GROUPS OF PROJECTIVITIES IN SPHERICAL BUILDINGS OF NON-SIMPLY LACED TYPE

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Abstract. In this article we describe the general and special projectivity groups for all irreducible residues of all thick, irreducible, spherical buildings of type $\mathsf{B_n}$, $\mathsf{C_n}$ and $\mathsf{F_4}$, and rank at least 3. This determines the exact structure and action of Levi subgroups of parabolic subgroups of groups of Lie type related to those buildings.

1. Introduction

In [5], all general and special projectivity groups for all irreducible residues of all thick, irreducible, spherical buildings of simply-laced type and rank at least 3 are determined. Previously, Norbert Knarr and the second author determined these groups for many spherical buildings of rank 2 in [14] and [24, Chapter 8], respectively. In this article, we determine the general and special projectivity groups for all irreducible residues of all thick irreducible buildings of type $\mathsf{B_n}$, $\mathsf{C_n}$ and $\mathsf{F_4}$ and rank at least 3. As described in [5], this will determine the exact structure and action of Levi subgroups of parabolic subgroups of simple groups of Lie type on the corresponding residues.

Buildings of type $\mathsf{B_n}$ and $\mathsf{C_n}$ correspond to polar spaces, and there is a large variety of such structures. The precise special and general projectivity groups sometimes heavily depend on the field and the associated pseudo-quadratic form (see below), but we provide as much general information as possible, in particular, we provide information about generating sets that should suffice to determine the exact groups for any given situation.

Referring for the notation and terminology to further sections, we summarise our results for type $\mathsf{B_n}$ as follows.

Theorem A. *Let Δ be a polar space of rank $r \geq 3$, and let U be a singular subspace of Δ. Then the following hold.*

- (i) *If U is a projective space of dimension $d \leq r-2$, then $\Pi^+_{\leq}(U)$ is the full linear type-preserving group of U and $\Pi_{\leq}(U)$ is the full linear group including (linear) dualities.*
- (ii) *If U is a maximal singular subspace, that is, a projective space of dimension $r-1$, then, if Δ is embeddable, $\Pi^+(U)$ is the linear group generated by homologies with factors in a certain norm set (see* Section 7.2.2*), whereas $\Pi(U)$ extends $\Pi^+(U)$ with a duality having companion field automorphism the involution of the pseudo-quadratic form defining Δ. If Δ is non-embeddable, then $\Pi^+(U)$ is the full projective group of the corresponding Cayley plane and $\Pi(U)$ extends this group with a standard polarity.*
- (iii) *If U has dimension at most $r-3$, then $\Pi^+_{\geq}(U)$ is generated by products of two reflections, that is, collineations that pointwise fix a geometric hyperplane. In the cases that $\Pi^+_{\geq}(R) \neq \Pi_{\geq}(R)$, we have that $\mathsf{Res}_\Delta(U)$ arises from a non-degenerate quadratic form with non-degenerate associated polarity and such that r minus the rank of $\mathsf{Res}_\Delta(U)$ is an odd integer. In that case, $\Pi_{\geq}(U)$ is generated by all reflections. If Δ is non-embeddable, then U is*

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a point and $\Pi^+(U) = \Pi(U)$ is the full group of direct similitudes of the corresponding quadratic form.

(*iv*) *If U has dimension $r-2$, then, if $\mathsf{Res}_\Delta(U)$ can be identified with an embeddable polar space of rank 1 in the natural way, the same conclusion holds as in the previous case. If Δ is non-embeddable, then $\Pi^+_{\geq}(U) = \Pi_{\geq}(U)$ is the full linear group of the corresponding projective line.*

Buildings of type $\mathsf{F_4}$ come in exactly five well specified flavours, and hence, it is possible to write down the exact special and general projectivity groups for all irreducible residues in each of these cases. We refer to Table 1 at the end of the paper for a complete enumeration.

In [5], the authors showed that, in the simply laced case, there is a general diagrammatic rule to predict when the special and general projectivity groups coincide *generically* (meaning, over all fields). The present paper shows that this rule does not hold anymore when the diagram is not simply laced. In particular, for buildings of type $\mathsf{F_4}$, the rule would imply that only for the residues of type $\{2,3,4\}$, $\{2.3\}$ and all rank 1 residues, the special and general projectivity groups generically coincide. In the split case, however, also residues of type $\{1,2,3\}$ show this behaviour, whereas we have rank 1 residues for which the special and general projectivity groups do not coincide (but the latter was to be expected as this phenomenon also exists in the non-simply laced rank 2 case, see [14] and [24, Chapter 8] that we mentioned before).

As a direct application of our results, we come to know the exact action of the (irreducible) Levi subgroups of the little projective group of a spherical building on the corresponding residues. (The little projective group of a spherical building is the group generated by all root elations and is usually simple.) Indeed, for that, it suffices to generalise [5, Proposition 3.2], which proves a connection between the little projective groups of Moufang spherical buildings of simply-laced type and the special projectivity groups of those buildings, to the non-simply laced case. In fact the proof given in [5] is valid for all Chevalley groups. However, the little projective group of a building of type $\mathsf{B_n}$, $\mathsf{C_n}$ or $\mathsf{F_4}$ is not always a Chevalley group (not even an algebraic group). We use a result proved in [16] to prove the same in the general case, see Theorem 3.1 below. This generalises the results of [5].

In [5, Section 8.1], a purely algebraic method, using the root lattice and the weight lattice of a Chevalley group, is given to determine the action of the (irreducible) Levi subgroups of the little projective group of the associated spherical building on the corresponding residues, and it is applied to the simply laced case. We think that this application could be extended to the non-simply laced case when the corresponding building is associated to an adjoint Chevalley group. Perhaps this can also be done, if the building is associated to an algebraic group (but we did not try to do so); in the remaining cases (when the building is associated to a classical group over a skew field, possibly infinite-dimensional over its centre, or to a group of mixed type $\mathsf{F_4}$), it is not obvious to us, how to extend that method. In any case, one of the requirements of the method explained in [5, Section 8.1] is a certain computation in rank 2 residues. Such a computation would be similar to the one performed in Section 7.2.2, for instance. Although certain shortcuts may arise to determine the special projectivity groups in some situations, if we extended the method of [5, Section 8.1] to the non-simply laced case, like in the split cases, we decided to treat all cases uniformly using geometric arguments. We refer the reader to [5] for more details on the algebraic approach.

