y_{\alpha,\lambda}$, for all $\alpha \in \phi$, $\lambda \neq 0 \in \mathbb{K}$. The group H acts on the Chevalley basis of $\mathfrak{L}_\mathbb{K}$, each element of H is an automorphism of $\mathfrak{L}_\mathbb{K}$, which acts trivially on $\mathfrak{H}_\mathbb{K}$ and transforms each root vector e_β into a multiple of itself. The coefficients arising in this way define naturally a character of the root lattice $\mathbb{Z}\phi$.*

Reference. This is [**Car72**, §6.4 & §7.1]. □

Definition & Facts C.3.0.12 (**Fundamental weights**, **weight lattices**).

- Each $\mathbb{K}$-character χ of $\mathbb{Z}\phi$ gives rise to an automorphism a_χ of $\mathfrak{L}_\mathbb{K}$, defined by
$$a_\chi(\alpha^\vee) = \alpha^\vee,\ a_\chi(e_\beta) = \chi(\beta)e_\beta.$$[2]
- The automorphisms of $\mathfrak{L}_\mathbb{K}$ of the form a_χ form a subgroup $\hat{H}$ of the full automorphism group of $\mathfrak{L}_\mathbb{K}$. The map $\chi \mapsto a_\chi$ is an isomorphism between the group of $\mathbb{K}$-characters of $\mathbb{Z}\phi$ and the group $\hat{H}$ (cf. [**Car72**, p. 98]).
- The group H is a subgroup of $\hat{H}$; each element of H has the form a_χ, for some $\mathbb{K}$-character χ.
- Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \subseteq \mathfrak{H}^*$ be a basis of the root lattice $\mathbb{Z}\phi$. Let $\{\alpha_1^\vee, \ldots, \alpha_m^\vee\} \subseteq \mathfrak{H}_\mathbb{R}$ be the corresponding basis of fundamental co-roots (cf. C.1.0.24). Define $\beta_1, \ldots, \beta_m \in \mathfrak{H}^*_\mathbb{R}$ to be the unique elements satisfying:
$$\beta_j(\alpha_i^\vee) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$
Then $\{\beta_1, \ldots, \beta_m\}$ forms a basis of $\mathfrak{H}^*_\mathbb{R}$, and is the dual basis to $\{\alpha_1^\vee, \ldots, \alpha_m^\vee\}$. Note that we used different dualities between $\mathfrak{H}$ and $\mathfrak{H}^*$ in the course of this chapter.

Object	Space	Dual Object	Dual Space
Duality between $\mathfrak{H}$ and $\mathfrak{H}^*$ via the Killing form			
α_i	$\mathfrak{H}^*$	$\alpha_i^\vee \leftrightarrow \dfrac{2t_{\alpha_i}}{\kappa(t_{\alpha_i}, t_{\alpha_i})}$	$\mathfrak{H}$
Duality between $\mathfrak{H}$ and $\mathfrak{H}^*$ via dual bases			
$\alpha_i^\vee$	$\mathfrak{H}$	β_i	$\mathfrak{H}^*$

[2] This is described in [**Car72**, p. 98], where instead of a_χ, the automorphisms are denoted by $h(\chi)$.

- One can see that the $\beta_1, \ldots, \beta_m$ form a basis for $\mathfrak{H}^*$ via the non-degeneracy of the Killing form (cf. [**Car72**, p. 98]). Put
$$\hat{\phi} = \{\beta_1, \ldots, \beta_m\}.$$
The elements of $\hat{\phi}$ are called the *fundamental weights of* $\mathfrak{L}$. Each fundamental root is a linear combination of elements of $\hat{\phi}$ with integer coefficients (cf. [**Car72**, p. 98]).
- The groups H and $\hat{H}$ normalize U^+ and U^-. Both U^+H and U^-H are subgroups of $\mathfrak{L}(\mathbb{K})$. The group $\hat{H}$ is a group of automorphisms of $\mathfrak{L}_{\mathbb{K}}$, but in general, $\hat{H}$ is not contained in $\mathfrak{L}(\mathbb{K})$. However, $\hat{H}$ normalizes $\mathfrak{L}(\mathbb{K})$ (cf. [**Car72**, p. 100]).
- Let $\mathbb{Z}\hat{\phi}$ be the set of linear combinations of elements of $\hat{\phi}$ with integer coefficients. Then $\mathbb{Z}\hat{\phi}$ is an additive free abelian group generated by the fundamental weights and it is called the *weight lattice* of ϕ (cf. [**Car72**, p. 99], where $\mathbb{Z}\hat{\phi}$ is denoted by Q, or [**Ste68**, pp. 42–47]).

Fact C.3.0.13. Similarly as before, a homomorphism from the additive group $\mathbb{Z}\hat{\phi}$ into the multiplicative group $\mathbb{K}^\times$, is called a $\mathbb{K}$-*character* of $\mathbb{Z}\hat{\phi}$. Every $\mathbb{K}$-character of $\mathbb{Z}\hat{\phi}$ gives rise to a $\mathbb{K}$-character of $\mathbb{Z}\phi$ by restriction, but not every $\mathbb{K}$-character of $\mathbb{Z}\phi$ is necessarily the restriction of some $\mathbb{K}$-character of $\mathbb{Z}\hat{\phi}$ ([**Car72**, p. 99]).

Observation C.3.0.14. We mentioned that every fundamental root can be written as a linear combination of fundamental weights with integer coefficients. Let
$$\hat{\phi} = \{\beta_1, \ldots, \beta_m\}$$
be the set of fundamental weights, $I := \{1, 2, \ldots, m\}$, and let α_j be a fundamental root such that
$$\alpha_j = \mu_{j,1}\beta_1 + \cdots + \mu_{j,m}\beta_m$$
for integers $\mu_{j,1}, \ldots, \mu_{j,m}$. Then
$$\mu_{j,i} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} =: A_{\alpha_j\alpha_i} \quad \text{(cf. [\textbf{Car72}, p. 98])}.$$
Instead of $A_{\alpha_j\alpha_i}$ we will sometimes write A_{ji} for short in cases where this is clear enough.

We have that $A = A(\phi) = (A_{ji})_{j,i\in I}$ is the Cartan matrix (cf. Definition 1.5.0.20) of ϕ, and
$$\alpha_j = \sum_{i\in I} A_{ji}\beta_i, \quad \text{for all } j \in I \quad \text{(cf. [\textbf{Hum72}, p. 68])}.$$

Note C.3.0.15. We have $A(\phi) = A(\phi^\vee)$ if ϕ is simply laced, because then all roots have the same length. If ϕ is not simply laced, then some α_i has a different length than some α_j and therefore $A_{\alpha_j\alpha_i} \neq A_{\alpha_j^\vee\alpha_i^\vee}$ and $A(\phi) \neq A(\phi^\vee)$.

Fact C.3.0.16. The subgroup H of $\mathfrak{L}(\mathbb{K})$ is exactly the subgroup consisting of all automorphisms a_χ of $\mathfrak{L}_{\mathbb{K}}$, for which χ is a $\mathbb{K}$-character of $\mathbb{Z}\phi$, which can be extended to a $\mathbb{K}$-character of $\mathbb{Z}\hat{\phi}$ ([**Car72**, Theorem 7.1.1]).

Identifying the elements $y_{\alpha_i,\lambda}$ of H with the characters induced by them, we see that we can extend each such character to a character of the weight lattice given by

$$\chi_{\alpha_i,\lambda}(\beta_j) = \lambda^{2(\alpha_i,\beta_j)/(\alpha_i,\alpha_i)} = \lambda^{A_{\alpha_i\beta_j}},$$

where $\beta_j \in \mathbb{Z}\hat{\phi}$ (cf. [**Car72**, p. 99]).

Thus, we have the following relations, where $\mathsf{Hom}(\mathbb{Z}\phi, \mathbb{K}^\times)^*$ denotes the set of all characters of $\mathbb{Z}\phi$ that extend to characters of $\mathbb{Z}\hat{\phi}$.

$$\begin{array}{ccccc} & & \mathsf{Hom}(\mathbb{Z}\hat{\phi}, \mathbb{K}^\times) & & \\ & & \wr\downarrow & & \\ H & \xrightarrow{\cong} & \mathsf{Hom}(\mathbb{Z}\phi, \mathbb{K}^\times)^* & \subseteq \mathsf{Hom}(\mathbb{Z}\phi, \mathbb{K}^\times) \xleftarrow{\cong} & \hat{H} \end{array}$$

Fact C.3.0.17. In the following, we list some information about the root lattice $\mathbb{Z}\phi$ and the weight lattice $\mathbb{Z}\hat{\phi}$ for adjoint Chevalley groups $\mathfrak{L}(\mathbb{K})$ of various Lie algebras $\mathfrak{L}$.

Type of $\mathfrak{L}$	A_n	B_n	C_n	D_{2n+1}	D_{2n}
$\mathfrak{L}(\mathbb{K})$	PSL_{n+1}	$\mathsf{PSO}_{2n+1} = \mathsf{SO}_{2n+1}$	PSp_{2n}	PSO_{4n+2}	PSO_{4n}
$\mathbb{Z}\hat{\phi}/\mathbb{Z}\phi$	$\mathbb{Z}_{n+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$[\mathbb{Z}\hat{\phi} : \mathbb{Z}\phi]$	$n+1$	2	2	4	4
Type of $\mathfrak{L}$	E_6	E_7	E_8	F_4	G_2
$\mathbb{Z}\hat{\phi}/\mathbb{Z}\phi$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$[\mathbb{Z}\hat{\phi} : \mathbb{Z}\phi]$	3	2	1	1	1

TABLE 1. The index $[\mathbb{Z}\hat{\phi} : \mathbb{Z}\phi]$ and the group $\mathbb{Z}\hat{\phi}/\mathbb{Z}\phi$ for adjoint Chevalley groups $\mathfrak{L}(\mathbb{K})$ of different Lie algebras $\mathfrak{L}$.

