A CHARACTERISATION OF LINES IN FINITE LIE INCIDENCE GEOMETRIES OF CLASSICAL TYPE

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Abstract. We consider any classical Grassmannian geometry Γ, that is, any projective or polar Grassmann space. Suppose lines have size $s+1$. Then we classify all sets of size $s+1$ with the property that no object of opposite type in the corresponding building, is opposite every point of the set. It turns out that such sets are either lines, or hyperbolic lines in suitable residues isomorphic to (symplectic) generalised quadrangles (where the restriction in parentheses only holds for rank at least 3; if Γ has rank 2 then any hyperbolic line of size $s+1$ does the job). This is a far-reaching extension of a famous and fundamental result of Bose & Burton in the 1960s.

1. Introduction

A Lie incidence geometry of classical type is a j-Grassmannian arising from either a projective space or a polar space. Hence, the point set of such geometry is the set of all (singular) subspaces of projective dimension $j-1$ of a given projective or polar space, and a line is the set of such subspaces containing a given subspace of dimension $j-2$, and contained in a given subspace of dimension j, if subspaces of the latter dimension exist (otherwise we drop this condition). Such geometries turn up in different circumstances, but in many cases they appear as geometries embedded in some projective space. In these cases, the lines of the Lie incidence geometry Δ are simply lines of the ambient projective space. However, there are cases in which lines of the projective space, that are entirely contained in Δ, exist, without forming a line of Δ. This has been described by Cohen & Cooperstein in [6]. In fact they classify all projective varieties that show such behaviour. Kasikova & Van Maldeghem [7] captured such "pseudo" lines under the name *geometric lines* and gave an abstract definition: *A geometric line is a set of points with the property that each object of opposite type is opposite either none, or all, except exactly one point of the set.* In the present paper, we introduce a "weaker" definition in the finite case and still prove that the sets we obtain are geometric lines: *A geometric line is a set of points of the same size as a line, such that no object in the geometry is opposite all points of the set (opposition in the sense of spherical buildings).*

Our motivation is two-fold. A first motivation is that – applied to ordinary projective spaces – this generalises in a very natural way the work of Bose & Burton [1] to arbitrary Grassmannians of projective and polar spaces. It is known that, in a polar space with $s+1$ points per line, any set of s points admits an opposite point. That means there exists a point not collinear to any given set of s points. This is not true for $s+1$, as lines are counterexamples. In this minimal case, it is natural to ask for a classification of all counterexamples. This can be phrased as a Segre-like or extremal problem as follows: If a set T of points does not admit a common opposite, then $|T|$ is at least $s+1$; what happens if equality occurs? At the same time, the answer to that question lays a solid basis for investigating the analogues of blocking sets in Lie incidence geometries (a *blocking set* in the classical sense being a set of points of a projective

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1991 *Mathematics Subject Classification.* 51E24 (primary), 51A05, 51A50 (secondary).

Key words and phrases. Classical geometries, Grassmannians, polar spaces.

The first author is funded by the Claussen-Simon-Stiftung and by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044 –390685587, Mathematics Münster: Dynamics–Geometry–Structure. This work is part of the PhD project of the first author.

space "blocked" by every hyperplane; note a hyperplane blocks a point if, and only if, it is not opposite to it).

A second motivation, which in fact initiated this work, was a problem arising in [4]. In order to decompose an arbitrary sequence of perspectivities into a sequence of projectivities of a certain prescribed type, we had to find a point opposite four arbitrary, given points of a certain Lie incidence geometry. This is easy, if the ground field has size at least 4, but over the field $\mathbb{F}_3$, this problem gave rise to exactly the question sketched in the previous paragraph with $s = 3$. The solution for $s = 3$ is not much simpler than the general case, and so we answer this question in full generality and for all possible Grassmannians, starting with the classical types here, and continuing with the exceptional geometries in [5]. The reason to not include the exceptional case in the present paper is that it requires different methods and it is handled in the framework of parapolar spaces, which is not needed in the classical case, where one only needs some background on projective and polar spaces.

2. Statement of the Main Result

We assume the reader is familiar with the basics on (finite) projective and polar spaces. We say that a polar space of rank $r \geq 2$ *has order* (s,t) if every line carries exactly $s+1$ points and every submaximal subspace (that is, a singular subspace of projective dimension $r-2$) is contained in exactly $t+1$ maximal singular subspaces (that is, singular subspaces of projective dimension $r-1$, sometimes also referred to as *generators*). Every singular subspace of a polar space of order (s,t) is a projective space of order s.

A *symplectic* polar space is the polar space associated to a non-degenerate alternating form, or, equivalently, to a symplectic polarity of a projective space. A *parabolic quadric* is a non-degenerate quadric in a projective space of even dimension at least 4. A *hyperbolic line* in a polar space Δ is a set of points collinear to every point collinear to two given non-collinear points. In other words: It is a set of points of the form $(\{x,y\}^{\perp})^{\perp}$ where x and y are two non-collinear points, and where $\perp$ denotes as usual the collinearity relation.

A crucial notion for the present paper is that of *opposition.* In a projective space, two subspaces are *opposite* if they are complementary, that is, they are disjoint and together generate the whole space. In a polar space two subspaces are *opposite* if no point of either of them is collinear to all points of both of them. They automatically have the same projective dimension.

We first phrase the assumptions of our main results uniformly in building theoretic terms, and only afterwards we specify to projective and polar spaces. Concerning the building theoretic notions, we refer to the literature, in particular the standard reference [8]. We just recall that we see buildings as simplicial complexes in which the maximal simplices are called *chambers* and the submaximal ones (or next-to-maximal ones) *panels.* A building is *thick* if every panel is contained in at least three chambers. It is called *spherical* if its apartments are finite (which is of course automatic if the whole building is finite, as we will assume). The automorphism group of a single apartment of a building is a Coxeter group, to which a Coxeter diagram can be attached, and we use Bourbaki labeling of the types [2]. For each thick building Δ , say of type X_n, and each type, say $i \in \{1,2,\ldots,n\}$, there is a unique point-line geometry where the points are the vertices of type i of Δ and the lines are the sets of vertices of type i completing a given panel, obtained from a chamber by removing the vertex of type i, to a chamber. This geometry is often called a *Lie incidence geometry (of type* $\mathsf{X}_{n,i}$).