The paper is structured as follows. In Section 2, we introduce terminology and notation, in particular, we define the geometries (polar spaces and metasymplectic spaces) that correspond to spherical buildings of types $\mathsf{B_n}$, $\mathsf{C_n}$ and $\mathsf{F_4}$ and review some basic properties. In Section 3 we recall some old general observations for spherical buildings that we will use in subsequent sections and make a new one. These observations are independent of the type. Then, in Section 4, we specialise to non-simply laced types and translate some reduction theorems, proved in [5], to these types. These reduction theorems come with some conditions, and we check in Section 5 that the conditions are satisfied in the cases we will need. In Section 6 we determine the projectivity groups for the upper residues in polar spaces, and in Section 7 we do the same for the lower residues. In Section 8 we determine the projectivity groups of the residues of points in a metasymplectic space (which are the residues of vertices of type 1 and type 4 in buildings of type $\mathsf{F_4}$), and we handle the other residues of metasymplectic spaces in Section 9. In Section 10 we put our results for metasymplectic spaces in a table.

2. Preliminaries

2.1. **Point-line geometries and partial linear spaces.** Throughout, we will work with incidence structures called polar spaces and metasymplectic spaces, which are instances of partial linear spaces. In this subsection, we introduce the general definitions we will need. Note that thick polar spaces correspond to buildings of type $\mathsf{B_n}$ and $\mathsf{C_n}$, and (thick) metasymplectic spaces will correspond to buildings of type $\mathsf{F_4}$.

Definition 2.1. A *point-line geometry* is a pair $\Delta = (\mathscr{P}, \mathscr{L})$ with $\mathscr{P}$ a set and $\mathscr{L}$ a set of subsets of $\mathscr{P}$. The elements of $\mathscr{P}$ are called *points*, the members of $\mathscr{L}$ are called *lines*. If $p \in \mathscr{P}$ and $L \in \mathscr{L}$ with $p \in L$, we say that the point p *lies on* the line L, and the line L *contains* the point p, or *goes through* p. If two points p and q are contained in a common line, they are called *collinear*, denoted $p \perp q$. If they are not contained in a common line, we say that they are *non-collinear*. For any point p and any subset $P \subset \mathscr{P}$, we denote

$$p^\perp := \{q \in \mathscr{P} \mid q \perp p\} \text{ and } P^\perp := \bigcap_{p \in P} p^\perp.$$

A *(thick) partial linear space* is a point-line geometry in which every line contains at least three points, and where there is a unique line through every pair of distinct collinear points p and q, which is then denoted with pq. We will usually omit the adjective "thick".

Example 2.2. Let V be a vector space of dimension at least 3. Let $\mathscr{P}$ be the set of 1-spaces of V, and let $\mathscr{L}$ be the set of 2-spaces of V, each of them regarded as the set of 1-spaces it contains. Then $(\mathscr{P}, \mathscr{L})$ is called a *projective space (of dimension* $\dim V - 1$*)*, and denoted as $\mathsf{PG}(V)$.

Definition 2.3. Let $\Delta = (\mathscr{P}, \mathscr{L})$ be a partial linear space.

(1) A *path of length* n in Δ from point x to point y is a sequence $(x = p_0, p_1, \ldots, p_{n-1}, p_n = y)$ of points of Δ such that $p_{i-1} \perp p_i$ for all $i \in \{1, \ldots, n-1\}$. It is called a *geodesic* when there exist no paths of Δ from x to y of length strictly smaller than n, in which case *the distance* between x and y in Δ is defined to be n, notation $d_\Delta(x, y) = n$.

(2) The partial linear space Δ is called *connected* when for any two points x and y, there is a path (of finite length) from x to y. If moreover the set $\{d_\Delta(x, y) \mid x, y \in \mathscr{P}\}$ has a supremum in $\mathbb{N}$, this supremum is called the *diameter* of Δ.

(3) A subset S of $\mathscr{P}$ is called a *subspace* of Δ when every line $L \in \mathscr{L}$ that contains at least two points of S, is contained in S. A subspace that intersects every line in at least a point, is called a *hyperplane*. A hyperplane is called *proper* if it does not consist of the whole point set. A subspace is called *convex* if it contains all points on every geodesic that connects any pair of points in S. We usually regard subspaces of Δ in the obvious way as subgeometries of Δ.

(4) A subspace S in which all points are collinear, or equivalently, for which $S \subseteq S^\perp$, is called a *singular subspace*. If S is moreover not contained in any other singular subspace, it is called a *maximal singular subspace*. A singular subspace is called *projective* if, as a subgeometry, it is a projective space (cf. Example 2.2). Note that every singular subspace is convex.

(5) For a subset P of $\mathscr{P}$, the *subspace generated by* P is denoted $\langle P \rangle_\Delta$ and is defined to be the intersection of all subspaces containing P. A subspace generated by three mutually collinear points, not on a common line, is called a *plane*. Note that, in general, this is not necessarily a singular subspace; however we will only deal with polar and metasymplectic

spaces, which implies that subspaces generated by pairwise collinear points are singular; in particular planes will be singular subspaces isomorphic to projective planes, that is, projective subspaces of dimension 2.

2.2. **Polar spaces.** We recall the definition of a polar space, and gather some basic properties. We take the viewpoint of Buekenhout–Shult [4]. All results in this section are well known.

Definition 2.4. A *polar space* is a point-line geometry in which every line contains at least three points and for every point x the set $x^\perp$ is a proper geometric hyperplane. We will only consider polar spaces of finite frank, that is, such that maximal singular subspaces are generated by a finite number of points. The minimal such number is called the *rank of the polar space.* A *submaximal* singular subspace is a hyperplane of a maximal singular subspace.

One can show that all singular subspaces are either empty, points, lines or projective spaces of finite dimension (see [21, Theorem 7.3.6 and Lemma 7.3.8] or [25, Theorem 1.3.7]). Consequently, a polar space is a partial linear space. A polar space is called *top-thin* if every submaximal singular subspace is contained in exactly two maximal singular subspaces. It is well known that a polar space is either top-thin, or each submaximal singular subspace is contained in at least three maximal singular subspaces (see [25, Theorem 1.7.1]). In the latter case the polar space is called *thick*. In the former case, the polar space corresponds to a building of type D_n, which have a simply laced diagram. Since these were treated in [5], we only consider the thick polar spaces.

Remark 2.5. If a partial linear space contains no points, or contains at least two points but no lines, it is automatically a polar space, of rank 0 or rank 1, respectively.

Definition 2.6. Let $\Delta = (\mathscr{P}, \mathscr{L})$ be a polar space, and let $\mathscr{M}$ be the set of maximal singular subspaces. For each submaximal singular subspace U, let $P(U)$ be the set of maximal singular subspaces containing U, and let $\mathfrak{M}$ be the collection of all such sets $P(U)$ for U ranging over all submaximal singular subspaces. Then $\Delta^* = (\mathscr{M}, \mathfrak{M})$ is a point-line geometry that we call a *dual polar space.* The *dual* of a dual polar space Δ^* is the original polar space Δ.