REFERENCE. This table combines a table on page 45 of [**Ste68**] and a table on page 99 of [**Car72**]. □

Proposition C.3.0.18. *Let $W = \langle\{s_{\alpha_i} \mid \alpha_i \in \Pi\}\rangle$ be the Weyl group generated by the fundamental reflections (cf.* [**Car72**, p. 108]*). The element n_{α_i} of $\mathfrak{L}(\mathbb{K})$ acts on $\mathfrak{H}$ in the same way as the element s_{α_i} of W and permutes the root spaces $\mathfrak{L}_\beta$, for $\beta \in \phi$, in the same way as s_{α_i}.*

REFERENCE. This is [**Car72**, Proposition 6.4.2]. □

Theorem C.3.0.19. *Let N be the subgroup of $G = \mathfrak{L}(\mathbb{K})$ generated by H and the elements n_α, for all $\alpha \in \phi$ and let W be the Weyl group of ϕ. There is a homomorphism from N to W with kernel H, under which n_α maps to w_α, for all $\alpha \in \phi$. The subgroup H is a normal subgroup of N and N/H is isomorphic to W.*

REFERENCE. This is [**Car72**, Theorem 7.2.2]. □

Corollary C.3.0.20. *We have $U \cap N = 1$, $UH \cap N = H$, $N \cap \hat{H} = H$ and $U\hat{H} \cap N = H$.*

REFERENCE. This is [**Car72**, Corollary 7.2.3 and Corollary 7.2.4]. □

Proposition C.3.0.21. *Let B be the subgroup UH of $G = \mathfrak{L}(\mathbb{K})$. Then B, together with N of Theorem C.3.0.19, forms a BN-pair for G.*

REFERENCE. This is [**Car72**, Proposition 8.2.1] □

Proposition C.3.0.22. *Let $\hat{G}$ be the group of automorphisms of $\mathfrak{L}_{\mathbb{K}}$ generated by G and $\hat{H}$. Then G is a normal subgroup of $\hat{G}$ and $\hat{G}/G \cong \hat{H}/H$*

REFERENCE. This is [**Car72**, Corollary 8.4.7]. □

C.4. The Levi Decomposition for Chevalley Groups

Parabolic subgroups of Chevalley groups can be written as semi-direct products of a *unipotent subgroup* and a *Levi subgroup* (cf. [**Car72**, p. 118]).

Definition C.4.0.1. Let G be a group with a BN-pair. A *parabolic subgroup* of G is a subgroup, which contains some conjugate gBg^{-1} of B (cf. [**Car72**, p. 111]).

Facts C.4.0.2. Let G be a group with a BN-pair and let $n \in N$. Let $w \in W$ be the image of n under the natural homomorphism and let

$$w = w_{i_1} w_{i_2} \dots w_{i_k},\ i_1, i_2, \dots \in I,$$

where k is the word length of w. Let J be the subset $\{i_1, i_2, \dots, i_k\}$ of I.

(i) Then the following three subgroups of G are equal:
- $\langle B, n\rangle$
- $\langle B, nBn^{-1}\rangle$
- $P_J = BN_JB$

(ii) Each parabolic subgroup of G is conjuagte to one of the subgroups
$$P_J = BN_JB.$$

(iii) Furthermore, the subgroups P_J are the only subgroups of G containing B.

(iv) Each subgroup P_J of G is equal to its normalizer.

(v) The subgroups P_J for distinct subsets J of I are all distinct.

(vi) We have $P_J \cap P_K = P_{J\cap K}$ and the subgroups P_J form a lattice isomorphic to the lattice of subsets of I.

REFERENCE. [**Car72**, Proposition 8.3.1, Theorem 8.3.2, Theorem 8.3.3 & Theorem 8.3.4] □

Fact C.4.0.3. Let ϕ_J be the set of roots of ϕ, which are integral combinations of roots in J and let $\overline{\phi_J}$ be the set $\phi \setminus \phi_J$. Then P_J is the subgroup of G generated by H and the root subgroups X_α for $\alpha \in \phi^+ \cup \phi_J$ (cf. [**Car72**, Proposition 8.5.1]).

Fact C.4.0.4. Let U_J be the subgroup of G generated by the root subgroups X_α for $\alpha \in \phi^+ \cap \overline{\phi_J}$. We have

$$U_J = \prod_{\alpha \in \phi^+ \cap \overline{\phi_J}} X_\alpha.$$

Compare [**Car72**, p. 119].

Theorem C.4.0.5. *Let L_J be the subgroup of G generated by H and X_α for $\alpha \in \phi_J$. Then the following hold.*

(i) $P_J = U_J L_J$ and $U_J \cap L_J = 1$.
(ii) P_J is the normalizer of U_J in G.

REFERENCE. This is [**Car72**, Theorem 8.5.2] □

Definition C.4.0.6 (**Levi Decomposition**, **Levi subgroups**)**.** The decomposition of P_J into the semi-direct product of U_J and L_J is called a *Levi decomposition*. The subgroup L_J and its conjugates in U_J are called the *Levi subgroups* of P_J and U_J.

L. Linear Algebraic Groups

In Appendix C we mention that Chevalley groups have BN-pairs, and in Section 2.7 we describe that groups with BN-pairs are associated to buildings. Chevalley groups can thus be described as groups of automorphisms of buildings. In Section 2.9, we talk about the fact that, for some Moufang buildings Δ, we can view their little projective group as the Chevalley group attached to Δ (e.g. if Δ is of simply-laced type). In that case — and we always restrict to that case — the Chevalley group is a simple group.

Chevalley groups can also be described from the perspective of *algebraic groups*. Jacques Tits classified all semi-simple algebraic groups in [**Tit66**] in 1966, and it is known that every connected reductive linear algebraic group over an algebraically closed field is uniquely determined by its root datum (cf. [**Car93**, §1.11] or [**MT11**, Theorem 9.13]).

Since the language of algebraic groups and root data provides a more convenient bridge to the theory of buildings, we will briefly review some of the most basic concepts in that setting. Our treatment will remain rudimentary, omitting substantial parts of the general theory and their proofs. The main references for this section are [**Bou68**], [**Bor69**], [**BT65**], [**Car93**], [**MT11**], [**Spr98**], and [**Ste68**].

L.1. Varieties, Linear Algebraic Groups, $\mathbb{F}$-groups, Radicals

Throughout this subsection, $\mathbb{K}$ will always denote an algebraically closed field.

Let $\mathbb{K}^n$ the vector space of n-tuples over $\mathbb{K}$. Recall that the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ gives rise to a ring of functions $f\colon \mathbb{K}^n \to \mathbb{K}$.

Definition L.1.0.1 (**Algebraic sets**, **Zariski topology**, **affine varieties**)**.** A subset $X \subset \mathbb{K}^n$ of the form

$$X = X(I) = \{(x_1, \ldots, x_n) \in \mathbb{K}^n \mid f(x_1, \ldots, x_n) = 0 \text{ for all } f \in I\}$$

for some ideal $I \subset \mathbb{K}[T_1, \ldots, T_n]$ is called an *algebraic set*. Taking complements of algebraic sets as open sets defines a topology on $\mathbb{K}^n$, the *Zariski topology* (cf. [**MT11**, §1.1]).

An *affine variety* is an algebraic set together with the induced Zariski topology (cf. [**MT11**, §1.1]).

Note L.1.0.2 (**Varieties**)**.** If we just talk about a *variety* (in Chapter 5), we simply mean the solution set of a system of polynomial equations, i.e., the common zeros of a given family of polynomials.

Definition L.1.0.3 (**Coordinate Algebras**)**.** For $X \subset \mathbb{K}^n$ an affine variety, let $I \subset \mathbb{K}[T_1, \ldots, T_n]$ denote the (radical) ideal of polynomials vanishing identically on X. The quotient ring

$$\mathbb{K}[X] = \mathbb{K}[T_1, \ldots, T_n]/I$$

is called the *coordinate algebra* or *algebra of regular functions* on X, since it can be naturally identified with the algebra of all polynomial functions on X with values in $\mathbb{K}$ (cf. [**MT11**, §1.1]).

Note L.1.0.4. If $X \subset \mathbb{K}^n$ and $Y \subset \mathbb{K}^m$ are affine varieties, their cartesian product $X \times Y$ is an algebraic set in $\mathbb{K}^{n+m}$, and possesses the structure of an affine variety. We always consider the product $X \times Y$ equipped with the Zariski topology (not with the product topology). Note that

$$\mathbb{K}[X \times Y] \cong \mathbb{K}[X] \otimes_{\mathbb{K}} \mathbb{K}[Y].$$

(cf. [**MT11**, §1.1])

Definition L.1.0.5 (**Morphisms of affine varieties**)**.** A map $\varphi\colon X \to Y$ between two affine varieties X, Y which can be defined by polynomial functions in the coordinates, is called a *morphism of affine varieties*. Note that morphisms are continuous in the Zariski topology. A morphism $\varphi\colon X \to Y$ induces functorially a $\mathbb{K}$-algebra homomorphism

$$\varphi^* : \mathbb{K}[Y] \longrightarrow \mathbb{K}[X], \qquad \varphi^*(f) := f \circ \varphi, \quad f \in \mathbb{K}[Y].$$

Note L.1.0.6. The coordinate algebra $\mathbb{K}[X]$ has the following properties (cf. [**Spr98**, §§1.3.1]).