Theorem A. *If in an irreducible thick finite classical spherical building Δ of type X_n, $n \geq 2$, the panels of cotype $\{i\}$ are s-thick (that is, every panel of cotype $\{i\}$ is contained in precisely $s+1$ chambers), then every set T of $s+1$ vertices of type i of Δ admits a common opposite vertex except precisely in the following four cases.*

(1) *The set T is a line in the corresponding Lie incidence geometry of type $\mathsf{X}_{n,i}$.*
(2) *Δ is the building corresponding to a symplectic polar space of rank at least 3 and T is a hyperbolic line in the link of a simplex of type $\{1,2,\ldots,i-1\}$ of the latter.*

(3) Δ *is the building corresponding to a parabolic quadric and* T *is a set of generators containing a common singular subspace* U *of codimension* 2 *in each member of* T *such that in the residue of* U*, which is a generalised quadrangle, the set* T *is a regulus (the set of lines intersecting two given disjoint lines).*
(4) Δ *is a generalised quadrangle of order* (s,s) *and* T *is either a hyperbolic line or a dual hyperbolic line.*

For projective spaces, this theorem can be stated as follows. Recall that the empty projective subspace has dimension -1 by convention.

Corollary 2.1. *Let* $0 \le k < n$ *be integers, and let* q *be a prime power. Let* T *be a set of* $q+1$ k*-dimensional subspaces of* $\mathsf{PG}(n,q)$*. If no* $(n-k-1)$*-space is disjoint from each member of* T*, then there exist a* $(k-1)$*-space* U *and a* $(k+1)$*-space* W *such that* T *coincides with the set of* k*-spaces containing* U *and contained in* W*.*

For polar spaces we have the following formulation (excluding the rather trivial case of an asymmetric grid).

Corollary 2.2. *Let* Γ *be a polar space of rank* r *at least* 2 *and order* (s,t)*. Let* T *be a set of either* $s+1$ *singular subspaces of* Γ *of dimension* $k \le r-2$*, or* $t+1$ *maximal singular subspaces if* $t>1$*, or* $s+1$ *maximal singular subspaces of the same natural system if* $t=1$*. Then there exists a singular subspace of dimension* k *opposite each member of* T*, except if*

(*i*) $k \le r-2$ *and all members of* T *contain a given* $(k-1)$*-dimensional subspace and are contained in a given* $(k+1)$*-dimensional singular subspace;*
(*ii*) $k = r-1$*,* Γ *is not hyperbolic and all members of* T *contain a given* $(r-2)$*-dimensional subspace;*
(*iii*) $k \le r-2$*,* Γ *is symplectic, and all members of* T *contain a given* $(k-1)$*-dimensional subspace in the residue of which they form a hyperbolic line;*
(*iv*) $k = r-1$*,* Γ *is either parabolic or hyperbolic, and all members of* T *contain a given* $(r-3)$*-dimensional subspace in the residue of which they form a regulus.*
(*v*) $r=2$*,* $s=t$*, and* T *is either a hyperbolic line of length* $s+1$*, or a regulus of size* $s+1$*.*

In the last case, T is half of a subquadrangle of order $(1,s)$ or $(s,1)$, respectively.

There is a rather intriguing corollary of our main results. We state it as our second main result. It makes a connection with a notion defined in [7], namely, a *geometric line*, that is, a set L of vertices of common type i such that for every vertex v of opposite type, either exactly one or all members of L are not opposite v.

Theorem B. *If in an irreducible thick finite classical spherical building* Δ *of type* X_n*,* $n \ge 2$*, the panels of cotype* $\{i\}$ *are* s*-thick (that is, every panel of cotype* $\{i\}$ *is contained in precisely* $s+1$ *chambers), then a set* T *of vertices of type* i *of* Δ *is a geometric line in the* i*-Grassmannian geometry (of type* $\mathsf{X}_{n,i}$*) if, and only if, it has size* $s+1$ *and does not admits a common opposite vertex.*

The funny thing is that there is no obvious direct and general connection between geometric lines and the sets T of $s+1$ vertices that we are considering. The proof of Theorem B consists of observing that both notions provide the same objects, both after proofs of some pages. Only in Lemma 3.7 where we treat the case of points in a polar space, we are able to use the notion of projective line to obtain our classification. But the proof is not as direct as one would like to. In particular, we also provide an alternative proof of the same result in the special case where the polar space is a related to a quadric, and this one is shorter and does not use the notion of projective line, see Lemma 3.5.

Another corollary is the following.

Corollary 2.3. *If in an irreducible thick finite classical spherical building* Δ *of type* X_n*,* $n \ge 2$*, the panels of cotype* $\{i\}$ *are* s*-thick (that is, every panel of cotype* $\{i\}$ *is contained in precisely* $s+1$ *chambers), then every set* T' *of* s *vertices of type* i *of* Δ *admits a common opposite vertex.*

Proof. We can always complete T' to a set T of $s+1$ vertices by adding a vertex such that T is not a set as in the conclusion of *Theorem A*. $\square$

Remark 2.4. There is s a notion of *split building*. Without defining this in general, we mention that, in the finite classical case, this concerns the projective spaces, the symplectic polar spaces (which are then said to be of type C_n), the parabolic polar spaces (type B_n) and the oriflamme complexes of hyperbolic polar spaces (type D_n). These have not only a Coxeter diagram attached, but more specialised a *Dynkin diagram*, where nodes correspond to fundamental roots of a root system. Then the cases in the conclusions of Theorem A, where the set T is not a line in the Lie incidence geometry of type $\mathsf{X}_{n,i}$, occur precisely when i represents a short root in the root system corresponding to the Dynkin diagram. This behaviour will sustain in the exceptional case.

3. Proofs of the main results

First we recall the following extension of Theorem 3.30 of [8]. For a proof, see Proposition 8.2 of [4].

Proposition 3.1. *If every panel of a spherical building is contained in at least $s+1$ chambers, then every set of s chambers admits an opposite chamber.*

We now consider the projective space and its Grassmannians.

3.1. Projective spaces—Type A_n.

The following lemma proves Theorem A for buildings of type A_n.