For two non-collinear points in a polar space, we call the set $\{x, y\}^{\perp\perp} = (\{x, y\}^\perp)^\perp$ a *hyperbolic line.*

2.3. **Metasymplectic spaces.** Let $\Delta = (\mathscr{P}, \mathscr{L})$ be a point-line geometry. Then we call Δ a *metasymplectic space*, if the following conditions are satisfied.

(1) If a point $p \in \Delta$ has distance 2 to some other point $q \in \Delta$, then either there exists a unique point $u \in \Delta$, such that $p \perp u \perp q$, or p and q are contained in a convex subspace of Δ. These convex subspaces are — viewed as point-line geometries on their own — isomorphic to thick polar spaces of rank 3 and we call them *symplecta*, or *symps* for short. If two points p and q span a symplecton, it is commonly denoted by $\xi(p, q)$. Every line is contained in a symplecton.
(2) For every point $p \in \Delta$, there exists at least one point $q \in \Delta$, such that p and q have distance 3.
(3) For every point p and every symp ξ not containing p, the set $p^\perp \cap \xi$ is never a point.

It follows from the Main Theorem of [20] that metasymplectic spaces are in one-to-one correspondence to (thick) buildings of type F_4, together with the choice of an extremal vertex of its diagram. We summarise some more properties of such spaces, while, at the same time, introducing some more terminology. In the following, let $\Delta = (\mathscr{P}, \mathscr{L})$ be a metasymplectric space.

(i) The maximal distance between two points is 3. Points at distance 3 are called *opposite* as they correspond to opposite vertices of the associated spherical building. If two points p and q are opposite, we write $p \equiv q$. Points p, q at distance 2 not contained in a symp are called *special*, and we write $p \bowtie q$. For points p, q at distance 2 that are contained in a symp, we say that p and q are *symplectic*, and we write $p \perp\!\!\!\perp q$. For a subset $S \subseteq \mathscr{P}$, we denote by $S^{\perp\!\!\!\perp}$ the set of all points symplectic to all points of S.

(ii) Let Ξ be the set of symps of Δ. Singular subspaces of Δ of dimension 2 are planes. For each plane π of a given symp, let $\Xi(\pi)$ be the set of symps of Ξ containing π. Let Π be the set of all such sets $\Xi(\pi)$, for π ranging over all planes of all symps of Δ. Then $\Delta^* = (\Xi, \Pi)$ is also a metasymplectic space. This is the *principle of duality.*

(iii) Under the duality mentioned in (ii), collinear points correspond to symps intersecting in a plane, symplectic points correspond to symps intersecting in just a point, special points correspond to disjoint symps, for which there exists a unique symp intersecting both in respective planes and opposite symps correspond to disjoint symps with the property that being symplectic defines a bijection between their point sets inducing an isomorphism.

(iv) Given a point $p \in \mathscr{P}$, we define $\mathscr{L}_p$ as the set of lines containing p and Π_p as the set of planes containing p. If we view each member π of Π_p as the set of all lines contained in π that go through p, then it follows from the principle of duality that the geometry $\mathsf{Res}_\Delta(p) = (\mathscr{L}_p, \Pi_p)$ is a thick dual polar space of rank 3. We call this geometry the *residue of Δ at p*. The set of lines through a certain point in a certain plane is called a *planar line pencil.*

(v) Given a point p and a symp ξ, there are the following three possibilities.

- (a) The point p belongs to ξ.
- (b) The point p is collinear to (all points of) a unique line $L = p^\perp \cap \xi$ of ξ and each point of ξ collinear to L is symplectic to p, while each point of ξ not collinear to L is special to p. WE say that p and ξ are *close*.
- (c) The point p is symplectic to a unique point $q \in \xi$ and each point of ξ collinear to q is special to p, while each other point of ξ is opposite p.

(vi) Given two opposite points $p, q \in \mathscr{P}$, let $E(p,q)$ be the set of points symplectic to both p and q. If we denote by $\Xi(p,q)$ the set of intersections of a symp containing at least two points of $E(p,q)$ with $E(p,q)$ itself, then $(E(p,q), \Xi(p,q))$ is a polar space isomorphic to the dual of $\mathsf{Res}_\Delta(p) \cong \mathsf{Res}_\Delta(q)$. It is called the *equator geometry (with poles p and q).* The poles are not necessarily unique, and the set of points which can be poles for a given equator geometry is called an *imaginary line.* It follows from this that there is a unique imaginary line through two given opposite points p, q and that it coincides with $(p^{\perp\!\!\!\perp} \cap q^{\perp\!\!\!\perp})^{\perp\!\!\!\perp}$. Below, in Section 2.6, we will mention precisely when an imaginary line contains more than two points.

2.4. **Opposition, residues, projections and groups of projectivities.** Projections and groups of projectivities in spherical buildings are defined in general, see Section 3 below. In this paragraph we specialise to buildings of type B_n and F_4, viewed as polar spaces and metasymplectic spaces.

Let Δ be a polar space of rank r. We call two singular subspaces U and V *opposite* if $\dim U = \dim V$ and no point of U is collinear to all points of V. This coincides with the notion of "opposition" in the corresponding building. If $d = \dim U \le r-1$, then we consider the set of all singular subspaces of dimension $d+1$ that contain U as the point set of a new geometry denoted $\mathsf{Res}_\Delta(U)$, or briefly $\mathsf{Res}(U)$, where each singular subspaces of dimension $d+2$ containing U defines in the natural way a line. That geometry is a polar space of rank $r-d-1$ (empty, if $d = r-1$, of rank 1, if $d = r-2$), called the *upper residue of U in Δ*. The geometry with point set U and lines those of U itself is called the *lower residue of U in* Δ; it is just the singular subspace U viewed as a projective space, and we simply denote it by U, for obvious reasons. Lower residues are always projective spaces; upper residues are polar spaces.

Let U and V be two opposite singular subspaces. By [25, Theorem 1.4.11], the map $U \,\overline{\wedge}\, V$ mapping any subspace $S \subseteq U$ to $S^\perp \cap V$ defines an isomorphism from U to the dual of V and is called a *(lower) perspectivity.* In general, we denote $S^\perp \cap V$ more systematically as $\mathsf{proj}_V(S)$, or, when we want to emphasize the origin, as $\mathsf{proj}_V^U(S)$. The composition of (a finite number of lower) perspectivities is a *lower projectivity.* When a lower projectivity has the same image as origin, then we call it a *lower self-projectivity.* The set of all lower self-projectivities of a subspace U forms a group denoted by $\Pi_{\leq}(U)$ and called the *general lower projectivity group of* U. The set of lower self-projectivities, which are the composition of an even number of lower perspectivities forms a normal subgroup of $\Pi_{\leq}(U)$ of index at most 2, called the *special lower projectivity group of* U and denoted as $\Pi^+_{\leq}(U)$. If U is a maximal singular subspace, then the upper residue is empty and so we denote the respective lower projectivity groups of U as $\Pi^+(U)$ and $\Pi(U)$.

