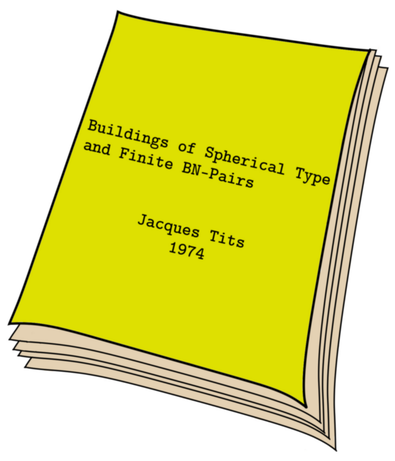
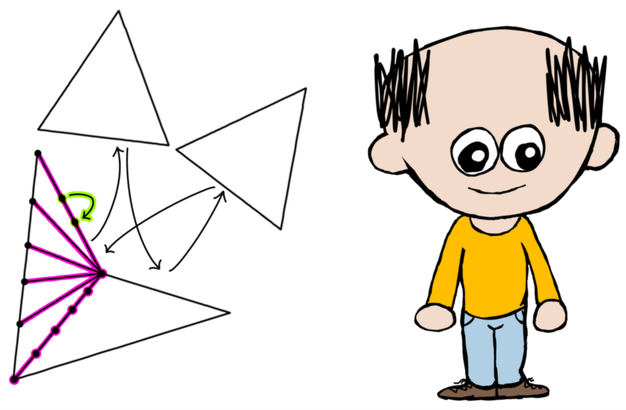
Moufang Property and Groups of Projectivities

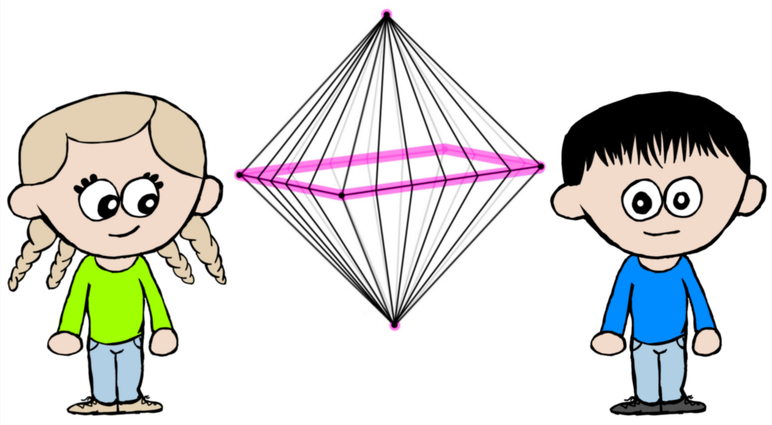
In his lecture notes from 1974, Jacques Tits showed that all thick, irreducible, spherical (t.i.s.) buildings of rank at least 3 have the Moufang property, using his own result about extensions of isometries.



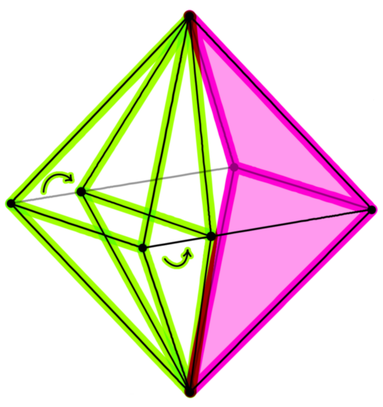
In 2022, Linus Kramer proposed that I could try to find a geometric proof for this result, by considering the corresponding Lie incidence geometries of these buildings, as a PhD project.



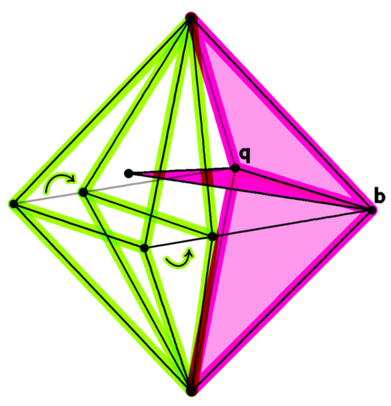
In 2023, I went to Ghent to look at the case of buildings of type B_n (polar spaces) together with Hendrik Van Maldeghem. We first investigated rank 2 polar spaces (generalized quadrangles) inside rank 3 polar spaces.



We were able to find a geometric proof for the fact that generalized quadrangles in rank 3 polar spaces are Moufang by constructing the desired automorphisms (root elations) directly using projections. We explored, whether we could use these, to define root elations for the whole space and we could! We found a geometric proof for the fact that rank 3 polar spaces are Moufang!