Lemma 3.2. *If no $(n-k-1)$-space is disjoint from each member of a set T of $q+1$ projective k-spaces of $\mathsf{PG}(n,q)$, then there exist a $(k-1)$-space U and a $(k+1)$-space W such that T coincides with the set of k-spaces containing U and contained in W.*

Proof. We first do the cases $n \le 3$ and then proceed by induction on n. If $k=0$, then the assertion follows directly from the main result of [1]. Dually, the case $k=n-1$ follows. Whence the case $n=2$. Now suppose $n=3$ and $k=1$. Suppose at least two members of T intersect in a point, say, $L_1 \cap L_2 = \{p\}$, $L_1, L_2 \in T$. Since we may suppose that T is not a planar line pencil, there is some point $x \in \langle L_1, L_2 \rangle$ not contained in any member of T. In the residue of x, then lines obtained from T form a set of at most q members (since L_1 and L_2 define the same line), and hence, by [1] again, we find a line K through x disjoint from all members of T. Now suppose every pair of members of T is disjoint. Pick a point x not on any member of T (that is possible since there are q^3+q^2+q+1 points and only $(q+1)^2$ on members of T). Pick $L_1, L_2 \in T$ arbitrarily. Then there exists a unique line K through x intersecting both L_1 and L_2 non-trivially. Since $K \setminus \{x\}$ contains q points, there exists a member $L_3 \in T$ not meeting K. Hence in the residue of x, not all lines corresponding to the members of T go through a common point. Hence there again exists a line through x not intersecting any member of T. This shows the case $n=3$.

We proceed by induction. By duality, we may assume $2k+1 \le n$. We consider two different cases.

(1) *Suppose each pair of members of T intersects in a $(k-1)$-space.* Then it is easy to see that there are again two cases.
   - (*i*) *The members of T contain a common $(k-1)$-space U.* Then we may assume they are not all contained in a common $(k+1)$-space. Intersecting the situation with a hyperplane H not containing U, we obtain $q+1$ $(k-1)$-spaces in $\mathsf{PG}(n-1,q)$ all going through the same $(k-2)$-space, but not contained in a common k-space. Applying induction we obtain an $((n-1)-(k-1)-1)$-space $Z \subseteq H$ disjoint from each member of T.

(*ii*) *The members of T are contained in a common $(k+1)$-space W.* Here, we may assume that they do not contain a common $(k-1)$-space. Since $k+1 \leq n-1$, we can apply induction and find a point $p \in K$ not contained in any member of T. Let C be an $(n-2-k)$-space complementary to W. Then $\langle p, W\rangle$ is an $(n-1-k)$-space disjoint from all members of T.

(2) *Some pair $\{A_1, A_2\}$ of members of T intersect in at most a $(k-2)$-space.* If either $2k+1 \leq n-1$, or not all pairs in T are disjoint, then we can find a point x outside the span $\langle A_1, A_2\rangle$ of two members $A_1, A_2 \in T$, with $k+2 \leq \dim\langle A_1, A_2\rangle \leq n-1$, and not lying in any member of T (use a simple count). It follows that we can apply induction in the residue of x and obtain an $(n-k-1)$-space through x disjoint from each member of T.

So we may assume that $2k+1 = n$ and all members of T are pairwise disjoint. Then the proof is similar to the last arguments of the case $(k, n) = (1, 3)$ above. □

3.2. **Opposition in polar spaces.** We now interrupt the proof to review some characterisations of oppositeness in polar spaces. Since some of our proofs will be inductive, we must recognise opposite singular subspaces from *locally opposite* subspaces. Recall that the *residue* $\mathsf{Res}_\Delta(U)$ of a singular subspace U in a polar space Δ is the polar space with point set the set of singular singular subspaces of Δ of dimension $1+\dim U$ containing U, and lines are defined by the singular subspaces of Δ of dimension $2+\dim U$ containing U in the natural way,

Definition 3.3. Two singular subspaces U, W of a polar space are called *locally opposite (at $U \cap W$)* if no point of $(U \cap W) \setminus (U \cap W)$ is collinear to all points of $U \cup W$. This is equivalent to U and W being opposite in $\mathsf{Res}_\Delta(U \cap W)$.

We now have the following local-to-global characterisation.

Lemma 3.4. *Let U, W be two singular subspaces of some polar space Δ. Let $A \subseteq U$ be subspace. Set $B := A^\perp \cap W$. Then the singular subspace S spanned by A and B is locally opposite U at A, and locally opposite W at B if, and only if, U and W are opposite in Γ.*

Proof. Suppose first that S is locally opposite U at A and locally opposite W at B. Then no point of $W \setminus B$ is collinear to all points of U since no such point is collinear to all points of A. No point of $B \subseteq S$ is collinear to all points of U since S is locally opposite U at A. No point of $U \setminus A$ is collinear to all points of B as such a point would otherwise be collinear to all points of S (recalling that A and B generate S), contradicting the local opposition of S and U at A. Finally, no point of A is collinear to all points of W since W is locally opposite S at B. Hence U and W are opposite.
The converse is proved similarly. □

We also notice that generators are opposite if, and only if, they are disjoint. Also, if U and W are singular subspaces of a polar space with the same dimension, then they are opposite if, and only if, no point of U is collinear to all points of W (see Corollary 1.4.7 of [9]). We will use these without reference.

3.3. **Oriflamme geometries—Type D_n.** The main purpose of this section is to prove Theorem A for the extremal type n for buildings of type D_n. We argue in the corresponding geometries, which are coined *oriflamme geometries* in [8]. They arise from hyperbolic polar spaces or rank r by forgetting the singular subspaces of dimension $r-2$ and splitting up the maximal singular subspaces in the two natural types (with natural incidence, except that two maximal singular spaces from different type are incident if they intersect in a subspace of dimension $r-2$).

To achieve this, we should first prove Theorem A for points of the oriflamme geometries. We provide a general proof for points of polar spaces of any type below (see Lemma 3.7). We nevertheless prove this separately for oriflamme geometries since this proof does not use the notion of geometric line and is simpler in nature. It holds for all polar spaces associated to quadrics, but we will not need this.

Lemma 3.5. *If every line of a hyperbolic quadric Q of rank at least 3 contains exactly $s+1$ points, then there exists a point non-collinear to each point of an arbitrary set T of $s+1$ (distinct) points, except if these points are contained in a single line.*

Proof. Let $T = \{p_0, \ldots, p_s\}$ be a set of $s+1$ distinct points of Q, and suppose first that p_0 is collinear with p_1. Since not all points $p_0, p_1, \ldots, p_s$ are contained in one line, we find a point $b \in p_0p_1$ not contained in T. Suppose now that p_0 and p_1 are not collinear, then, since $\{p_0, p_1\}^{\perp\perp} = \{p_0, p_1\}$, we find a point $b \in \{p_0, p_1\}^{\perp}$ not collinear to p_2.