We can now do the same for upper residues. Let U and V still be opposite singular subspaces. The map $U \barwedge V$ (it will always be clear from te context whether lower or upper residues is meant) mapping a subspace $U' \supseteq U$ of dimension $1+\dim U$ to the unique subspace $V' \supseteq V$ of dimension $1+\dim V$ intersecting U' in a point, and also denoted by $\mathsf{proj}_V(U')$, induces an isomorphism from $\mathsf{Res}(U)$ to $\mathsf{Res}(V)$, called an *upper perspectivity.* The composition of (a finite number of upper) perspectivities is an *upper projectivity.* When an upper projectivity has the same image as origin, then we call it an *upper self-projectivity.* The set of all upper self-projectivities of a subspace U forms a group denoted by $\Pi_{\geq}(U)$ and called the *general upper projectivity group of* U. The set of upper self-projectivities, which are the composition of an even number of upper perspectivities, forms a normal subgroup of $\Pi_{\geq}(U)$ of index at most 2, called the *special upper projectivity group of* U and denoted by $\Pi^+_{\geq}(U)$. If U is a point, then the lower residue is trivial and so, since there is no confusion, we denote the respective upper projectivity groups of U as $\Pi^+(U)$ and $\Pi(U)$.

Now let Δ be a metasymplectic space. Recall from above that the residue of Δ at a point is always a dual polar space of rank 3. Viewed in the dual Δ^*, the residue at a point p is just the rank 3 polar space given by the symplecton corresponding to p. For a line L, the *upper residue* $\mathsf{Res}_\Delta(L)$ is the point-line geometry with point set the set of planes of Δ containing L, and line set the set of sets of such planes lying in a common given symp through L. This is always a projective plane, which corresponds to the *lower residue* of L in the dual Δ^*. The *lower residue* of L in Δ is the projective line L itself, viewed as just a set of points (hence viewed as a polar space of rank 1). Similarly, the upper residue $\mathsf{Res}_\Delta(\pi)$ of a plane π is the polar space of rank 1 with point set the set of symplecta through π. The lower residue of π is just the projective plane π.

Let p, p' be two opposite points. Then for each symplecton ξ containing p, there exists a unique symplecton ξ' through p' intersecting ξ in a unique point. The map $p \barwedge p'$ mapping $\xi \mapsto \xi'$ induces an isomorphism of $\mathsf{Res}_\Delta(p)$ to $\mathsf{Res}_\Delta(p')$, called a *perspectivity.* We define in exactly the same way as above *projectivities*, *self-projectivities* and the *special* and *general* projectivity group of p, denoted $\Pi^+(p)$ and $\Pi(p)$, respectively.

Now let L, L' be two opposite lines, that is, each point x of L is not opposite a unique point x' of L' and vice versa. The map $L \barwedge L'$ given by $x \mapsto x'$ is again a *lower perspectivity*, and we similarly as before define *lower projectivities*, *lower self-projectivities*, and the *lower special* and *general projectivity groups* of L, denoted as $\Pi^+_{\leq}(L)$ and $\Pi_{\leq}(L)$, respectively. For each symplecton ξ containing L, there exists a unique plane π' through L' all points of which are close to ξ. The map $L \barwedge L'$ (again, it will always be clear from the context whether this is between lower or upper residues) given by $\xi \mapsto \pi'$ induces an isomorphism from $\mathsf{Res}_\Delta(L)$ to $\mathsf{Res}_\Delta(L')$ (not preserving types), called an *upper perspectivity.* Again this gives rise to *upper projectivities*, *upper self-projectivities*, and the *upper special* and *general projectivity groups* of L, denoted as $\Pi^+_{\geq}(L)$ and $\Pi_{\geq}(L)$, respectively. For a plane π of Δ, the *upper (lower) special* and *general projectivity groups* of π in Δ, denoted $\Pi^+_{\geq}(\pi)$ ($\Pi^+_{\leq}(\pi)$) and $\Pi^+_{\geq}(\pi)$ ($\Pi^+_{\leq}(\pi)$), respectively, are the lower (upper) special and general projectivity groups of π as a line in the dual Δ^*.

Now let p, p' be opposite points and ξ, ξ' opposite symps, with $p \in \xi$ and $p' \in \xi'$. Let L be a line with $p \in L \subseteq \xi$. There is a unique line $L'' \in \xi'$, which is the projection of L onto ξ', and there is a unique line $L' \ni p'$ intersecting L'' in some point. It follows from [22, 3.19.5] that the map $\{p,\xi\} \barwedge \{p',\xi'\}$ given by $L \mapsto L'$ induces an isomorphism from $\mathsf{Res}_\Delta(\{p,\xi\}) =: \mathsf{Res}_\xi(p)$

to $\mathsf{Res}(\{p',\xi'\}) =: \mathsf{Res}_{\xi'}(p')$, called a *perspectivity.* This defines, as before, *(self-)projectivities* and the *special* and *general projectivity group* of $\{p,\xi\}$, denoted as $\Pi^+(\{p,\xi\})$ and $\Pi(\{p,\xi\})$, respectively.

Finally, let P be a planar line pencil in a polar space or a metasymplectic space. Such a pencil is determined by its vertex x and a plane π and consists of all lines through x in π. Similarly as in the previous paragraph one defines *perspectivities* $P \,\overline{\wedge}\, P'$ and (self-)projectivities, and the *special* and *general projectivity groups* of P, which we denote by $\Pi^+(P)$ (or $\Pi^+(\{x,\pi\})$) and $\Pi(P)$ (or $\Pi(\{x,\pi\})$), respectively, and which are permutation groups of P.

2.5. **Classification of polar spaces.** Polar spaces of rank at least 3 have been classified and we will give an overview in the following. For that purpose, recall from [22, Section 8.3] (see also [25, Section 3.2]) that a *polarity* of a projective space is a symmetric relation $\Box$ between the points, such that, (with obvious notation,) $p^\Box$ is a hyperplane for each point p. A polarity is non-degenerate, if $p^\Box$ is always a proper hyperplane. For a subspace S, we define $S^\Box$ to be the intersection of all $x^\Box$, for $x \in S$. An *absolute subspace* for $\Box$ is a subspace U with the property that $U \subseteq U^\Box$. The *radical* of a polarity $\Box$ is the image under $\Box$ of the full point set. The radical is then the intersection of all $x^\Box$, for x varying over all points. A polarity is called a *null-polarity*, if each point is absolute. Most polar spaces we will review are embeddable into some projective space. We also include the rank 1 and 2 polar spaces that arise as residues in higher rank polar spaces. In the following, we assume that the reader is familiar with the basic algebraic notions of bilinear and quadratic forms. The null set of a quadratic form is a *quadric.* If a subspace U is disjoint from that quadric, then the quadratic form is said to be *anisotropic over U*. For a $\mathbb{K}$-vector space and bilinear form $b\colon V \times V \to \mathbb{K}$, the *associated polarity* $\Box$ is defined as $p\Box q$ if $b(v,w)=0$, where p and q are the 1-spaces of V generated by v and w, respectively.