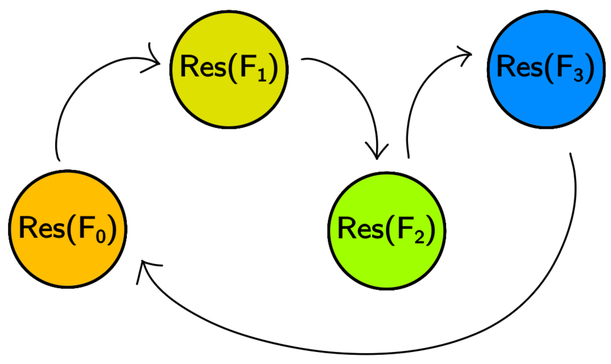


Looking at our constructions, we saw that our automorphisms fix more points than we needed, meaning we gained some more insight into the fix point structure of root elations.



All planes through **q** and **b** get fixed pointwise.

Additionally, we observed that we could write root elations as products of an even number of projections, starting and ending in the same residue, which made us wonder, whether or not this could be done for all t.i.s. buildings.



In 1988, Norbert Knarr defined groups of projectivities and groups of even projectivities for generalized polygons Γ and determined them in the finite case. He showed:

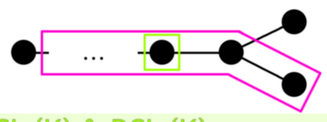
Theorem[Knarr, 1988] If Γ is Moufang, then for every point or line x of Γ, the special group of projectivities $\Pi^+(x)$, coincides with the stabiliser of x in the little projective group $\mathrm{Aut}^+(\Gamma)$ of Γ.
(That is the group generated by all root elations.)

In 2024, Hendrik Van Maldeghem, Jeroen Schillewaert and me showed:

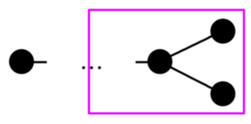
Theorem If F is a non-maximal simplex in a Moufang spherical building Δ, then $\Pi^+(F)$ coincides with the faithful permutation group induced by the stabiliser $\mathrm{Aut}^+(\Delta)_F$ of F in $\mathrm{Aut}^+(\Delta)$ on the residue $\mathrm{Res}_\Delta(F)$ of F in Δ.

And we determined all groups of projectivities for irreducible residues of t.i.s. buildings of simply-laced type and rank at least 3.

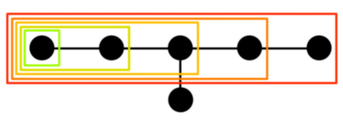
$PGL_2(K)$ & $PGL_2(K)$, $PGL_{r+1}(L)$ & $PGL_{r+1}(L)$ for cotyp(F) of type A_r



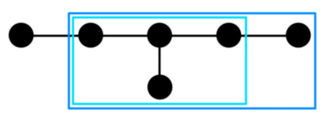
$PGL_2(K)$ & $PGL_2(K)$,
$PGL_{r+1}(K)$ & $PGL_{r+1}(K).2$ for Res(F) of type A_r,
r in [2, r-2], cotyp(F)≠{n-2, n-1, n},
$PGL_n(K,2)$ & $PGL_n(K,2).2$ for Res(F) of type A_{n-1}, n even
$PGL_n(K)$ & $PGL_n(K)$ for Res(F) of type A_{n-1}, n odd



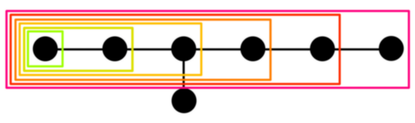
$PGO^\circ_{2n-4r}(K)$ & $PGO^\circ_{2n-4r}(K)$ for Res(F) of type D_{n-2r}, $4 \le n-2r \le n-1$,
$PGO^\circ_{2n-4r+2}(K)$ & $PGO_{2n-4r+2}(K)$ for Res(F) of type D_{n-2r+1}, $3 \le n-2r+1 \le n-1$



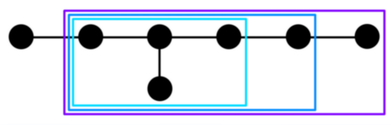
$PGL_2(K)$ & $PGL_2(K)$, $PGL_3(K)$ & $PGL_3(K).2$, $PGL_4(K)$ & $PGL_4(K)$,
$PGL_5(K)$ & $PGL_5(K)$, $PSL_6(K,3)$ & $PSL_6(K,3)$



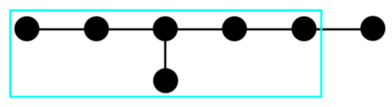
$\overline{PGO_8}^\circ(K)$ & $\overline{PGO_8}(K)$, $\overline{PGO_{10}}^\circ(K)$ & $\overline{PGO_{10}}^\circ(K)$