(a) $\mathbb{K}[X]$ is a $\mathbb{K}$-algebra of *finite type*, i.e., there exists a finite subset $\{f_1, \ldots, f_r\}$ of $\mathbb{K}[X]$ such that $\mathbb{K}[X] = \mathbb{K}[f_1, \ldots, f_r]$.
(b) $\mathbb{K}[X]$ is *reduced*, i.e., 0 is the only nilpotent element of $\mathbb{K}[X]$.

Definition L.1.0.7 (**Affine algebras**)**.** A $\mathbb{K}$-algebra with the two properties (a) and (b) of Note L.1.0.6 is called an *affine* $\mathbb{K}$*-algebra*. If $\mathbb{A}$ is an affine $\mathbb{K}$-algebra, then there exists an algebraic subset $X \subset \mathbb{K}^r$ such that

$$\mathbb{A} \cong \mathbb{K}[X].$$

We call $\mathbb{K}[X]$ the *affine algebra* of X (cf. [**Spr98**, §§1.3.1]).

Definition L.1.0.8 ($\mathbb{F}$**-rational points**)**.** Let $\mathbb{A} = \mathbb{K}[X]$ be an affine algebra, and let $\mathbb{F}$ be an arbitrary subfield of $\mathbb{K}$. An $\mathbb{F}$*-structure* on X is an $\mathbb{F}$-subalgebra $\mathbb{A}_0 \subset \mathbb{A}$ which is of finite type over $\mathbb{F}$ and such that the homomorphism induced by multiplication

$$\mathbb{K} \otimes_{\mathbb{F}} \mathbb{A}_0 \;\longrightarrow\; \mathbb{K}[X]$$

is an isomorphism of $\mathbb{K}$-algebras. We then write $\mathbb{A}_0 = \mathbb{F}[X]$. The set $X(\mathbb{F})$ of $\mathbb{F}$-*rational points* for our given $\mathbb{F}$-structure is the set of $\mathbb{F}$-homomorphisms $\mathbb{F}[X] \to \mathbb{F}$.

Definition L.1.0.9 (**Affine** $\mathbb{F}$**-varieties**)**.** An affine variety over $\mathbb{K}$ with an $\mathbb{F}$-structure will be called an *affine* $\mathbb{F}$*-variety*.

Note L.1.0.10. An instance of an affine $\mathbb{F}$-variety is the affine n-space A^n ($n \geq 0$), whose coordinate algebra is $\mathbb{K}[T_1, \ldots, T_n]$. If X is an affine $\mathbb{F}$-variety, its set $X(\mathbb{F})$ of $\mathbb{F}$-rational points can be viewed as the set of $\mathbb{F}$-morphisms $A^0 \to X$ (cf. [**Spr98**, §§1.4.9]).

Definition L.1.0.11 (**Linear algebraic groups**)**.** Let $\mathbb{K}$ be an algebraically closed field. A *linear algebraic group* over $\mathbb{K}$ is a group G, that is also an affine variety over $\mathbb{K}$ such that the group operations

$$\mu\colon G \times G \to G, \quad (g,h) \mapsto gh, \quad \text{and} \quad \iota\colon G \to G, \quad g \mapsto g^{-1}$$

are morphisms of varieties defined over $\mathbb{K}$ (cf. [**Car93**, §1.2], [**MT11**, Definition 1.1]).

Definition L.1.0.12 ($\mathbb{F}$**-group**)**.** Let $\mathbb{K}$ be an algebraically closed field, let $\mathbb{F} \subseteq \mathbb{K}$ an arbitrary subfield, and let G be a linear algebraic group over $\mathbb{K}$. We say that G is an $\mathbb{F}$-*group* if G is an $\mathbb{F}$-variety, if the morphisms μ and ι are defined over $\mathbb{F}$, and if the identity element e is an $\mathbb{F}$-rational point. If G is an $\mathbb{F}$-group, the set $G(\mathbb{F})$ of $\mathbb{F}$-rational points has a canonical group structure (cf. [**Spr98**, SS2.1.1]).

Definition L.1.0.13 (**Solvable**, **connected**)**.** A linear algebraic group G over a field $\mathbb{K}$ is called

- *solvable*, if G has a series of normal subgroups
$$G = G^{(0)} \supset G^{(1)} \supset \cdots \supset G^{(d)} = 1,$$
such that $G^{(i+1)} = [G^{(i)} : G^{(i)}]$, for each i ([**Car93**, pp. 14–15]),
- *connected*, if G cannot be written as the union of two disjoint non-empty proper closed subsets ([**MT11**, §1.3]).

Definition L.1.0.14 (**Radicals**, **unipotent radicals**)**.** Let G be a linear algebraic group. The product of all closed connected solvable normal subgroups of G is called the *radical* $R(G)$ of G. The set of closed connected unipotent normal subgroups of G has a unique maximal element, called the *unipotent radical* $R_u(G)$ of G. In other words: The radical is the maximal closed connected solvable normal subgroup and the unipotent radical is the maximal closed connected normal unipotent subgroup. We have $R_u(G) \subseteq R(G)$ (cf. [**Car93**, p. 16] or [**MT11**, Definition 6.13]).

Definition L.1.0.15 (**reductive**, **semi-simple**)**.** A linear algebraic group G is called

- *reductive*, if its unipotent radical $R_u(G)$ is trivial ([**MT11**, Definition 6.14]).
- *semi-simple*, if it is connected and its radical $R(G)$ is trivial ([**MT11**, Definition 6.14]).

Every semi-simple group is reductive but the converse is not true (for a reference, see [**Car93**, p. 16]; for an example, see Example L.1.0.17).

Definition L.1.0.16 (**Morphisms of linear algebraic groups**)**.**
A map $\varphi\colon G \to G'$ of linear algebraic groups is a *morphism of linear algebraic groups*, if it is a group homomorphism and also a morphism of varieties, that is, the induced map $\varphi^*\colon \mathbb{K}[G] \to \mathbb{K}[G']$ is a $\mathbb{K}$-algebra homomorphism (cf. [**MT11**, Definition 1.3]).

Example L.1.0.17. We can identify the general linear group $\mathsf{GL}_n(\mathbb{K})$ with the subset

$$\{(A, y) \in \mathbb{K}^{n\times n} \times \mathbb{K} \mid \det(A)y = 1\}.$$

If we do that, then we can regard $\mathsf{GL}_n(\mathbb{K})$ as a closed subset of $\mathbb{K}^{n\times n} \times \mathbb{K}$ and as an affine variety. Thus, $\mathsf{GL}_n(\mathbb{K})$ is an example of a linear algebraic group (cf. [**Car93**, p. 6] or [**MT11**, Example 1.2]).

Additive linear algebraic groups. We consider the subgroup of $\mathsf{GL}_2(\mathbb{K})$, which consists of all matrices of the form

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad \lambda \in \mathbb{K}.$$

This is a closed subgroup of $\mathsf{GL}_2(\mathbb{K})$ and, as a group, isomorphic to the additive group of $\mathbb{K}$, since

$$\begin{pmatrix} 1 & \lambda_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda_1 + \lambda_2 \\ 0 & 1 \end{pmatrix}.$$

A linear algebraic group isomorphic to this subgroup of $\mathsf{GL}_2(\mathbb{K})$ is called *additive*, and is often denoted by $\mathbb{G}_a$.

Multiplicative linear algebraic groups. The group $\mathsf{GL}_1(\mathbb{K})$, consists of all matrices of the form (λ), for $\lambda \neq 0$. It is, as a group, isomorphic to the multiplicative group $\mathbb{K}^\times$. A linear algebraic group isomorphic to $\mathsf{GL}_1(\mathbb{K})$ is called *multiplicative*, and is often denoted by $\mathbb{G}_m$.

L.2. Tori and Characters

Throughout this subsection, $\mathbb{K}$ will always denote an algebraically closed field.

Definition L.2.0.1 (**Tori**, **maximal tori**)**.** A linear algebraic group is called a *torus*, if it is isomorphic to a direct product

$$\mathbb{G}_m \times \cdots \times \mathbb{G}_m$$

of copies of the multiplicative group $\mathbb{G}_m$. It is then isomorphic (cf. [**MT11**, §3.2]) to some group Diag_n, for some $n \geq 0$, defined as follows ([**MT11**, p. 6]).

$$\mathsf{Diag}_n := \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \text{ invertible} \right\} = \{\mathrm{diag}(a_1, \ldots, a_n) \mid a_i \neq 0 \text{ for } 1 \leq i \leq n\}$$

For $T \cong \mathsf{Diag}_n$, we call n the *dimension of the torus* T.

A *maximal torus* of a linear algebraic group G is a closed subgroup of G, which is a torus, and not properly contained in any larger tori (cf. [**Car93**, p. 15], [**MT11**, p. 28]).

A linear algebraic group H is isomorphic to the semi-direct product of two closed subgroups X and N, if $N \lhd H$, $N \cap X = 1$, and the product map $N \times X \to H$ is an isomorphism of linear algebraic groups (cf. [**Bor69**, §1.11], [**Hum75**, §8.4]).

Notation L.2.0.2. For a linear algebraic group G, we define

$$G_u := \{g \in G \mid g \text{ is unipotent}\}.$$

Fact L.2.0.3. If G is a connected solvable linear algebraic group, then all maximal tori of G are conjugate and, if T is any such maximal torus, then G is the semi-direct product $G_u T$ and the normalizer $N_G(T)$ and the centralizer $C_G(T)$ coincide. In particular, all maximal tori are of the same maximal dimension.