2.5.1. *Symplectic polar spaces.* These are polar spaces arising from non-degenerate null-polarities, that is, non-degenerate polarities in finite-dimensional projective spaces, such that all points of the projective space are absolute points. The dimension of the projective space is always odd, say $2n-1$, and the rank of the polar space is n. The coordinatising field is always commutative. These polar spaces are completely and uniquely determined by giving the collinearity relation between arbitrary points of the projective space. In standard form, this is given by the alternating bilinear form

$$f((x_{-n},\ldots,x_{-1},x_1,\ldots,x_n),(y_{-n},\ldots,y_{-1},y_1,\ldots,y_n)) = x_{-n}y_n - y_{-n}x_n + \cdots + x_{-1}y_1 - y_{-1}x_1.$$

2.5.2. *Orthogonal polar spaces.* These are polar spaces arising from non-degenerate quadratic forms of finite Witt index at least 1. Recall that a quadratic form is non-degenerate, if it is anisotropic over the radical of the associated bilinear form. In characteristic different from 2, this just means that the associated bilinear form is non-degenerate (has trivial radical). The polarity associated to the bilinear form is also non-degenerate. In characteristic 2, however, we distinguish between non-degenerate quadratic forms with degenerate associated bilinear form (and call them *inseparable*), and those with non-degenerate associated bilinear form (and call them *separable*). We extend this terminology to the other characteristics by calling every non-degenerate quadratic form in characteristic not 2 *separable.* We further extend this terminology to the orthogonal polar spaces themselves in the obvious way. In characteristic 2, the associated polarity of a separable orthogonal polar space is a non-degenerate null-polarity. The one of an inseparable orthogonal polar space is a degenerate null-polarity. Separable orthogonal polar spaces admit, up to isomorphism, a unique embedding in some projective space. The difference of the dimension of the ambient projective space with twice the rank, plus one, will be called the *anisotropic corank* of the polar space. It is the (maximal) vector dimension of an anisotropic form that is needed to describe the corresponding quadric. Quadrics with anisotropic corank equal to 0 are *hyperbolic quadrics* and correspond to buildings of type D_n. If the anisotropic corank is 1, we speak about *parabolic quadrics* and *parabolic polar spaces.*

Since hyperbolic quadrics are related to buildings of type D_n, and we determined the projectivity groups of buildings of type D_n in [5], we will not be much concerned with them in this article, except that they will show up in some proofs, when we extend the ground field (which we explain now in some more detail). According to Theorems 3.4.3 and 4.4.4 of [25], the standard equation of a quadric in $\mathsf{PG}(V)$, with V a $\mathbb{K}$-vector space, that corresponds to a polar space of rank r, is

$$X_{-r}X_r + \cdots + X_{-2}X_2 + X_{-1}X_1 = f_0(v_0, v_0),$$

where $V = V' \oplus V_0$, with $\dim V' = 2r$, $\{e_{-r}, \ldots, e_{-1}, e_1, e_2, \ldots, , e_r\}$ is a basis of V', and a vector $v \in V$ is written as $v = x_{-r}e_{-r} + \cdots + x_{-1}e_{-1} + x_1e_1 + x_2e_2 + \cdots x_re_r + v_0$, with $v_0 \in V_0$, and f_0 is an anisotropic quadratic form in V_0. In the finite dimensional case, if $\dim V_0$ is even, then the above equation becomes the equation of a hyperbolic quadric over a *splitting field*, that is, an overfield of $\mathbb{K}$ over which f_0 becomes completely reducible (one can take the algebraic or quadratic closure of $\mathbb{K}$). We will use this and note that hyperbolic quadrics have two natural systems of maximal singular subspaces. Two maximal singular subspaces belong to the same system if and only if the intersection has even codimension in each of them (the *codimension of* $U \cap V$ *in* U is $\dim U - \dim(U \cap V)$).

2.5.3. *Pseudo-quadratic polar spaces.* These are polar spaces arsing from σ-quadratic forms, where σ is an involution of a skew field $\mathbb{L}$ (and $\sigma \neq \mathsf{id}$). We briefly describe these. Let V be a right vector space over $\mathbb{L}$. Let $g : V \times V \to \mathbb{L}$ be a (σ, id)-linear form (meaning g is additive in both variables and $g(vk, w\ell) = k^\sigma g(v, w)\ell$, for all $k, \ell \in \mathbb{L}$ and $v, w \in V$). We define the following:

$$\begin{aligned}
&\mathbb{L}_\sigma := \{t - t^\sigma \mid t \in \mathbb{L}\}\\
&f\colon V \times V \to \mathbb{L},\ (v, w) \mapsto g(v, w) + g(w, v)^\sigma\\
&\mathfrak{q}\colon V \to \mathbb{L}/\mathbb{L}_\sigma,\ v \mapsto g(v, v) + \mathbb{L}_\sigma
\end{aligned}$$

Then f is a Hermitian sesquilinear form (meaning it is (σ, id)-linear and $f(v, w)^\sigma = f(w, v)$) and we denote its radical by R_f, that is, $R_f = \{v \in V \mid f(v, w) = 0, \forall w \in V\}$. We say that f *is associated to* $\mathfrak{q}$. Suppose $\mathfrak{q}$ is anisotropic over R_f, that is, $\mathfrak{q}(v) = 0$ if, and only if, $v = 0$, for all $v \in R_f$. Let X be the set of vectors $v \in V$ for which $\mathfrak{q}(v) = 0$. Suppose the subspaces of V of maximal dimension, that are contained in X, have finite dimension r. Then the point set $\mathscr{P} = \{\langle v\rangle \mid v \in X\}$ of $\mathsf{PG}(V)$, together with the lines induced from $\mathsf{PG}(V)$, is a polar space of rank r, called a *pseudo-quadratic polar space*. This description also makes sense for $\sigma = 1$ (and all embeddable polar spaces, except for the symplectic ones in characteristic distinct from 2, can be described like this). We call it the *pseudo-quadratic description*.