In any case, the point b has the property that the number of lines joining b to a collinear point in T is at most s. Applying Proposition 3.1 we find a line L through b such that no point of T collinear to b is collinear to all points of T. Consequently every point p_i of T is collinear to a unique point p'_i of L. Since $p'_0 = p'_1$ and $|L| = s+1$, there is at least one point $q \in L$ not collinear to any member of T. □

Lemma 3.6. *If every line of a hyperbolic quadric Q with Witt index $d \geq 3$ contains exactly $s+1$ points, then there exists a maximal singular subspace opposite each member of an arbitrary set T of $s+1$ (distinct) maximal singular subspaces of common type, except if these maximal singular subspaces contain a common singular subspace of codimension 2 in each.*

Proof. For $d = 3$, this is Lemma 3.2, Case $\mathsf{PG}(3, s)$. For $d = 4$, triality yields the result using Lemma 3.5. So suppose $d \geq 5$. We argue by induction on d and consider various possibilities.

(*i*) *All members of T contain a common point p.* In this case we apply induction in the residue $\mathsf{Res}_\Delta(p)$ at p and obtain a maximal singular subspace M through p intersecting every member of T in just p. Let p' be a point of Δ opposite p and let M' be the unique maximal singular subspace of Δ containing p' and adjacent to M, that is, intersecting M in a subspace of dimension $n-2$. Then clearly M' is disjoint from each member of T.

(*ii*) *At least two members of T intersect, but they do not all share a common point.* Set $T = \{V_0, V_1, \ldots, V_s\}$. Let p be a point of Δ contained in at least two members of T, say V_0, V_1, but not in all of them, say $p \notin V_2$. Let T' be the set of maximal singular subspaces through p obtained by taking those of T containing p, and taking for each member V_i not containing p an arbitrary maximal singular subspace V'_i containing p and intersecting V_i in a subspace of codimension 2. If all members of T' intersected in a common subspace of dimension $d-3$, then by replacing V'_2 with another maximal singular subspace through p intersecting V_2 in subspace of dimension $d-3$ (distinct from $V_2 \cap V'_2$), we obtain a new set T' not having that property. Hence the induction hypothesis applied to $\mathsf{Res}_\Delta(p)$ yields a maximal singular subspace W through p intersecting each member of T' exactly in p, and hence, since T_i and T'_i have a subspace of dimension $d-3$ in common and W intersects each member of T in a subspace of even dimension (which cannot contain a plane), W intersects each member V_i of T in exactly one point p_i. Since $p_0 = p_1$, the number of such points is at most s and so we find a hyperplane H of W disjoint from all $V_i \in T$. The unique maximal singular subspace U distinct from W and containing H has the opposite type of the V_i and hence is disjoint from all of them (since it cannot intersect any of them in at least a line as this would imply that H intersects a member of T nontrivially).

(*iii*) *Each pair of members of T is opposite.* A simple count yields a point b not contained in any member of T. Let T' be the set of maximal singular subspaces through b intersecting a given member of T in a hyperplane. Suppose two members V'_0, V'_1 of T' intersect in a subspace U of dimension $d-3$. Then the corresponding members V_0, V_1, respectively, of T intersect U in a subspace of dimension $d-4$, and those have mutually a subspace in common of dimension at least $d-5 \geq 0$, a contradiction to the disjointness of V_0 and V_1. Hence induction yields a maximal singular subspace M through b intersecting each member of T' in just $\{b\}$. Since all points collinear to b of the union of the members of T are contained in the members of T', and all points of M are collinear to b, we conclude that M is disjoint from each member of T. □

3.4. **Polar spaces—Types B_n, C_n and D_n.** We first prove Theorem A for type 1 vertices, that is, points of the corresponding polar space. Wd distinguish between rank at least 3 and rank 2.

Lemma 3.7. *If every line of a polar space Δ of rank at least 3 contains exactly $s+1$ points, then there exists a point non-collinear to each point of an arbitrary set T of $s+1$ (distinct) points, except if these points are contained in a single line, or if they form a hyperbolic line in case Δ is a symplectic polar space.*

Proof. Suppose no point of Δ is non-collinear to each point of T, that is, each point of Δ is collinear to some member of T. We prove that T is a geometric line in the sense of [7] and then the result follows from Lemmas 4.7 and 4.8 of [7].

So we have to prove that every point of Δ is collinear to exactly one or all points of T. Since we assume that each point is collinear to at least one point of T, it suffices to show that, if some point is collinear to at least two points of T, then it is collinear to all points of T. Suppose for a contradiction that some point p is collinear to at least two points x, y of T, and opposite the point $z \in T$. We claim that we may assume that $p \notin T$. Indeed, suppose it is. Then we may as well assume $p = y$. Let L be the line through x and y. Then there are at most s points of T contained in L, hence there is a point $q \in L \setminus T$. The number of lines through q containing a line that also contains a point of T is at most s, hence Exercise 2.11(iii) of [9] in $\mathsf{Res}_\Delta(q)$ yields a line $K \ni q$ not collinear to any point of T. Since x and y project to the same point on K, there is some point on K opposite each point of T, a contradiction. The claim is proved. But now exactly the same argument replacing q with p (and using the fact that z is not collinear to p to obtain the assertion that there are at most s lines through p containing a point of T) again leads to a contradiction. Hence the lemma is proved. □

Now for rank 2 polar spaces.

Lemma 3.8. *If every line of a polar space Δ of rank 2 contains exactly $s+1$ points, and every point is contained in exactly $t+1$ lines, then there exists a point non-collinear to each point of an arbitrary set T of $s+1$ (distinct) points, except if these points are contained in a single line, or if they form a long hyperbolic line, that is, $T = T^{\perp\perp} = \{y, z\}^\perp$ for every pair of distinct points y, z of $T^\perp$ and $s = t$.*

Proof. Let Δ have order (s, t).

Suppose that no point is opposite all points of T. Suppose first that T contains (at least) two collinear points, say x_1, x_2. Let the line L through x_1 and x_2 contain ℓ points of T. If $\ell = s+1$, then we are done, so assume $2 \le \ell \le s$. Let p be a point on $L \setminus T$. If there was some line K through p containing no point of T, then, since x_1 and x_2 project onto the same point p of K, there exists a point on K opposite each member of T, a contradiction. Hence

$$\ell + t(s+1-\ell) \le |T| = s+1,$$

implying $t \le 1$, a contradiction.