Reference. This is [**MT11**, Theorem 4.4]. □

Definition L.2.0.4 (**Character** and **co-character groups**)**.** Let G be a linear algebraic group. We define the *group $X(G)$ of characters of G* and the *group $X^\vee(G)$ of co-characters of G* as follows (cf. [**MT11**, Definition 3.4]).

$$X(G) := \mathsf{Hom}(G, \mathbb{G}_m), \quad X^\vee(G) := \mathsf{Hom}(\mathbb{G}_m, G)$$

The group $X(G)$ is an abelian group and for two characters χ_1 and $\chi_2 \in X(G)$, we write

$$(\chi_1 + \chi_1)(g) := \chi_1(g)\chi_2(g), \text{ for } g \in G.$$

If G is commutative, then $X^\vee(G)$ is an abelian group as well with respect to

$$(\chi_1 + \chi_1)(x) := \chi_1(x)\chi_2(x), \text{ for } \chi_1, \chi_s \in X^\vee(G),\ x \in \mathbb{G}_m.$$

Note L.2.0.5. Let $T \cong \mathsf{Diag}_n$ be an n-dimensional torus. Then

$$X(T) \cong \mathbb{Z}^n \cong X^\vee(T) \quad \text{(cf. [**MT11**, p. 23])}.$$

For a character $\chi \in X(T)$ and a co-character $\gamma \in X^\vee(T)$, the composite

$$\chi \circ \gamma \colon \mathbb{G}_m \to \mathbb{G}_m, \quad t \mapsto \chi(\gamma(t)), \quad \text{for } t \in T,$$

is an endormorphism of $\mathbb{G}_m$ and we have

$$\chi(\gamma(t)) = t^\lambda, \quad \text{for some } \lambda \in \mathbb{Z}.$$

We put $\langle \chi, \gamma \rangle := \lambda$.

Proposition L.2.0.6. *Let T be a torus with character group $X(T)$ and co-character group $X^\vee(T)$. Then:*

Any homomorphism $X(T) \to \mathbb{Z}$ is of the form $\chi \mapsto \langle \chi, \gamma \rangle$, for some $\gamma \in X^\vee(T)$.
Any homomorphism $X^\vee(T) \to \mathbb{Z}$ is of the form $\gamma \mapsto \langle \chi, \gamma \rangle$, for some $\chi \in X(T)$.

This defines group isomorphisms

$$X^\vee(T) \cong \mathsf{Hom}(X(T), \mathbb{Z}) \quad \textit{and} \quad X(T) \cong \mathsf{Hom}(X^\vee(T), \mathbb{Z}).$$

Reference. This is [**MT11**, Proposition 3.6]. □

Definition L.2.0.7 (**Perfect pairing**)**.** In the situation of Proposition L.2.0.6, we also say that the map

$$\langle\ ,\ \rangle \colon X(T) \times X^\vee(T) \to \mathbb{Z}$$

is a *perfect pairing* between $X(T)$ and $X^\vee(T)$.

Definition L.2.0.8 (**Bases of co-character groups**)**.** Let S be a finite set. For each $s \in S$ let $\chi_s \colon T \to \mathbb{K}^\times$ be a homomorphism. We suppose that $\bigcap_{s\in S} \ker(\chi_s) = 1$. A *base of the system* $(T, (\chi_s)_{s\in S})$ is a tuple $\Omega = (\omega_s)_{s\in S}$ of co-characters of T such that $\chi_t \circ \omega_s$ is the identity on $\mathbb{K}^\times$ if $t = s$ and trivial if $t \neq s$.

Note L.2.0.9. Let $\Omega = (\omega_s)_{s\in S}$ be a base of a system $(T, (\chi_s)_{s\in S})$. Then each co-character $\chi^\vee$ can be written uniquely as

$$\chi^\vee = \sum_{s\in S} \langle \chi^\vee, \chi_s \rangle \, \omega_s,$$

and we have $X^\vee(T) \cong \mathbb{Z}^{|S|}$. For a cocharacter $\chi^\vee \in X^\vee(T)$, the group

$$H_{\chi^\vee} := \chi^\vee(\mathbb{K}^\times) \leq T$$

is called the *one-parameter group* associated with $\chi^\vee$. A subgroup $H \leq T$ is called *admissible* if H is generated by all one-parameter groups contained in it.

The admissible subgroups are in one-to-one correspondence with the subgroups of $X^\vee(T)$.

Definition L.2.0.10 (**$\mathbb{F}$-torus**)**.** Let $\mathbb{F}$ be an arbitrary subfield of $\mathbb{K}$. An $\mathbb{F}$-*torus* is an $\mathbb{F}$-group which is a torus. An $\mathbb{F}$-torus is called $\mathbb{F}$-*split* if it is $\mathbb{F}$-isomorphic to some Diag_n, for some $n \geq 0$ (cf. [**Spr98**, p. 47]).

Note L.2.0.11. An $\mathbb{F}$-torus is $\mathbb{F}$-split if and only if all its characters are defined over $\mathbb{F}$ ([**Spr98**, Proposition 3.2.12(i)]).

L.3. A Root Datum for G

Throughout this subsection, $\mathbb{K}$ will always denote an algebraically closed field.

As mentioned in the introduction of this chapter, it is known that, given any root datum, there exists a unique connected reductive group G over $\mathbb{K}$, which gives rise to this root datum (cf. [**Car93**, §1.9]). In the following, we will introduce Borel subgroups, the Weyl group and root system of G and see, how we can obtain a root datum for G.

Definition L.3.0.1 (**Borel subgroups**)**.** If G is a connected linear algebraic group over an algebraically closed field $\mathbb{K}$, then a *Borel subgroup* of G is defined as a maximal closed connected solvable subgroup of G ([**Car93**, §1.7]).

Facts L.3.0.2. Let G be a connected linear algebraic group over an algebraically closed field $\mathbb{K}$. Any maximal torus of G lies in some Borel subgroup of G. Any two Borel subgroups of G are conjugate. Inside a Borel subgroup, any two maximal tori are conjugate, and in G any two maximal tori of G are conjugate. The union of all Borel subgroups of G is G. Each Borel subgroup is its own normalizer. If G is reductive, then the intersection of any two Borel subgroups of G contains a maximal torus.

Reference. See [**Car93**, §1.7] and [**MT11**, Corollary 11.19]. □

Theorem L.3.0.3. *Let G be a connected reductive linear algebraic group over an algebraically closed field $\mathbb{K}$ with Borel subgroup B and $N := N_G(T)$ for some maximal torus $T \subseteq B$. Then the groups B and N form a BN-pair for G.*

REFERENCE. See [**BT65**], [**Tit64**] or [**MT11**, Theorem 11.16]. □

The Weyl group. The following is described in [**Car93**, §1.9] and [**MT11**, §8.1]. Let G be a connected group over an algebraically closed field $\mathbb{K}$ and T a maximal torus of G. Let $N(T)$ and $C(T)$ be the normalizer and the centralizer of T in G, respectively. Then $W(T) := {}^{N(T)}/_{C(T)}$ is a finite group, called the *Weyl group of* T. It is uniquely determined up to isomorphism, since any two maximal tori in G are conjugate. The *abstract* group W, isomorphic to each $W(T)$, is called the *Weyl group of* G. If G is reductive, we have $C(T) = T$. In the following, we will assume G to be reductive and we will just write X for the character group $X(T)$ and $X^\vee$ for the co-character group $X^\vee(T)$. For $w \in W$ and $\chi \in X$, we define $w.\chi \in X$ by

$$(w.\chi)(t) = \chi(t^w), \quad \text{for } t \in T.$$

For $w \in W$, $\gamma \in X^\vee$, we define $w.\gamma \in X^\vee$ by

$$w.\gamma(c) :=^w (\gamma(c)) := \gamma(c)^{w^{-1}}, \quad \text{for } c \in \mathbb{G}_m.$$

With that, W acts on both X and $X^\vee$. These actions are faithful and compatible with the pairing $X \times X^\vee \to \mathbb{Z}$.

For all $w \in W$, $\chi \in X$ and $\gamma \in X^\vee$ we have with [**MT11**, Lemma 8.3]:

$$\langle w.\chi, \gamma \rangle = \langle \chi, w^{-1}.\gamma \rangle.$$

The roots and root subgroups. The following is described in [**Car93**, pp. 18–19]. Let G be a connected reductive group, T a maximal torus of G and let B be a Borel subgroup of G containing T. Then B can be written as a semi-direct product $B = UT$, where U denotes the unipotent radical $R_u(B)$. The group G has a unique Borel subgroup B^-, containing T such that $B \cap B^- = T$. The subgroups B and B^- are called *opposite Borel subgroups*. Putting $U^- = R_u(B^-)$, we have $B^- = U^-T$ and $U \cap U^- = \{1\}$. Both U and U^- are maximal unipotent subgroups of G. We consider the minimal proper subgroups of U and U^- that are normalized by T. These are all connected unipotent groups of dimension 1 and isomorphic to the additive group $\mathbb{G}_a$. The torus T acts on each of them by conjugation. We get a homomorphism $T \to \mathsf{Aut}(\mathbb{G}_a)$. The automorphisms of $\mathsf{Aut}(\mathbb{G}_a)$ are of the form $\lambda \mapsto \mu\lambda$, for some $\mu \in \mathbb{K}$, $\mu \neq 0$. Thus, $\mathsf{Aut}(\mathbb{G}_a)$ is isomorphic to $\mathbb{G}_m$. That means, each of the 1-dimensional unipotent groups determines an element of $\mathsf{Hom}(T, \mathbb{G}_m) = X$. The elements of X arising in this way are called the *roots* and they are all non-zero elements of X and distinct 1-dimensional unipotent subgroups give rise to distinct roots. The roots form a finite subset ϕ of X, which is independent of the choice of the Borel subgroup B containing T. For each root $\alpha \in \phi$, the 1-dimensional unipotent subgroup giving rise to it is denoted by X_α, and the X_α are called the root subgroups of G. The roots arising from root subgroups in U^- are the negatives of the roots arising from root subgroups in U.

We have $\boldsymbol{G = \langle T, (X_\alpha)_{\alpha \in \phi} \rangle}$ (cf. [**Bor69**, p. 189], [**Car93**, p. 19]).