In the commutative case, the Hermitian sesquilinear form f determines the polar space (without using the associated pseudo-quadratic form; its points correspond to vectors $v \in V$ for which $f(v, v) = 0$) and we call the polar spaces *Hermitian*. Sometimes the form $V \to V$ with $v \mapsto f(v, v)$ is called a *Hermitian form*.

2.5.4. *Non-embeddable polar spaces.* There are two different kinds of non-embeddable polar spaces. Non-embeddable polar spaces of one kind are top-thin and arise as the line-Grassmannian of a 3-dimensional projective space over a non-commutative skew field. Since the projectivity groups of projective spaces are all well known (see for instance [5]), we will not need to consider these in the sequel.

The other non-embeddable polar spaces are thick and each of them arises as the fixed point structure of an involution in a building of type E_7. In fact, by the main result of [22, Chapter 9], for each non-Desarguesian Moufang plane π, there exists a unique polar space of rank 3, whose planes are isomorphic to π. We will not need an explicit construction of those polar spaces and refer to [9] for an elementary one. We will use some properties of these polar space derived in [17], where thick non-embeddable polar spaces are called *Freudenthal-Tits polar spaces*. However, we will need the standard description of the Moufang plane associated to a Cayley division algebra $\mathbb{O}$ over a field $\mathbb{K}$. We denote that plane as $\mathsf{PG}(2,\mathbb{O})$. Recall that $\mathbb{O}$ is an 8-dimensional, alternative, quadratic, non-associative composition division algebra with a standard involution $x \mapsto \overline{x}$, such that, for all $x \in \mathbb{O}$, we have $x\overline{x} \in \mathbb{O}$ and $x + \overline{x} \in \mathbb{O}$. An affine plane can be described as the set of ordered pairs $(x, y) \in \mathbb{O} \times \mathbb{O}$, with lines of two types: (1) For $m, k \in \mathbb{O}$, the line $[m, k]$ contains the points $(x, mx + k)$, for all $x \in \mathbb{O}$; (2) for $x \in \mathbb{O}$, the line $[x]$ contains the points (x, y), for all $y \in \mathbb{O}$. We will need this description in Section 7.2.4.

2.6. **Classification of metasymplectic spaces.** According to [22, Chapter 10], buildings of type F_4 are in one-to-one correspondence with pairs $(\mathbb{K}, \mathbb{A})$, where $\mathbb{K}$ is a field and $\mathbb{A}$ is a quadratic, alternative (composition) division algebra over $\mathbb{K}$. Usually, one labels the F_4-diagram linearly in such a way that the type set $\{1, 2\}$ corresponds to the residues isomorphic to $\mathsf{PG}(2,\mathbb{K})$, and $\{3, 4\}$ to those isomorphic to $\mathsf{PG}(2,\mathbb{A})$ (this is also called the *Bourbaki labelling*). We denote such building as $\mathsf{F}_4(\mathbb{K}, \mathbb{A})$. The metasymplectic spaces $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$ take the vertices of type 1 as points, and hence, its planes are defined over $\mathbb{K}$, whereas $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ takes the vertices of type 4 as points and has planes defined over $\mathbb{A}$. Quadratic alternative division algebras come in exactly five flavours. Note in advance that each such algebra is equipped with a *standard involution*, that is, and involutive anti-automorphism σ, such that $xx^\sigma \in \mathbb{K}$ and $x + x^\sigma \in \mathbb{K}$. The form $x \mapsto xx^\sigma$ is the *norm form*.

(1) $\mathbb{A} = \mathbb{K}$. Then $\mathsf{F}_4(\mathbb{K}, \mathbb{K})$ is the so-called *split building of type* F_4. The symps of $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{K})$ are parabolic polar spaces over $\mathbb{K}$; those of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{K})$ are symplectic polar spaces over $\mathbb{K}$.
(2) $\mathbb{A}$ is a separable quadratic extension of $\mathbb{K}$. The symps of $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$ are orthogonal polar spaces with anisotropic corank 2. The anisotropic quadratic form needed to describe those, is exactly the norm form of the Galois extension. The symps of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ are Hermitian polar spaces, naturally embedded into $\mathsf{PG}(5,\mathbb{A})$, and associated to the standard involution of $\mathbb{A}$ (which is the Galois involution).
(3) $\mathbb{A}$ is a quaternion division algebra over $\mathbb{K}$. The symps of $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$ are orthogonal polar spaces with anisotropic corank 4. The anisotropic quadratic form needed to describe these is exactly the norm form of the quaternion division algebra. The symps of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ are pseudo-quadratic polar spaces, naturally embedded into $\mathsf{PG}(5,\mathbb{A})$, and associated to the standard involution σ in $\mathbb{A}$.
(4) $\mathbb{A}$ is a Cayley division algebra (or octonion division algebra) over $\mathbb{K}$. The symps of $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$ are orthogonal polar spaces with anisotropic corank 8. The anisotropic quadratic form needed to describe these is exactly the norm form of the Cayley division algebra. The symps of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ are the thick non-embeddable polar spaces associated to $\mathbb{A}$.
(5) $\mathbb{A}$ is an inseparable extension of $\mathbb{K}$ in characteristic 2. The symps of both $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$ and $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ are inseparable orthogonal polar spaces.

In general, we denote a symp of $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$ as $\mathsf{B}_{3,1}(\mathbb{K}, \mathbb{A})$, and one of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ as $\mathsf{C}_{3,1}(\mathbb{A}, \mathbb{K})$. The corresponding dual polar spaces, which are point residues in $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ and $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$, respectively, are denoted as $\mathsf{B}_{3,3}(\mathbb{K}, \mathbb{A})$, and $\mathsf{C}_{3,3}(\mathbb{A}, \mathbb{K})$, respectively.

In Section 2.3 above, we defined the equator geometry of two opposite points of a metasymplectic space. We can now define the extended equator geometry for the metasymplectic space $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$. Let p, q be two opposite points of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$. Then the union of all equator geometries $E(x, y)$, for x, y opposite points varying over $E(p, q)$, together with all lines in these geometries, and in $E(p, q)$, is called the *extended equator geometry*, denoted as $\widehat{E}(p, q)$. Note that is contains p and q. It follows from [18] that $\widehat{E}(p, q)$ is a polar space whose residues are isomorphic to $E(p, q)$ (which are isomorphic themselves to $\mathsf{B}_{3.1}(\mathbb{K}, \mathbb{A})$, and hence, we denote such polar space as $\mathsf{B}_{4,1}(\mathbb{K}, \mathbb{A})$).