Now suppose every pair of points in T is opposite. Pick $x_1, x_2 \in T$ arbitrarily. Let $z \in \{x_1, x_2\}^\perp$ be arbitrary and note $z \notin T$. Then, as in the previous paragraph, every line through z contains at least (and hence exactly, by our assumption that no pair of points of T is collinear) one point of T. Hence $s = t$ and $T \subseteq z^\perp$. By the arbitrariness of z we obtain $T \subseteq \{x_1, x_2\}^{\perp\perp}$. Hence $|\{x_1, x_2\}^{\perp\perp}| = s+1 = t+1$ and the arbitrariness of x_1, x_2 implies $T = T^{\perp\perp}$. □

Note that the only Moufang quadrangles with long hyperbolic lines are the symplectic quadrangles, as follows from Theorem 1.4 of [3].

Remark 3.9. The inequality $t \le 1$ obtained in the previous proof cannot be sharpened since in case $t = 1$, every set of points intersecting each "horizontal line"' of a given $(s+1) \times (s+1)$ grid in exactly one point, does not admit an opposite in the grid and is not necessarily of the type described in the lemma.

Now we handle the case of maximal singular subspaces. In that case, the parabolic quadrics (these conform to the split buildings of type B_n) play an exceptional role in Theorem A. Hence we treat them first.

Lemma 3.10. *If in a parabolic quadric Q of Witt index n, $n \geq 2$, every line contains precisely $s+1$ points, then every set T of $s+1$ maximal singular subspaces admits an opposite maximal singular subspace, except if they form a line or hyperbolic line in the corresponding dual polar space $\Gamma(Q)$.*

Proof. Embed Q naturally in a hyperbolic quadric Q' of Witt index $n+1$. Each member M of T is contained in a unique maximal singular subspace M' of Q' of given type. The set of all such subspaces M' admits an opposite maximal singular subspace W' if they do not form a line in the corresponding half spin geometry, that is, by [7], if T is not a line or a hyperbolic line in $\Gamma(Q)$. Then $W = W' \cap Q$ is a maximal singular subspace of Q disjoint from every member of T. □

Now we can treat the other cases. It is convenient to first collect the rank 2 case from what we already showed above.

Lemma 3.11. *If every line of a polar space Δ of rank 2 contains exactly $s+1$ points, and every point is contained in exactly $t+1$ lines, then there exists a line disjoint from each member of an arbitrary set T of $t+1$ (distinct) lines, except if these lines contain a common point, or if they form one regulus of a full subgrid and $s = t$.*

Proof. This is just the dual of Lemma 3.8. □

We can now show the general case.

Lemma 3.12. *Let Δ be a finite polar space of order (s,t), with both s and t at least 2. Suppose Δ does not correspond to a parabolic quadric. Then every set T of $t+1$ maximal singular subspaces admits an opposite maximal singular subspace, except if they contain a common singular subspace of codimension 1 (that is, except if they form a line in the corresponding dual polar space $\Gamma(Q)$).*

Proof. We prove this by induction on the rank n of Δ, where Lemma 3.11 serves as our base, noting that a parabolic quadric is characterised by $s = t$ and admitting a full embedded grid. The proof has the same structure as the one of Lemma 3.6, except that only some dimensions are different.

So, let $n \geq 3$. We again distinguish some cases.

- (i) *The members of T contain a common point p.* This is the same as the corresponding case in the proof of Lemma 3.6.
- (ii) *At least two members of T intersect, but they do not all share a common point.* Set $T = \{V_0, V_1, \ldots, V_t\}$. Let p be a point of Δ contained in a maximum number r of members of T and note that our assumptions imply that $2 \leq r \leq t$. We may therefore assume $p \in V_0 \cap V_1$. Let T' be the set of maximal singular subspaces through p obtained by taking those of T containing p, and taking the unique ones through p intersecting a member of T, that does not contain p, in a hyperplane. Then $|T'| = t+1$. Suppose all members of T' contain a common singular subspace U of dimension $n-2$. Let $V_i \in T \setminus T'$. Then V_i intersects U in a subspace of dimension $n-3$, which is in particular not empty, since $n \geq 3$. Let $q \in V_i \cap U$. Then q is contained in each member of $T' \cap T$, and in $V_i \in T \setminus T'$, contradicting the maximality of r. Hence we can apply induction in $\mathsf{Res}_\Delta(p)$ and obtain a maximal singular subspace M through p intersecting each member of T in at most the point p. Let H be a hyperplane of M not containing p. For each member $V_i \in T$, there is a unique maximal singular subspace $V_i' \supseteq H$ intersecting V in a point. Since $V_0' = V_1'$, at most t distinct maximal singular subspaces through H are not disjoint from some member

of T, hence there is at least one maximal singular subspace through H disjoint from all members of T.

(*iii*) *Every pair of members of T is disjoint.* Here the fact that s and t are not necessarily equal, and that the Witt index can still be 3, complicates the corresponding argument in the proof of Lemma 3.6. So we provide a detailed modified reasoning. Let p be a point not contained in any member of T (that p exists follows again from an easy count: the number of points of Δ is strictly more than $q+1$ times the number of points of a maximal singular subspace). Let T' be as above: it is the set of maximal singular subspaces through p intersecting a member of T in a hyperplane. Suppose the members of T' share a subspace U of dimension $n-2$. Then two arbitrary members of T intersect U in subspaces of dimension $n-3$, which have to intersect nontrivially if $n \geq 4$, contradicting the assumed disjointness of members of T. Hence $n = 3$.