Let α and $-\alpha$ be two opposite roots. The subgroup $\langle X_\alpha, X_{-\alpha}\rangle$ is a simple group and isomorphic to either $\mathsf{SL}_2(\mathbb{K})$ or $\mathsf{PGL}_2(\mathbb{K})$ (cf. [**Spr98**, Theorem 7.2.4]) and we have a homomorphism $\varphi\colon \mathsf{SL}_2(\mathbb{K}) \to \langle X_\alpha, X_{-\alpha}\rangle$ such that

$$\varphi\left\{\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}\right\} = X_\alpha, \quad \varphi\left\{\begin{pmatrix}1 & 0\\ * & 1\end{pmatrix}\right\} = X_{-\alpha}, \quad \text{and} \quad \varphi\left\{\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\right\}$$

is a 1-dimensional subgroup of T. The co-root $\alpha^\vee$ of α is given by the homomorphism $\mathbb{G}_m \to T$ such that

$$\alpha^\vee(\lambda) = \varphi\left\{\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\right\}.$$

Each element of W permutes the set of roots ϕ in X and the set of co-roots $\phi^\vee$ in $X^\vee$. Now, for each root α, we consider the element

$$\varphi\left\{\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right\} \in \langle X_\alpha, X_{-\alpha}\rangle.$$

This element lies in $N(T)$ and gives rise to an element $w_\alpha \in W = {}^{N(T)}\!/_{C(T)}$. That element w_α acts on X by

$$w_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee\rangle \alpha, \quad \chi \in X,$$

and on $X^\vee$ by

$$w_\alpha(\gamma) = \gamma - \langle \alpha, \gamma\rangle \alpha^\vee, \quad \gamma \in X^\vee.$$

We have $w_\alpha = w_{-\alpha}$ and $w_\alpha^2 = 1$, and the elements w_α, for all $\alpha \in \phi$, generate W.

The quadruple $\boldsymbol{(X, \phi, X^\vee, \phi^\vee)}$ is the root datum for G (cf. [**Car93**, p. 19], [**MT11**, Proposition 9.11]).

The Lie algebra. Without wanting to go into any details, we mention that one can show that a Lie algebra $\mathfrak{L}(G)$ associated to G is isomorphic to the *tangent space* of G at $1 \in G$. For an affine variety X the tangent space of X at $x \in X$ is defined by

$$T_x(X) := \{\delta\colon \mathbb{K}[X] \to \mathbb{K} \text{ linear } \mid \delta(fg) = f(x)\delta(g) + \delta(f)g(x), \text{ for } f, g \in \mathbb{K}[X]\}.$$

The tangent space at any group element $g \in G$ is isomorphic to the tangent space at $1 \in G$ of G, and it is the $\mathbb{K}$-vector space of point derivations at x [**MT11**, p. 45]).

If, for $\chi \in X$, we put

$$\mathfrak{L}_\chi = \{v \in \mathfrak{L}(G) \mid (\mathsf{ad}\, t)(v) = \chi(t)v, \text{ for all } t \in T\},$$

where ad denotes the adjoint representation $\mathsf{ad} : G \to \mathsf{GL}(\mathfrak{L}(G))$, as defined in [**MT11**, §7.2], then we can write

$$\mathfrak{L}(G) = \bigoplus_{\chi \in X} \mathfrak{L}_\chi\,.$$

The set of roots of G with respect to T can then be described as the set of non-zero characters with non-zero eigenspace occurring in the above decomposition (cf. [**MT11**, Definition 8.1]):

$$\Phi(G) := \{\chi \in X(T) \mid \chi \neq 0,\ \mathfrak{L}_\chi \neq \{0\}\}.$$

We refer the interested reader to [**MT11**, Ch. 7 & 8].

Theorem L.3.0.4. *Let G be a connected reductive linear algebraic group, T a maximal torus of G, $\mathfrak{L}(G)$ the Lie algebra of G, and let Φ denote the root system of G with respect to T. Then:*

(i) $\mathfrak{L}(G) = \mathfrak{L}(T) \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{L}_\alpha$, *with* $\dim(\mathfrak{L}_\alpha) = 1$, *for all* $\alpha \in \Phi$, *and* $\mathfrak{L}(T) = \mathfrak{L}_0$.

(ii) For each $\alpha \in \Phi$ there exists a morphism of linear algebraic groups

$$u_\alpha \colon \mathbb{G}_a \to G,$$

which induces an isomorphism onto its image $u_\alpha(\mathbb{G}_a)$ such that

$$tu_\alpha(c)t^{-1} = u_\alpha(\alpha(t)c), \quad \text{for all } t \in T,\ c \in \mathbb{K}.$$

If $\tilde{u}_\alpha$ is another morphism with the same properties, then there exists a unique $a \in \mathbb{K}^\times$ such that

$$\tilde{u}_\alpha(c) = u_\alpha(ac) \quad \text{for all } c \in \mathbb{K}.$$

(iii) $U_\alpha := \mathsf{Im}(u_\alpha)$ is the unique one-dimensional connected unipotent subgroup of G normalized by T such that $\mathfrak{L}(U_\alpha) = \mathfrak{L}_\alpha$.

(iv) For $w \in W = {}^{N_G(T)}\!/_T$ with a representative $n \in N_G(T)$, we have

$$nU_\alpha n^{-1} = U_{w\alpha}.$$

Reference. This is [**MT11**, Theorem 8.17]. □

L.4. Isogeny Classes

Throughout this subsection, $\mathbb{K}$ will always denote an algebraically closed field.

The Dynkin diagram. The following is described in [**Car93**, p. 23]. We consider a connected semi-simple linear algebraic group G over an algebraically closed field with a system of simple roots Φ.[1] We can associate a Dynkin diagram to G (cf. 1.5.0.24). The Dynking diagram is uniquely determined by G, independent of the choice of the maximal torus T and Borel subgroup B containing T. Also, ϕ and $\phi^\vee$ are determined by the Dynkin diagram. However, the way in which ϕ and $\phi^\vee$ occur as subsets of X and $X^\vee$, respectively, is not determined by the Dynkin diagram.

Definition L.4.0.1 (**Isogenies**)**.** We say that two simple groups are *isogenous* if their Dynkin diagrams are the same. In that case, we also say that they are in the same *isogeny class* or share a common *isogeny type* (cf. [**Car93**, p. 25]).

Let T be a maximal torus of a semi-simple linear algebraic group G over an algebraically closed field $\mathbb{K}$, and let ϕ be the root system of G with respect to T. We can identify $\mathbb{G}_m$ with its group $\mathbb{K}^\times$ of $\mathbb{K}$-points (cf. [**AB08**, p. 443]). Then — as mentioned before — each root $\alpha \in \phi$ gives rise to a one-dimensional unipotent subgroup X_α, and each such subgroup corresponds to an element in

$$X = \mathsf{Hom}(T, \mathbb{G}_m) \cong \mathsf{Hom}(T, \mathbb{K}^\times).$$

[1] If G is semi-simple, both $|X : \mathbb{Z}\phi|$ and $|X^\vee : \mathbb{Z}\phi^\vee|$ are finite (cf. [**Car93**, p. 23]).

Put

$$\Omega := \mathsf{Hom}(\mathbb{Z}\phi^\vee, \mathbb{Z}).$$

Restriction gives a natural injective homomorphism

$$X = \mathsf{Hom}(X^\vee, \mathbb{Z}) \to \mathsf{Hom}(\mathbb{Z}\phi^\vee, \mathbb{Z}) = \Omega.$$

The group $\Omega/\mathbb{Z}\phi$ is finite, does not depend on X, anad is called the *fundamental group of* ϕ (cf. [**MT11**, p. 70]).

If we fix a Dynkin diagram, then every simple group G with that Dynkin diagram determines a subgroup X such that $\Omega \supseteq X \supseteq \mathbb{Z}\phi$. Heuristically, the location of X between Ω and $\mathbb{Z}\phi$ determines G up to isomorphism. It is possible for distinct subgroups between Ω and $\mathbb{Z}\phi$ to arise from isomorphic simple groups (cf. [**Car93**, p. 25]). In every isogeny class, there are only finitely many simple groups. Among the groups in an isogeny class, we have that, if $X = \mathbb{Z}\phi$, then G is called *adjoint*, and if $X = \Omega$, then G is called *simply-connected* (cf.[**Bor69**, §24, pp. 268–269], [**MT11**, pp. 70–71]).

L.5. Finite Groups of Lie Type

Definition L.5.0.1 (**Frobenius maps**, **Steinberg endomorphisms**)**.** Let G be a linear algebraic group over an algebraically closed field of characteristic p. Then G is isomorphic to a closed subgroup of $\mathsf{GL}_n(\mathbb{K})$, for some n. Let $q = p^e$, $e \geq 1$, and define

$$F_q \colon \mathsf{GL}_n(\mathbb{K}) \to \mathsf{GL}_n(\mathbb{K}), \quad (a_{ij}) \mapsto (a_{ij}^q).$$

Then F_q is a homomorphism. We call a homomorphism $F \colon G \to G$ a *standard Frobenius map*, if there exists an injective homomorphism

$$i \colon G \to \mathsf{GL}_n(\mathbb{K}), \quad i(F(g)) = F_q(i(g)), \quad \text{for some } q = p^e, \text{ and all } g \in G.$$

A homomorphism $F \colon G \to G$ is called a *Frobenius map* (cf. [**Car93**, §1.17]), or *Steinberg endomorphism* (cf. [**MT11**, Definition 21.3]), if some power of F is a standard Frobenius map.

Facts L.5.0.2. Any Steinberg endomorphism F is

- bijective,
- a surjective homomorphism of linear algebraic groups,
- an isomorphism of abstract groups.