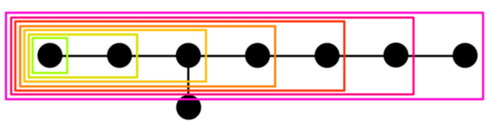
$PGL_2(K)$ & $PGL_2(K)$, $PGL_3(K)$ & $PGL_3(K).2$, $PGL_4(K)$ & $PGL_4(K).2$,
$PGL_5(K)$ & $PGL_5(K).2$, $PSL_6(K,2)$ & $PSL_6(K,2).2$ for cotyp(F)={2,4,5,6,7}
and $PGL_6(K)$ & $PGL_6(K).2$ for 2 not in cotyp(F), $PGL_7(K)$ & $PGL_7(K).2$



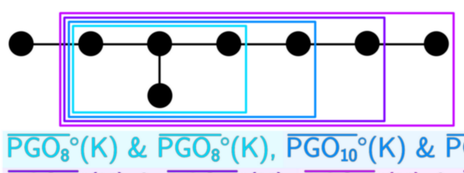
$\overline{PGO_8}^\circ(K)$ & $\overline{PGO_8}^\circ(K)$, $\overline{PGO_{10}}^\circ(K)$ & $\overline{PGO_{10}}(K)$,
$\overline{P\Omega_{12}}(K)$ & $\overline{P\Omega_{12}}(K)$



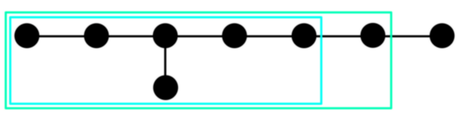
$PGE_6(K)$ & $PGE_6(K).2$



$PGL_2(K)$ & $PGL_2(K)$, $PGL_3(K)$ & $PGL_3(K).2$, $PGL_4(K)$ & $PGL_4(K).2$, $PGL_5(K)$ & $PGL_5(K).2$,
$PGL_6(K)$ & $PGL_6(K).2$, $PGL_7(K)$ & $PGL_7(K).2$, $PGL_8(K)$ & $PGL_8(K).2$



$\overline{PGO_8}^\circ(K)$ & $\overline{PGO_8}^\circ(K)$, $\overline{PGO_{10}}^\circ(K)$ & $\overline{PGO_{10}}(K)$,
$\overline{PGO_{12}}^\circ(K)$ & $\overline{PGO_{12}}^\circ(K)$, $\overline{PGO_{14}}^\circ(K)$ & $\overline{PGO_{14}}(K)$

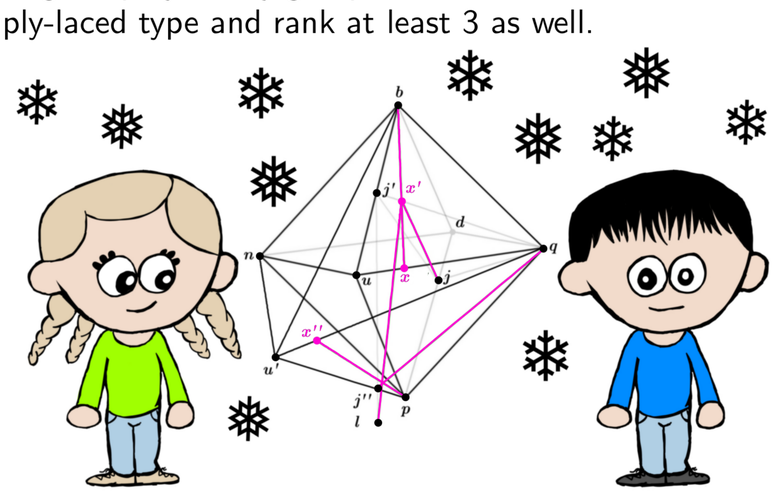


$PGE_6(K)$ & $PGE_6(K).2$, $PGE_7(K)$ & $PGE_7(K)$

Turning our heads to group theory, we saw that this determines the exact structure and action of Levi subgroups of parabolic subgroups of groups of Lie type related to t.i.s. buildings of simply-laced type.

Levi Decomposition
The parabolic subgroups of $\mathrm{Aut}^+(\Delta)$ can be written as semidirect products of a unipotent subgroup and a Levi subgroup.

Equipped with our constructions from 2023, Hendrik and I are looking forward to determining the projectivity groups for irreducible residues of t.i.s. buildings of non-simply-laced type and rank at least 3 as well.



Sira Busch ▪ Universität Münster ▪ s_busc16@uni-muenster.de ▪ September 2024