Let $\pi_0, \pi_1, \pi_2 \in T$ be arbitrary (but distinct) and select $p_0 \in \pi_0$ arbitrarily. Let T' be the collection of planes through p_0 intersecting a member of $T \setminus \{\pi_0\}$ in a line, completed with π_0. Then π_0 is locally opposite each other member of T' at p_0. Induction yields a plane α through p_0 disjoint from each member of $T \setminus \{\pi_0\}$ and intersecting π_0 in just p_0. Select a plane β through p_0 intersecting π_0 in a line L_0 and α in a line L. Pick two points b_1 and b_2 on $L \setminus \{p_0\}$. Let, for $i = 1, 2$, the set T_i be defined as the set of planes through b_i intersecting some member of T in a line. Then $\beta \in T_i$. If, for some $i \in \{1, 2\}$, no line in β through b_i intersects each member of T, then the members of T_i do not contain a common line, we can apply the induction hypothesis on T_i and obtain a plane opposite each member of T. Hence we may assume that, for $i = 1, 2$, some line L_i through b_i in β intersects each member of T. Since π_1 and π_2 are disjoint, at least one of them does not contain the intersection point $L_1 \cap L_2$; say π_1 does not. But then π_1 has two points in common with $L_1 \cup L_2 \supseteq \beta$, hence intersects β in a line, which evidently intersects π_0 in some point, contradicting the disjointness of π_0 and π_1. □

We use the previous result for maximal singular subspaces to prove the general result for singular subspaces of arbitrary dimension, delaying the proof for symplectic polar spaces to later.

Lemma 3.13. *Let Δ be finite thick polar space of rank $n \geq 3$, order (s,t), and suppose Δ is not symplectic. Let $T = \{\alpha_0, \ldots, \alpha_s\}$ be a set of $s+1$ different singular subspaces of common dimension $\ell \leq n-2$ in Δ such that they do not form a pencil in a singular space of dimension $\ell+1$ (and we also assume $\ell \geq 1$). Then there exists a singular subspace opposite to all of $\alpha_0, \ldots, \alpha_s$ in Δ.*

Proof. Let, for given order (s,t) and rank n, the polar space Δ and $\alpha_0, \ldots, \alpha_s$ be a smallest (with respect to $n-\ell$) counterexample to the lemma.

Select arbitrarily singular subspaces β_i of dimension $\ell+1$ (these exists since $\ell \leq n-2$) containing α_i, for all i. We can easily choose them in such a way that they do not have a subspace of dimension $\ell-1$ in common, and, if Δ is parabolic and $\ell = n-2$, that they do form a hyperbolic line in the dual polar space. Then it follows, either since we are dealing with the smallest counterexample (if $\ell \leq n-3$), or because of Lemma 3.10 and Lemma 3.12 (if $\ell = n-2$) that we can find a singular subspace β of dimension $\ell+1$ opposite each of β_i, $i \in \{0, 1, \ldots, s\}$. It follows that there is a unique point p_i in β collinear to all points of α_i. If $\{p_0, p_1, \ldots, p_s\}$ is not a line, then by Lemma 3.2 we find a subspace $\alpha \subseteq \beta$ of dimension ℓ not containing any of the points $p_0, p_1, \ldots, p_s$. Then α is opposite each member of T and we are done.

So we may assume that $\{p_0, p_1, \ldots, p_s\}$ is a line L. The set of points of β_i collinear to all points of L is a subhyperplane H_i contained in α_i (as a hyperplane of the latter). We now claim that all points of $\alpha_i \setminus H_i$ are collinear to all points of $\alpha_j \setminus H_j$, for distinct $i, j \in \{0, 1, \ldots, s\}$. Indeed, we may assume $(i,j) = (0,1)$. Let $x_0 \in \alpha_0 \setminus H_0$ and $x_1 \in \alpha_1 \setminus H_1$. Then $x_0^\perp \cap \beta =: K_0$ and $x_1^\perp \cap \beta =: K_1$ are two hyperplanes of β none of which containing L, but containing p_0 and p_1, respectively. It follows that $N := K_0 \cap K_1$ is a subspace of dimension $\ell - 1 \geq 0$ disjoint from L. The line $N^\perp \cap \beta_i$ intersects α_i in a unique point x_i as otherwise N belongs to the perp of a point of H_i, contradicting the facts that also L belongs to the perp, that L and N are complementary, and β and β_i are opposite (and note that the notation x_i is in conformity with the definitions of x_0 and x_1 above), We may hence view $x_0, x_1, \ldots . x_s$ as points of $\mathsf{Res}_\Delta(N)$. If they do not constitute a line in $\mathsf{Res}_\Delta(N)$, then by Lemma 3.7 we can find a point opposite all of them, meaning using Lemma 3.4, we can find a subspace α of dimension ℓ containing N opposite each of α_i, $i \in \{0, 1, \ldots, s\}$. Hence all of $x_0, x_1, \ldots, x_s$ are collinear and the claim is proved.

The previous claim now easily implies that all members of T are contained in a common singular subspace, say M, which we may assume to be maximal. Since by assumption, they do not form a Grassmann line in U, Lemma 3.2 yields a subspace $U \subseteq M$ of dimension $n-2-\ell$ disjoint from all α_i. Let M' be a maximal singular subspace of Δ opposite M. Then $U^\perp \cap M$ has dimension ℓ and is opposite each member of T.

This concludes the proof of the lemma. □

There remains to deal with symplectic polar spaces. It turns out that type 2 elements, that is, lines, cannot be included in the general proof, so we treat them separately. However, the final proof is inductive and the result for lines in rank 3 is needed to do the general case, which is then used to do the lines for higher rank. This explains the rather peculiar conditions in the next lemma, which shall become clear in the proof of Proposition 3.16 below.

Lemma 3.14. *Let Δ be a symplectic polar space of rank at least 3 and order (s, s). Assume that every set of $s+1$ singular planes, not contained in a 3-space of the underlying projective space if they contain a common line, admits a common opposite plane. Then a set T of $s+1$ lines of Δ admits a common opposite in Δ if, and only if, T is not a line pencil in some plane of the underlying projective space.*

Proof. Let T be a set of $s+1$ lines of the symplectic polar space Δ of rank r naturally embedded in $\mathsf{PG}(2r-1, s)$, $r \geq 3$. Suppose T is not a line pencil in some plane of $\mathsf{PG}(2r-1, s)$. We show that there exists a line of Δ opposite all members of T. As usual, we set $T = \{L_0, L_1, \ldots, L_s\}$.