Reference. All can be found in [**Car93**, §1.17] or [**MT11**, §21.1]. □

Definition L.5.0.3 (**Finite Groups of Lie type**)**.** Let G be a semi-simple linear algebraic group over an algebraically closed field. If $F \colon G \to G$ is a Steinberg endomorphism, we define G^F by

$$G^F = \{g \in G \mid F(g) = g\}.$$

Then G^F is finite and called a *finite group of Lie type* (cf. [**MT11**, Definition 21.6]).

Definition L.5.0.4 (**Split BN-pairs**)**.** A linear algebraic group G over an algebraically closed field has a *split BN-pair*, if G has closed subgroups B and N satisfying the following axioms:

(1) B, N form a BN-pair in G.
(2) $B = U(B \cap N)$ is the semi-direct product of a closed normal unipotent group U and a closed commutative subgroup $B \cap N$, all of whose elements are semi-simple.
(3) $\bigcap_{n \in N} nBn^{-1} = B \cap N$.

Definition & Facts L.5.0.5. Let G be a semi-simple linear algebraic group over an algebraically closed field. We say a subgroup $H \subseteq G$ is *F-stable*, if $F(H) = H$.

(a) G has an F-stable Borel subgroup and an F-stable maximal torus.
(b) Every F-stable Borel subgroup of G contains an F-stable maximal torus of G.
(c) An F-stable maximal torus of G is called *maximally split*, if it lies in an F-stable Borel subgroup of G, and a maximal torus of G^F is called *maximally split*, if it has the form T^F, for some maximally split torus $T \subseteq G$ (cf. [**Car93**, p. 33]).
(d) Any two maximally split tori of G^F are conjugate in G^F (cf. [**Car93**, pp. 33–34]). However, it is not generally true that any two maximal tori of G^F are conjugate in G^F.
(e) The unipotent radical U is F-stable.
(f) Let B be an F-stable Borel subgroup of G and T an F-stable maximal torus of G contained in B. Let $N := N_G(T)$. Then N is F-stable, and since $B \cap N = T$, we have $B^F \cap N^F = T^F$ (cf. [**Car93**, p. 34], [**MT11**, Theorem 24.1, Theorem 24.10]).

We adopt the notation of (f). Since F acts on N and T, it also acts on $W = {}^N\!/_T$ by $F(nT) = F(n)T$. Let W^F be the subgroup of F-stable elements of W. We have ${}^{N^F}\!/_{T^F} \cong W^F$. The root subgroups X_α, $\alpha \in \phi^+$, get permuted by F. Let ρ be a permutation of the positive roots such that $F(X_\alpha) = X_{\rho(\alpha)}$. We have $\rho(\Phi) = \Phi$. If Φ is indexed by a set I, then we consider an orbit J of ρ on I and define $W_J = \langle s_i \mid i \in I \rangle$. Let ϕ_J be the subset of ϕ such that $\phi_J = W_J(\Phi_J)$, let $\phi_J^+ = \phi_J \cap \phi^+$ and $\phi_J^- = \phi_J \cap \phi^-$. Then there is a unique element s_J of W_J, which satisfies $s_J(\phi_J^+) = \phi_J^-$. The group W^F is the subgroup of W generated by the elements s_J for all ρ-orbits J on I, and W^F is generated by the s_J as a Coxeter group.

One can show that B^F and N^F form a split BN-pair for G^F.

Reference. All can be found in [**Car93**, §§1.17–1.18] or [**MT11**, §§22.1–25.1]. □

Note L.5.0.6. If G is a connected reductive group over an algebraically closed field, then, with notation as in Appendix L.3, G has a split BN-pair and $B \cap N = T$, $B = UT$, where $U = R_u(B)$. Furthermore, one can find an element $n \in N$ such that $nBn^{-1} = B^-$ and $B \cap B^- = T$ (cf. [**Car93**, p. 23]).

Note L.5.0.7. We can also define an action of F on the character and co-character groups of the maximally split torus T as follows (cf. [**MT11**, §22.1]).

$$F\colon X \to X, \quad F(\chi)(t) := \chi(F(t)), \text{ for } \chi \in X, t \in T,$$
$$F\colon X^\vee \to X^\vee, \quad F(\gamma)(c) := F(\gamma(c)), \text{ for } \gamma \in X^\vee, c \in \mathbb{K}^\times,$$

One can show that (by linearity) the action of F on X extends to an action on the vector space $X_\mathbb{R} = X \otimes_\mathbb{Z} \mathbb{R}$, and the determinant of F on $X_\mathbb{R}$ is a positive integral power of p. (cf. [**Car93**, p. 35]).

It is also possible to define a root system for the group G^F. For details we refer to [**Car93**, §1.18]. More importantly, the permutation ρ of the simple roots, determined by the Steinberg endomorphism, gives rise to a symmetry of the Dynkin diagram of G, if arrows are disregarded, and the type of G, together with ρ, determine the type of G^F (cf. [**Car93**, §1.19]). Moreover, one can show that the *complete root datum* $(X, \Phi, X^\vee, \Phi^\vee, W\phi)$ determines G^F up to isomorphism. For details in this context, we refer to [**MT11**, p. 195].

Now, let G still be a connected reductive linear algebraic group over an algebraically closed field $\mathbb{K}$, F a Steinberg endomorphism, and G^F the corresponding finite group of Lie type. Then there exists a positive integer n such that F^n is a standard Frobenius map, and F^n is an integral multiple of the identity on X. Let δ be the smallest such n. We have

$$F^\delta|_X = k\mathsf{id},$$

where $k > 1$ is some power of $p = \mathrm{char}(\mathbb{K})$ (cf. [**MT11**, Lemma 22.1], [**Car93**, p. 35]). We define the positive real number q by $k = q^\delta$, and we define $F_0\colon X_\mathbb{R} \to X_\mathbb{R}$ by $F = qF_0$. Then $F_0^\delta = \mathsf{id}$. Thus we have $F = qF_0$ where $q > 1$, F_0 has finite order, and q is uniquely determined (cf. [**Car93**, pp. 35, 38]).

Fact L.5.0.8. A finite group G^F of Lie type, where G is a simple linear algebraic group over an algebraically closed field, is determined up to isomorphism by the following three invariants:

- the Dynkin diagram with ρ-action,
- the isogeny type,
- and the number q.

Reference. This is explained in [**Car93**, §1.19]. Compare also with Proposition 22.2, Theorem 22.5, and §22.2 (in particular page 193) of [**MT11**]. □

Definition & Facts L.5.0.9 (**Split forms**, **twisted groups**, **Suzuki & Ree groups**)**.** The groups G^F, for which ρ acts trivially on the Dynkin diagram, are called *split forms* of G. The groups G^F, for which the Dynkin diagram has only single bonds, and for which ρ acts non-trivially, are called *twisted groups*, or *quasi-split forms* of G. The remaining groups G^F are the Suzuki and Ree groups (cf. [**Car93**, §1.19]).

L.6. Over Arbitrary Fields

We can also obtain a root datum for a connected $\mathbb{F}$-reductive group. Again, we will not go into details and refer to [**Spr98**, §15.3] for an in-depth description.

Let G be a connected $\mathbb{F}$-reductive group. We fix a maximal $\mathbb{F}$-split $\mathbb{F}$-subtorus S of G, and denote by ${}_{\mathbb{F}}X$ and ${}_{\mathbb{F}}X^{\vee}$ the group of characters and cocharacters of S respectively. We assume that S is non-trivial.

One can obtain a Lie algebra $\mathfrak{L}(G)$ for G similar to before and we refer to [**Spr98**, 4.1.8, 4.4.8 & p. 209] for details.

If H is a subgroup of G normalized by S, we denote by $R(H,S)$ the set of non-zero weights of S in $L(H)$. The characters in $R(G,S)$ are the *roots of G relative to S*, or the *$\mathbb{F}$-roots of G*. They form a subset ${}_{\mathbb{F}}R(G,S)$ (or ${}_{\mathbb{F}}R$ for short) of ${}_{\mathbb{F}}X$. The *Weyl group of G relative to $\mathbb{F}$* is the finite group ${}_{\mathbb{F}}W = {}^{N_G(S)}/_{Z_G(S)}$, and we view it as a group of automorphisms of ${}_{\mathbb{F}}X$. One can show that, if ${}_{\mathbb{F}}R$ is non-empty, ${}_{\mathbb{F}}R$ is a root system with Weyl group ${}_{\mathbb{F}}W$. We denote the co-root system by ${}_{\mathbb{F}}R^{\vee}$.

Then $\boldsymbol{({}_{\mathbb{F}}X, {}_{\mathbb{F}}R, {}_{\mathbb{F}}X^{\vee}, {}_{\mathbb{F}}R^{\vee})}$ is a root datum (cf. [**Spr98**, Theorem 15.3.8]).

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Bibliography

[AB08] Peter Abramenko and Kenneth S. Brown. *Buildings*. Springer New York, NY, 2008. doi:10.1007/978-0-387-78835-7.

[APS12] J. Armstrong, M. Povero, and S. Salamon. Twistor lines on cubic surfaces, 2012. URL: https://arxiv.org/abs/1212.2851.

[AS76] Michael Aschbacher and Gary M. Seitz. Involutions in Chevalley groups over fields of even order. *Nagoya Mathematical Journal*, 63:1–91, 1976. doi:10.1017/S0027763000017438.

[BB66] Raj Chandra Bose and Robert Corry Burton. A characterization of flat spaces in a finite geometry and the uniqueness of the hamming and the MacDonald codes. *J. Combin. Theory*, 1:96–104, 1966.

[BC13] Francis Buekenhout and Arjeh M. Cohen. *Diagram Geometry – Related to Classical Groups and Buildings*, volume Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 57. Springer Berlin, Heidelberg, 2013. doi:10.1007/978-3-642-34453-4.