We now take a closer look at imaginary lines. Let p, q be two opposite points. By [15, Proposition 2.10.5], the imaginary line through p and q coincides with the hyperbolic line through p and q in the polar space $E(x, y)$, for any pair of opposite points x, y in $E(p, q)$. Hence, imaginary lines have size 2 in $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ when $\mathbb{A}$ is not an inseparable extension of $\mathbb{K}$, including the case $\mathbb{A}=\mathbb{K}$ when $\operatorname{char}\mathbb{K}=2$. In alll other cases, imaginary lines have size at least 3.

3. Some general observations for buildings of non-simply laced type

The present section is the only section, where we use some pure building theory. We briefly introduce the notions that we will need to use. For more background we refer to the book by Abramenko & Brown [1].

A (thick) spherical building is a (thick) simplicial chamber complex, such that every pair of simplices is contained in a finite thin chamber subcomplex, called an *apartment*, and every two apartments are isomorphic through an isomorphism fixing every vertex in their intersection. A *panel* is a simplex, which can be completed to a chamber by properly adding precisely one vertex. *Adjacent chambers* are chambers sharing a panel. The adjacency graph on the set of chambers induces a numerical distance functiion on the set of chambers. For each pair of adjacent chambers C, C' of an apartment there exists a unique *folding*, that is, an idempotent morphism with the property that every chamber is the image of zero or precisely two chambers, mapping C' to C. The image of a folding is called a *root*. If α is the root determined by the folding $C' \mapsto C$, then we denote by $-\alpha$ the *opposite root*; namely the one determined by the opposite folding $C \mapsto C'$. The intersection $\alpha \cap (-\alpha)$ is called the *boundary* of both α and $-\alpha$ and denoted $\partial\alpha$. The *interior* of α is $\alpha \setminus \partial\alpha$.

For a simplex F, the residue $\mathsf{Res}(F)$ is the simplicial complex induced on the set of vertices $v \notin F$ such that $F \cup \{v\}$ is a simplex. If we work in a building Δ, then, for clarity, $\mathsf{Res}(F)$ is sometimes also denoted by $\mathsf{Res}_\Delta(F)$. The *type* of F is the set of types of its members; the *cotype* is the the complementary set of types (with respect to the whole type set). The *type* of the residue $\mathsf{Res}(F)$ is the cotype of F. If F and F' are opposite simplices, then for each chamber C containing F there exists a chamber C' containing F' at minimal distance. The mapping $C \setminus F \mapsto C' \setminus F'$ induces an isomorphism from $\mathsf{Res}(F)$ to $\mathsf{Res}(F')$ (see [22, Theorem 3.28]), which we call a *perspectivity*. As in the previous section, this gives rise to the notions *(even) projectivity, self-projectivity*, the *special projectivity group* $\Pi^+(F)$, and the *general projectivity group* $\Pi(F)$.

Chambers in an apartment are called *opposite*, if they are never contained in the same root. This is independent of the chosen apartment and therefore, we say that chambers are opposite in a building, if they are opposite in an apartment. If the building is thick, then this is equivalent to saying that there exists a unique apartment containing the two chambers. All spherical buildings of rank at least 3 are *Moufang*, that is, for each root α, the group U_α fixing every chamber having a panel in the interior of α, acts transitively on the set of apartments containing α.

The aim of this section is to extend [5, Theorem A] to spherical Moufang buildings, which are not necessarily of simply laced type. We will prove the following theorem.

Theorem 3.1. *Let F be a simplex of a Moufang spherical building Δ. Let $\mathsf{Aut}^+(\Delta)$ be the automorphism group of Δ generated by the root groups. Then $\Pi^+(F)$ is permutation equivalent to the action of the stabiliser $\mathsf{Aut}^+(\Delta)_F$ of F in $\mathsf{Aut}^+(\Delta)$ on the residue $\mathsf{Res}_\Delta(F)$ of F in Δ.*

The proof of this theorem, given in [5] for the simply laced case, requires that the unipotent radical of a parabolic subgroup in a Moufang spherical building pointwise stabilises the corresponding residue, and acts transitively on the simplices opposite the given residue. For the simply laced case, this follows from the Levi decomposition of parabolic subgroups in Chevalley groups. In general, we can use [16, Proposition 24.21]:

Lemma 3.2. *Let Δ be a spherical Moufang building and let F be a simplex of Δ. Let G_F be the stabiliser of F in $\mathsf{Aut}^+(\Delta)$. Then there exists a subgroup $U_F \leq G_F$ which acts sharply transitively on the set $F^{\equiv}$ of simplices opposite F, and which pointwise fixes $\mathsf{Res}_\Delta(F)$.*

For completeness we describe how the subgroup U_F is constructed.

Let F' be a simplex in Δ opposite F. Choose an apartment Σ containing F and F' and a chamber C in Σ containing F. For a root α of Σ, let U_α be the corresponding root group; that is the group of automorphisms g of Δ that fixes all chambers that have a panel in the interior of α.

Then U_F is the group generated by all root groups U_α corresponding to roots α, such that α contains C, and F is in α, but not in the boundary $\partial\alpha$ of α.

Now exactly the same arguments as in [5] lead to Theorem 3.1. We also recall the following general rule, see [5, Observation 3.1].

Proposition 3.3. *Let Δ be a spherical building over the type set I and let $J \subseteq I$ be self-opposite. Let F be a simplex of type J. Then $\Pi^+(F) = \Pi(F)$ if, and only if, the identity in $\Pi(F)$ can be written as the product of an odd number of perspectivities.*

We will also use [5, Lemma 7.1], which we state now.

Lemma 3.4. *Let Δ be a spherical building over the type set I and let F_K be a simplex of type $K \subseteq I$. Let $K \subseteq J \subset I$ and let F_J be a simplex of type J containing F_K. Let $\Pi^+_K(F_J)$ be the special projectivity group of $F_J \setminus F_K$ in $\mathsf{Res}_\Delta(F_K)$. Then $\Pi^+_K(F_J) \leq \Pi^+(F_J)$.*

The following result follows directly from the fact that two chambers are always contained in an apartment, and each chamber of an apartment has a unique opposite chamber in that apartment.

Lemma 3.5. *In a thick spherical building Δ, given a pair (C, C') of distinct chambers, there exists a chamber D opposite C and not opposite C'.*

Finally we recall [5, Proposition 8.2]. A building is said to have *thickness at least* $t+1$, if every panel is contained in at least $t+1$ chambers.

Proposition 3.6. *If a spherical building has thickness at least $t+1$, then there exists a chamber opposite t arbitrarily given chambers. In particular, there exists a vertex opposite t arbitrarily given vertices of the same self-opposite type.*

4. General reduction theorems

In this section, we prove some results that reduce the computation of the projectivity groups to rather special cases. We also establish when the special and general projectivity groups generically coincide. We begin with the latter.

4.1. **Special versus general projectivity groups in polar spaces.** It is clear that the general projectivity group of the lower residue of a singular subspace contains dualities and the special group does not. Hence, these are always different.