We include L_i in a plane α_i in such a way that the α_i do not contain a common line (which can be easily accomplished). Let β be a plane opposite all α_i, $i \in \{0, 1, \ldots, s\}$. Set $m_i := L_i^\perp \cap \beta$. If $\{m_i | i \in \{0, 1, \ldots, s\}\}$ is not a line, then we can find a line L in β not containing any of the m_i and hence opposite all of the L_i. So the m_i constitute a line M. Let $b \in \beta \setminus M$ be arbitrary. Then $b_i := b^\perp \cap L_i$ is a unique point. Suppose the lines bb_i do not form a line pencil in a plane of $\mathsf{PG}(2r-1, s)$. Then we can find a line L through b locally opposite all of the bb_i. Then L is opposite all of the L_i by Lemma 3.4. Hence we may assume that the lines bb_i form a line pencil in a plane π_b of $\mathsf{PG}(2r-1, s)$. Suppose now that for two choices of $b \in \beta \setminus M$, the points b_i are contained in a common line K_b of $\mathsf{PG}(2r-1, s)$. Let b, c be those two points and adapt the same notation for c as we introduced for b. Without loss of generality, we may assume that the line bc contains the point m_0. Then the lines L_i, $i \in \{1, 2, \ldots, s\}$ are contained in the plane γ of $\mathsf{PG}(2r-1, s)$ spanned by K_b and K_c. The point $b_0 = c_0$ is also contained in γ.

Suppose first that also L_0 is contained in γ. Then, since we assumed that T is not a line pencil, it is easy to see that γ is a singular plane, and it contains a point x not on any of the L_i. Then the line $x^\perp \cap \gamma'$, with γ' a plane opposite γ in Δ is opposite each member of T.

Suppose now that L_0 is not contained in γ. Let $z := L_1 \cap L_2$. Then $z \perp b_0$ as $z \perp \{b_1, b_2\} \subseteq K_b \ni b_0$. We can select a singular plane β_z containing zb_0 and such that L_0 is not collinear to β_z, and β_z is not in a common singular 3-space with γ if singular.

In $\beta_z \setminus zb_0$ we can find a point y not collinear to b_1. It follows that y is not collinear to any of the b_i, $i \in \{1, 2 \ldots, s\}$. The lines joining y with the unique projection point of y on the L_i do not form a line pencil in any plane as two of these lines coincide (namely, the line yz joins y with the point z of both L_1 and L_2). Hence we can find a line L through y locally opposite all these lines. Again, by Lemma 3.4, L is opposite each member of T.

Hence we may assume that for at most one point $b \in \beta \setminus M$, the points $b_0, b_1, \ldots, b_s$ are on one line (and we denote that point, if it exists, from now on with b^*). Note that we may also assume that L_0 and L_1 do not intersect. Indeed, if all members of T pairwise intersect, then either they are contained in a plane, and we treated that case above, or they all contain a common point p. In the latter case the result follows from considering the residue at p (indeed, we can then select a line K locally opposite each member of T; then select a point $q \in K \setminus \{p\}$ and a line L locally opposite K at q. The line L is opposite each member of T by Lemma 3.4).

First let $s > 2$. Choose points $b, c \in \beta \setminus M$ with $m_2 \in bc$ and $b^* \notin \{b, c\}$. As before, the lines L_0 and L_1 are contained in $\langle \pi_b, \pi_c \rangle$, which is a 4-space U_2 of $\mathsf{PG}(2r-1, s)$. If we choose $d \in \beta \setminus M$ with $m_3 \in bd$ and $d \neq b^*$, then we obtain likewise that L_0 and L_1 are contained in a 4-space U_3 of $\mathsf{PG}(2r-1, s)$ spanned by π_b and π_d. Suppose first these two 4-spaces coincide. Then $U_2 = U_3$ contains β. The polar space induced in U_2 is degenerate, but, as we have opposite lines (L_0 and some line in β), the radical is a point, which must coincide with the intersection of any pair of singular planes, contradicting $m_0 \neq m_1$. Hence $U_2 \neq U_3$ and these intersect in a 3-space containing L_0, L_1 and b. However, L_0 and L_1 already span $U_2 \cap U_3$, and by the arbitrariness of b, the 3-space $\langle L_0, L_1 \rangle$ contains β, which is ridiculous as β is disjoint from $L_0 \cup L_1$.

Now let $s = 2$. Similar arguments as in the previous paragraph show that $W := \langle L_0, L_1, L_2 \rangle$ contains β. Then again $\dim W = 3$ leads to the contradiction that $L_0 \cap \beta$ is nonempty and if $\dim W = 4$, then we have a degenerate symplectic polar space induced in W, leading to the same contradiction as before. Hence $\dim W = 5$. Also as before, W is a non-degenerate symplectic polar space. We coordinatise W as follows (using obvious shorthand notation). The two points on L_0 not collinear to all points of M are labelled 100000 and 010000. Likewise those on L_1 and L_2 by 001000, 000100 and 000010, 000001, respectively. We may assume that the points 100000, 001000 and 000010 are together in a plane π_b, $b \in \beta$, and then b has labels 101010. Similarly we have the points 100101, 011001 and 010110 in $\beta \setminus M$. It follows that $m_0 = 001111$, $m_1 = 110011$ and $m_2 = 111100$. Then the point 111111 is contained in each of the planes $\langle m_i, L_i \rangle$, $i = 0, 1, 2$, and hence collinear in Δ with all points of $L_0 \cup L_1 \cup L_2$, which spans W, contradicting non-degeneracy.

The proof is complete. □

Lemma 3.15. *Let Δ be a symplectic polar space of rank $r \geq 4$ and order (s, s), and let $i \in \mathbb{N}$ be such that $2 \leq i \leq r-2$. Assume that every set of $s+1$ singular subspaces of dimension $i+1$, not contained in a common $(i+2)$-dimensional subspace of the underlying projective space if they contain a common i-dimensional singular subspace, admits a common opposite. Suppose also that* Theorem A *is true for symplectic polar spaces of rank at most $r-1$. Then a set T of $s+1$ i-dimensional singular subspaces of Δ admits a common opposite in Δ if, and only if, all members T are not contained in a common $(i+1)$-dimensional subspace of the underlying projective space if they contain a common $(i-1)$-dimensional singular subspace of Δ.*

Proof. Let T be a set of $s+1$ singular subspaces of dimension i of the symplectic polar space Δ of rank r naturally embedded in $\mathsf{PG}(2r-1, s)$, $r \geq 4$. Suppose all members of T are not contained in an $(i+1)$-dimensional subspace of $\mathsf{PG}(2r-1, s)$ if they share a common $(i-1)$-dimensional singular subspace of Δ. We show that there exists a singular i-dimensional subspace of Δ opposite all members of T. We set $T = \{U_0, U_1, \ldots, U_s\}$.