[BCN89] Andries E. Brouwer, Arjeh M. Cohen, and Arnold Neumaier. *Distance-regular Graphs*. Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, Band 18. Springer Berlin, 1989.

[Bor69] Armand Borel. *Linear Algebraic Groups*. Graduate Texts in Mathematics. Springer New York, 1969.

[Bou68] Nicolas Bourbaki. *Groupes et algèbres de Lie*. Hermann, Paris, 1968.

[BP98] Francis Buekenhout and M. Parker. The number of nets of the regular convex polytopes in dimension ≤ 4. *Discrete Mathematics*, 186(1–3):69–94, 1998.

[BR04] Albrecht Beutelspacher and Ute Rosenbaum. *Projektive Geometrie – Von den Grundlagen bis zu den Anwendungen*. Friedr. Vieweg und Sohn Verlag, Wiesbaden, 2 edition, 2004.

[Bri85] Michel Brion. Représentations exceptionnelles des groupes semi-simples. *Ann. Sci. École Norm. Sup. (4)*, 18(2):345–387, 1985.

[Bro89] Kenneth S. Brown. *Buildings*. Springer New York, NY, 1989. doi:10.1007/978-1-4612-1019-1.

[Bro91] Andries E. Brouwer. A non-degenerate generalized quadrangle with lines of size four is finite. In *Adv. Finite Geom. and Designs, Proceedings Third Isle of Thorn Conference on Finite Geometries and Designs, Brighton 1990 (ed. J. W. P. Hirschfeld et al.)*, pages 47–49. Oxford University Press, 1991.

[BS74] Francis Buekenhout and Ernest E. Shult. On the foundations of polar geometry. *Geom. Dedicata*, 3:155–170, 1974.

[BSV25] Sira Busch, Jeroen Schillewaert, and Hendrik Van Maldeghem. Groups of projectivities and Levi subgroups in spherical buildings of simply laced type. *Journal of Group Theory*, 2025.

[BT65] Armand Borel and Jacques Tits. *Groupes réductifs*. Number 27. 1965.

[BTV02] Leen Brouns, Joseph A. Thas, and Hendrik Van Maldeghem. m-clouds in generalized hexagons. *Discrete Math.*, 255:25–33, 2002.

[Bue95] Francis Buekenhout. *Handbook of Incidence Geometry*. Elsevier B.V., 1995. doi:10.1016/B978-0-444-88355-1.X5000-2.

[Bus25a] Sira Busch. On the Moufang condition for polar spaces. *submitted*, 2025.

[Bus25b] Sira Busch. A point-line approach to the nonexistence of buildings of type H_3 and H_4. *submitted*, 2025.

[BV22] Andries E. Brouwer and Hendrik Van Maldeghem. *Strongly regular graphs*. Cambridge University Press, 2022.

[BV25a] Sira Busch and Hendrik Van Maldeghem. Groups of projectivities in spherical buildings of non-simply laced type. *manuscript*, 2025.

[BV25b] Sira Busch and Hendrik Van Maldeghem. Lines and opposition in finite Lie incidence geometries of exceptional type. *manuscript*, 2025.

[BV26] Sira Busch and Hendrik Van Maldeghem. A characterisation of lines in finite Lie incidence geometries of classical type. *Discrete Mathematics*, 349(2), 2026. doi:10.1016/j.disc.2025.114711.

[Car72] Roger W. Carter. *Simple Groups of Lie Type*. John Wiley & Sons, 1972.

[Car93] Roger W. Carter. *Finite Groups of Lie Type*. John Wiley & Sons, 1993.

[CC98] Arjeh M. Cohen and Bruce N. Cooperstein. Lie incidence systems from projective varieties. *Proc. Amer. Math. Soc.*, 126:2095–2102, 1998.

[Che54] Claude Chevalley. *The Algebraic Theory of Spinors*. Columbia University Press, New York Chichester, West Sussex, 1954.

[Che55] Claude Chevalley. Sur certains groupes simples. *Tohoku Math. J.*, 7(1–2):14–66, 1955.

[CI07] Arjeh M. Cohen and Gábor Ivanyos. Root shadow spaces. *European Journal of Combinatorics*, 28(5):1419–1441, 2007.

[CM17] Matt Clay and Dan Margalit. Groups. In *Office Hours with a Geometric Group Theorist*, pages 3–17. Princeton University Press, 2017.

[Coh86] Arjeh M. Cohen. *Point-line characterizations of buildings*, volume 1181, pages 191–206. Proc. Como (L.A Rosati, ed.), Springer Berlin, 1984, 1986.

[Cox34] Harold Scott MacDonald Coxeter. Discrete groups generated by reflections. *Annals of Mathematics*, 35(3):588–621, 1934.

[Cox35] Harold Scott MacDonald Coxeter. The complete enumeration of finite groups of the form $r_i^2 = (r_i r_j)^{k_{ij}} = 1$. *Journal of the London Mathematical Society*, s1-10(1):21–25, 1935.

[CS90] Arjeh M. Cohen and Ernest E. Shult. Affine polar spaces. *Geometriae Dedicata*, 35:43–76, 1990.

[Cuy06] Hans Cuypers. The geometry of k-transvection groups. *Journal of Algebra*, 300(2):455–471, 2006. doi:10.1016/j.jalgebra.2005.12.012.

[Die43a] Jean A. Dieudonné. Les déterminants sur un corps non commutatif. *Bull. Soc. Math. France*, 71:27–45, 1943.

[Die43b] Jean A. Dieudonné. *Sur les groupes classiques*, volume 1040. Hermann, Paris, 1943.

[Die63] Jean A. Dieudonné. *La Géometrie des Groupes Classiques*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 5, 2nd ed. Springer Berlin, Göttingen, Heidelberg, 1963.

[DSV18] Anneleen De Schepper, N. S. Narasimha Sastry, and Hendrik Van Maldeghem. Split buildings of type F_4 in buildings of type E_6. *Abh. Math. Sem. Univ. Hamburg*, 88:97–160, 2018.

[DSV22] Anneleen De Schepper, N. S. Narasimha Sastry, and Hendrik Van Maldeghem. Buildings of exceptional type in buildings of type E_7. Dissertationes Math., 2022.

[DSV24] Anneleen De Schepper, Jeroen Schillewaert, and Hendrik Van Maldeghem. On the generating rank and embedding rank of the hexagonic Lie incidence geometries. *Combinatorica*, 44:355–392, 2024.

[DSVV24] Anneleen De Schepper, Jeroen Schillewaert, Hendrik Van Maldeghem, and Magali Victoor. Construction and characterisation of the varieties of the third row of the Freudenthal–Tits magic square. *Geom. Ded.*, 218, 2024.

[DTV25] Bart De Bruyn, Koen Thas, and Hendrik Van Maldeghem. *Projective Geometry over Arbitrary Fields*. (in progress), 2025.

[DV14] Bart De Bruyn and Hendrik Van Maldeghem. Non-embeddable polar spaces. *Münster J. Math.*, 7:557–588, 2014.

[DV16] Bart De Bruyn and Hendrik Van Maldeghem. Dual polar spaces of rank 3 defined over quadratic alternative division algebras. *J. Reine Angew. Math.*, 715:39–74, 2016.

[DV18] Anneleen De Schepper and Hendrik Van Maldeghem. Maps related to polar spaces preserving a Weyl distance or an incidence condition. *J. Combin. Theory Ser. A*, 160:332–408, 2018.

[DV23] Anneleen De Schepper and Hendrik Van Maldeghem. On inclusions of exceptional long root geometries of type E. *Innovations in Incidence Geometry*, 20(2–3), 2023.

[FH64] Walter Feit and Graham Higman. The nonexistence of certain generalized polygons. *Journal of Algebra*, 1:114–131, 1964.

[FS73] Paul Fong and Gary M. Seitz. Groups with a (B, N)-pair of rank 2. I. *Invent. Math.*, 21:1–57, 1973.

[FS74] Paul Fong and Gary M. Seitz. Groups with a (B, N)-pair of rank 2. II. *Invent. Math.*, 24:191–239, 1974.

[Gov97] Eline Govaert. A combinatorial characterization of some finite classical generalized hexagons. *J. Combin. Theory Ser.*, 80:339–346, 1997.

[Gru83] Theo Grundhöfer. Über Projectivitätengruppen affiner und projektiver Ebenen unter besonderer Berücksichtigung von Moufangebenen. *Geom. Dedicata*, 13:435–458, 1983.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, 2002.

[Hau92] Barbara Hausmann. Tits–Gebäude. Diplomarbeit, Februar 1992.

[Hig83] Graham Higman. Admissible graphs. In *Finite geometries (Pullman, Wash., 1981)*, volume 82 of *Lecture Notes in Pure and Appl. Math.*, pages 211–222. Dekker, New York, 1983.

[Hol01] Leanne D. Holder. On blocking sets of conics. PhD thesis, 2001.

[Hum72] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Springer New York, 1972. doi:10.1007/978-1-4612-6398-2_3.

[Hum75] James E. Humphreys. *Linear algebraic groups*. Graduate Texts in Mathematics, No. 21. Springer-Verlag, New York-Heidelberg, 1975.

[Hum90] James E. Humphreys. *Reflection Groups and Coxeter Groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990. doi:10.1017/CBO9780511623646.

[HV04] Fabienne Haot and Hendrik Van Maldeghem. Some characterizations of Moufang generalized quadrangles. *Glasgow Math. J.*, 46:335–343, 2004.

[Jan19] Paulien Jansen. Tits polygons and long root geometries. Master's Thesis, May 2019.

[Joh90] Peter M. Johnson. Polar spaces of arbitrary rank. *Geom. Dedicata*, 35:229–250, 1990.

[JSV25] Paulien Jansen, Jeroen Schillewaert, and Hendrik Van Maldeghem. Embedded metasymplectic spaces of type $F_{4,4}$. *manuscript*, 2025.

[JV23] Paulien Jansen and Hendrik Van Maldeghem. Parapolar spaces of infinite rank. *Adv. Stud. Euro-Tbil. Math. J.*, 16:191–203, 2023.

[Kle90] Felix Klein. *Vorlesungen über die Theorie der elliptischen Modulfunctionen, ausgearbeitet und vervollständigt von Dr. Robert Fricke*, volume 1. B.G. Teubner, Leipzig, 1890.

[Kna88] Norbert Knarr. Projectivities of generalized polygons. *Ars Combinatoria*, 25B:265–275, 1988.

[KS02] Anna Kasikova and Ernest E. Shult. Point-line characterizations of Lie geometries. *Advances in Geometry*, 2:147–188, 2002.

[KV13] Anna Kasikova and Hendrik Van Maldeghem. Vertex opposition in spherical buildings. *Des. Codes Cryptogr.*, 68:285–318, 2013.

[LV25] Linde Lambrecht and Hendrik Van Maldeghem. Automorphisms and opposition in spherical buildings of exceptional type III. metasymplectic spaces. *Selecta Math.*, 2025.

[MPW15] Bernhard Mühlherr, Holger P. Petersson, and Richard M. Weiss. *Descent in Buildings*. Princeton University Press, 2015.

[MR95] Bernhard Mühlherr and Mark A. Ronan. Local to global structure in twin buildings. *Invent. Math.*, 122:71–81, 1995.

[MT11] Gunter Malle and Donna Testerman. *Linear Algebraic Groups and Finite Groups of Lie Type*. Number 133. Cambridge Studies in Advanced Mathematics, 2011.

[Müh90] Bernhard Mühlherr. A geometric approach to non-embeddable polar spaces of rank 3. *Bull. Soc. Math. Belg. Sér. A*, 42(3):577–594, 1990.

[MV99] Bernhard Mühlherr and Hendrik Van Maldeghem. Exceptional Moufang quadrangles of type F_4. *Canad. J. Math.*, 51(2):347–371, 1999.

[MV26] Bernhard Mühlherr and Hendrik Van Maldeghem. Hexagonic spaces. *in progress*, 2026.

[NPVV25] Yannick Neyt, James Parkinson, Hendrik Van Maldeghem, and Magali Victoor. Automorphisms and opposition in spherical buildings of exceptional type, IV: The E_7 case. *manuscript*, 2025.

[NV25] Yannick Neyt and Hendrik Van Maldeghem. Polar kangaroos of type E_6. *Quaestiones Mathematicae*, pages 1–27, 2025. doi:10.2989/16073606.2025.2581625.

[Pas09] Antonio Pasini. On polar spaces of infinite rank. *J. Geom.*, 91:84–118, 2009.

[Pic59] Günter Pickert. Bemerkungen über die projektive Gruppe einer Moufang-Ebene. *Illinois J. Math.*, 3:169–173, 1959.

[PT84] Stanley E. Payne and Joseph A. Thas. *Finite Generalized Quadrangles*, volume 110 of *Research Notes in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[PV24] Antonio Pasini and Hendrik Van Maldeghem. An essay on Freudenthal-Tits polar spaces. *J. Algebra*, 656:367–393, 2024.

[Ram87] Annamalai Ramanathan. Equations defining Schubert varieties and Frobenius splitting of diagonals. *Inst. Hautes Études Sci. Publ. Math.*, (65):61–90, 1987.

[Roc10] Claudio Rocchini. Rockini home page. https://www.rockini.name/, 2010. last accessed: November 21, 2025.

[Ron80] Mark A. Ronan. A geometric characterization of Moufang hexagons. *Invent. Math.*, 57:227–262, 1980.

[Ron09] Mark A. Ronan. *Lectures on Buildings*. The University of Chicago Press, Chicago, IL, 2009.

[RT87] Mark A. Ronan and Jacques Tits. Building buildings. *Math. Ann.*, 278:291–306, 1987.

[Sch99] Andreas E. Schroth. How to draw a hexagon. *Discrete Mathematics*, 199(1–3):161–171, 1999.

[Shu08] Ernest E. Shult. On characterizing the long-root geometries. *Adv. Geom.*, 10:353–370, 2008.

[Shu10] Ernest E. Shult. *Points and Lines*. Springer Berlin, Heidelberg, 2010. doi:10.1007/978-3-642-15627-4.

[Spr98] Tonny Albert Springer. *Linear Algebraic Groups*. Number 2. Birkhäuser Boston, Basel, Berlin, (1981) 1998.

[Ste68] Robert Steinberg. *Lectures on Chevalley Groups*. Yale University, Notes prepared by John Faulkner and Robert Wilson, 1968.

[Ste20] Yegor Stepanov. On an octonionic construction of the groups E_6 and ${}^2E_6$. PhD Thesis, May 2020.

[SV] N. S. Narasimha Sastry and Hendrik Van Maldeghem. Metasymplectic spaces fully embedded in the long root subgroup geometry of a building of type E_8. *in progress*.

[Ten04] Katrin Tent. Split BN-pairs of rank 2: the octagons. *Adv. Math.*, 181:308–320, 2004.

[Tim01] Franz Timmesfeld. *Abstract Root Subgroups and Simple Groups of Lie-Type*. 2001. doi:10.1007/978-3-0348-7594-3.

[Tit55] Jacques Tits. Sur certaines classes d'espaces homogènes de groupes de Lie. *Acad. Roy. Belg. Cl. Sci. Mém. Collect. 8°*, 29(2):5–268, 1955.

[Tit57] Jacques Tits. Sur la Géométrie des R-espaces. *Journal de mathématiques pures et appliquées*, 36:17–38, 1957.

[Tit59] Jacques Tits. Sur la trialité et certains groupes qui s'en déduisent. *Inst. Hautes Études Sci. Publ. Math*, 2:13–60, 1959.

[Tit64] Jacques Tits. Algebraic and abstract simple groups. *Ann. of Math.*, 80:313–329, 1964.

[Tit66] Jacques Tits. Classification of algebraic semisimple groups. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 33–62. Amer. Math. Soc., Providence, RI, 1966.

[Tit74] Jacques Tits. *Buildings of Spherical Type and Finite BN-Pairs*. Springer Berlin, Heidelberg, 1974. doi:10.1007/978-3-540-38349-9.

[Tit76] Jacques Tits. Non-existence de certains polygones généralisés, I. *Invent. Math.*, 36:275–284, 1976.

[Tit77] Jacques Tits. Endliche Spiegelungsgruppen, die als Weylgruppen auftreten. *Inventiones Math.*, 43:283–295, 1977.

[Tit79] Jacques Tits. Reductive groups over local fields. *Proceedings of Symposia in Pure Mathematics*, 33(part 1):29–69, 1979.

[Tit81] Jacques Tits. A local approach to buildings. In *The Geometric Vein*, pages 519–547, New York, NY, 1981. Springer New York.

[Tit92] Jacques Tits. Twin buildings and groups of Kac-Moody type. In *Groups, combinatorics and geometry*, volume 165 of *London Math. Soc. Lecture Note Ser.*, pages 249–286. Cambridge Univ. Press, 1992.

[TTV11] Beukje Temmermans, Joseph A. Thas, and Hendrik Van Maldeghem. Domesticity in projective spaces. *Innov. Incid. Geom.*, 12:141–149, 2011.

[TV01] Katrin Tent and Hendrik Van Maldeghem. On irreducible (B, N)-pairs of rank 2. *Forum Math.*, 13:853–862, 2001.

[TV03] Katrin Tent and Hendrik Van Maldeghem. Moufang polygons and irreducible spherical BN-pairs of rank 2. I. *Adv. Math.*, 174:254–265, 2003.

[TW02] Jacques Tits and Richard M. Weiss. *Moufang Polygons*. Springer Berlin, Heidelberg, 2002. doi:10.1007/978-3-662-04689-0.

[Van95] Hendrik Van Maldeghem. The non-existence of certain regular generalized polygons. *Arch. Math.*, 64:86–96, 1995.

[Van98] Hendrik Van Maldeghem. *Generalized Polygons*. Springer Basel AG, 1998.

[Van12] Hendrik Van Maldeghem. Symplectic polarities in buildings of type E_6. *Des. Codes Cryptogr.*, 65:115–125, 2012.

[Van24] Hendrik Van Maldeghem. *Polar Spaces*. Number 3 in Münster Lectures in Mathematics. Europ. Math. Soc. Press, 2024.

[vCV96] John van Bon, Hans Cuypers, and Hendrik Van Maldeghem. Hyperbolic lines in generalized polygons. *Forum Math.*, 8:343–362, 1996.

[Vel60] Ferdinand Douwe Veldkamp. Polar geometry I-V. Indag. Math. 21 (1959) 512-551, Indag. Math. 22 (1960) 207-212., 1959-1960. MR0125472 (23 A2773).

[VV19] Hendrik Van Maldeghem and Magali Victoor. Some combinatorial and geometric constructions of spherical buildings. *Surveys in Combinatorics, Cambridge University Press (ed. A. Lo et al.), London Math. Soc. Lect. Notes Ser.*, 456:237–265, 2019.

[VV22] Hendrik Van Maldeghem and Magali Victoor. On severi varieties as intersections of a minimum number of quadrics. *Cubo*, 24, 2022.

[Wei04] Richard M. Weiss. *The Structure of Spherical Buildings*. Princeton University Press, 2004.

[Wie24] Torben Wiedemann. Root graded groups. Dissertation, Februar 2024. doi:http://dx.doi.org/10.22029/jlupub-18373.