First we assume that all members of T are contained in a common $(i+1)$-dimensional subspace W of $\mathsf{PG}(2r-1, s)$. By our assumption, not all members of T share the same $(i-1)$-dimensional subspace. This implies that the radical of the polar space induced in W has dimension strictly larger than $i-1$; hence W is a singular subspace. Our assumption and Lemma 3.2 imply that we can find a point $x \in W$ not contained in any member of T. Let W' be a singular subspace of dimension $i+1$ opposite W. Then $x^\perp \cap W'$ is a singular subspace of dimension i opposite each member of T.

Henceforth we may assume that not all members of T are contained in the same subspace of $\mathsf{PG}(2r-1, s)$ of dimension $i+1$.

Include every member of T in a singular subspace of dimension $i+1$ in such a way that not all of them are contained in a common subspace of dimension $i+2$ of $\mathsf{PG}(2r-1,s)$. By assumption we can find a subspace W opposite all of these $(i+1)$-dimensional subspaces. Set $\{m_i\} = W \cap U_i^\perp$. If the m_i do not form a line, then Lemma 3.2 yields a singular i-space $U \subseteq W$ not containing any of the m_i and hence opposite all members of T. Hence we may assume that the m_i form a line M. Set $H_i = U_i \cap M^\perp$. Then H_i is a hyperplane of U_i.

Assume now that two members of T, say U_0 and U_1, do not share a common $(i-1)$-dimensional singular subspace. Set $D = U_0 \cap U_1$. Let H be a hyperplane of U_0 not containing D, and hence distinct from H_0. Then $K := H^\perp \cap W$ is a line through m_0. Pick $x \in K \setminus \{m_0\}$ arbitrarily. Define $U_i^x := \langle x, x^\perp \cap U_i\rangle$. Suppose there exists an i-dimensional singular subspace U through x locally opposite all U_i^x. Then, by Lemma 3.4, U is opposite each member of T. Our hypotheses imply that the U_i^x are contained in an $(i+1)$-dimensional subspace A_x of $\mathsf{PG}(2r-1,s)$ and all U_i^x share a common $(i-1)$-space B_x. The singular subspace B_x contains a hyperplane of H and a hyperplane of $x^\perp \cap U_1$. As D is not contained in H, these two hyperplanes do not coincide, and hence they generate B_x. Now we do the same construction with $y \in K \setminus \{x, m-)\}$ and obtain the similarly defined subspaces A_y and B_y. It is elementary to check that $x^\perp \cap U_i \neq y^\perp \cap U_i$, for $i \in \{1,2,\ldots,s\}$. Hence $A := \langle A_x, A_y\rangle$ contains U_i for all $i \in \{1,2,\ldots,s\}$. The intersection $A_x \cap A_y$ contains H. Also, since H does not contain D, it does not contain the intersection $x^\perp \cap y^\perp \cap U_1$ (which is a singular subspace of dimension $(i-2)$). It follows that A_x and A_y share at least a subspace of dimension i (generated by H and $x^\perp \cap y^\perp \cap U_1$). Hence $\dim A \in \{n+1, n+2\}$. Suppose $\dim A = n+1$. Then $A_x = A_y$ and contains the line $\langle x, y\rangle$, which necessarily intersects U_1 for dimension reasons. But this contradicts the fact that W is opposite some singular $(i+1)$-space containing U_1. Consequently $\dim A = n+2$. We can now do the same thing with another hyperplane H' of U_0 not containing D and distinct from H_0, and obtain the similarly defined subspace A' of dimension $n+2$ and the line $\langle x', y'\rangle$ of W, with $m_0 \in \langle x', y'\rangle$. Since both A and A' share the subspace $\langle H, U_1\rangle$ of dimension at least $i+1$, we have $\dim\langle A, A'\rangle \in \langle n+2, n+3\rangle$. If $A = A'$, then A contains the plane $\langle x, x', m_0\rangle$, which necessarily intersects U_1 in at least a point since $\dim A = n+2$ and $\dim U_1 = i$. This is again a contradiction. It follows that $A_0 := A \cap A'$ has dimension $i+1$ and contains all of $U_1, U_2, \ldots, U_s$, plus H and m_0. If A_0 were singular, then $m_0 = m_1$, a contradiction. Hence A_0 is not singular and all U_i, $i \in \{1,2,\ldots,s\}$, share a common $(i-1)$-space V_0. There is a unique i-space U_0^* through V_0 in A_0 distinct from U_i for any $i \in \{1,2,\ldots,s\}$. The space U_0^* contains H and m_0.

Now we can find a singular $(i+1)$-space W^* containing U_0^* with the property that not all of its points are collinear to all points of U_0. In W^*, we can then find a point z not collinear to all points of U_1 (because $U_1^\perp$ cuts out a hyperplanes of W^*). Since a point of Δ is collinear to either all points, or a hyperplane of points of W^*, we see that $z^\perp$ intersects each U_i in a hyperplane of U_i, and for $i = 1,2,\ldots,s$, that hyperplane is necessarily V_0. Hence there exists an i-space U through z locally opposite the two spaces $\langle z, H\rangle$ and $\langle z, V_0\rangle$ at z, and U is opposite each member of T by Lemma 3.4.

Hence we may assume that each pair in T intersects in an $(i-1)$-space. Then either that $(i-1)$-space is unique, say V^*, or all members of T are contained in some $(i+1)$-space, say W^*. In the former case, our hypotheses permit to find an i-space U through V^* locally opposite each member of T, and then the projection U' of U onto a singular subspace opposite V^* is opposite each member of T at V^*. In the latter case we are back to the situation of the first paragraph of this proof, which we already handled.

This completes the proof of the lemma. □

We can now prove Theorem A for symplectic polar spaces.

Proposition 3.16. *Let Γ be a symplectic polar space of rank r at least 2 and order (s,s). Let T be a set of $s+1$ singular subspaces of Γ of dimension $k \leq r-2$. Then there exists a singular subspace of dimension k opposite each member of T, except if either T is a line* *of the corresponding k-Grassmannian geometry, or all members of T contain a given $(k-1)$-dimensional subspace in the residue of which they form a hyperbolic line.*

Proof. We prove this by induction on r, and for given r we use induction on $r-i$.

Let first $r=3$. Then $i=1$ and the assertion holds by Lemma 3.14, Lemma 3.12 and Lemma 3.10 (we need the latter in characteristic 2).

Now assume $r \geq 4$. Then the assertion holds for $i \geq 2$ by Lemma 3.15, and it holds for $i=1$ by Lemma 3.14.

The proof is complete. □

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