Concerning the projectivity groups of the upper residues, the almost completely opposite situation holds. Indeed, we have the following result.

Proposition 4.1. *Let Δ be a thick polar space of rank at least 3. Let U be a non-maximal singular subspace of odd dimension at least 1. Then $\Pi^+_\geq(U) \equiv \Pi_\geq(U)$. The same conclusion holds, if Δ is not a separable orthogonal polar space and U has arbitrary dimension (but is non-maximal).*

Proof. Pick a singular subspace U' opposite U. If Δ is non-embeddable, then U and U' are points or lines and $(U^\perp \cap U'^\perp)^\perp$ is a thick polar space Γ of rank 1 or 2, respectively (this follows from Proposition 5.11 of [17]). Hence, there exists a point or line U'' in Γ opposite both U and U'. The projectivity of upper residues $U \barwedge U' \barwedge U'' \barwedge U$ is the identity, showing the assertion in this case.

Now suppose that Δ is embedded in $\mathsf{PG}(V)$, with V minimal; that is, the associated polarity is non-degenerate. In this case, the subspace Γ of Δ, induced by the subspace of $\mathsf{PG}(V)$ spanned by U and U', is a polar space distinct from a hyperbolic space of odd rank. Therefore, Γ contains a singular subspace U'' opposite both U and U'. Since the associated polarity in $\mathsf{PG}(V)$ is non-degenerate, we have $U'' \subseteq (U^\perp \cap U'^\perp)^\perp$ and so $U \barwedge U' \barwedge U'' \barwedge U$ is the identity. Now the assertion follows from Proposition 3.3. $\square$

4.2. **Special versus general projectivity groups in metasymplectic spaces.**

Proposition 4.2. *Let p be a point of a metasymplectic space Γ. Then $\Pi(p) = \Pi^+(p)$ as soon as there exists an imaginary line in Γ of size at least 3 and containing p.*

Proof. Let p, q, r be three points of the same imaginary line. Then all points symplectic to both p and q are also symplectic to r, which implies that $p \barwedge q \barwedge r \barwedge p$ fixes each symplecton through p, and is hence the identity. Now apply Proposition 3.3 $\square$

From [5, Lemma 5.2] it now follows, for a simplex S of type $\{1\}, \{1,4\}$ or $\{1,3,4\}$ of $\mathsf{F}_4(\mathbb{K}, \mathbb{A})$, with $\mathbb{K}$ a field and $\mathbb{A}$ an alternative quadratic division algebra over $\mathbb{K}$, that $\Pi(F) = \Pi^+(F)$. For simplices of type $\{1,4\}$ we will come back to this in an explicit way in the proof of Lemma 9.1.

Also, note that the condition stated in Proposition 4.2 is not necessary, as for split buildings the conclusion will hold without the condition being satisfied, see the first line of row (B3) in Table 1.

4.3. **Reduction to the product of three perspectivities.** In principle, to determine the projectivity groups, one has to consider arbitrarily long sequences of perspectivities. However, the following results will lead to the fact that projectivity groups are generated by self-projectivities which are products of at most four perspectivities in a particular sequence.

We say that a set Π of automorphisms of a polar space Δ is *geometric*, if its members are characterised by their fix structure. Formally, this means that an automorphism belongs to Π if, and only if, its fix set is a member of a certain given set of subsets of the point set of Δ, closed under the action of the full automorphism group of Δ. We will mainly apply this notion for fix structures being hyperplanes or subhyperplanes. We now phrase [5, Lemma 8.1] for our situation of polar spaces.

Lemma 4.3. *Let Δ be a polar space of rank r and let j, $0 \le j < r$, be an arbitrary natural number. If $j > 0$, then suppose that for each quadruple of singular subspaces of dimension j containing at least one opposite pair, there exists a singular subspace of dimension j opposite all the members of the given quadruple. If $j = 0$, suppose the same conclusion holds for each quadruple of points with the property that the pairwise intersections of the perps are not all the same. Let F, F', F'' be three pairwise opposite singular subspaces of dimension j and denote by θ_0 the projectivity $F \barwedge F' \barwedge F'' \barwedge F$ of upper residues, if $j < r-1$, and of lower residues, if $j = r-1$. Denote by $\Pi_3(F)$ the set of all corresponding self-projectivities of F of length 3 and suppose that $\Pi_3(F)$ is geometric. Then $\Pi(F) = \langle \Pi_3(F) \rangle$ and $\Pi^+(F) = \langle \theta_0^{-1}\theta \mid \theta \in \Pi_3(F) \rangle$.*

The conditions on the quadruple of singular subspaces do not appear in [5, Lemma 8.1]. However, in the proof, each considered quadruple (F_0, F_1, F_2, F_3) is part of a chain of perspectivities $F_0 \barwedge F_1 \barwedge F_2 \barwedge F_3$. Hence, F_0 is opposite F_1, which is opposite F_2, and F_2 is opposite F_3. Also, if $j = 0$, and if p_0, p_1, p_2 are points, such that $p_0^\perp \cap p_1^\perp = p_1 \cap p_2^\perp$, then $p_0 \barwedge p_1 \barwedge p_2 = p_0 \barwedge p_2$. So we may shorten the chain without altering the projectivity defined by the chain.

4.4. **Reduction to the product of four perspectivities.** Now we phrase [5, Lemma 8.17] in terms of polar spaces and our situation. In the subsequent remark, we improve on the conditions. But first a definition.

Definition 4.4. Let Δ be a polar space of rank $n \geq 2$. Let S be a set of singular subspaces of dimension s in Δ. We define the *s-space-graph* $\Gamma(S) = (S, \sim)$, with $\sim$ denoting the adjacency, as follows:

(V) The vertices are the elements of S.

(E)
- For $s = 0$, we draw an edge between two vertices in $\Gamma(S)$, if the corresponding points are collinear in Δ.
- For $s \in [1, n-2]$, we draw an edge between two vertices in $\Gamma(S)$, if the corresponding subspaces in Δ intersect in a subspace of dimension $s-1$ and are both contained in a common subspace of dimension $s+1$.
- For $s = n-1$, we draw an edge between two vertices in $\Gamma(S)$, if the corresponding maximal subspaces in Δ intersect in a subspace of dimension $s-1$.

Lemma 4.5. *Let Δ be a polar space of rank r and let j, $0 \leq j < r$, be an arbitrary natural number. Suppose that for each pair of singular subspaces F, F' of dimension j, the graph $\Gamma(S)$, where S is the set of singular subspaces opposite both F and F', is connected. Suppose also that there exists a singular subspace opposite any given set of three singular subspaces of dimension j. Let F be a given singular subspace of dimension j. Denote by $\Pi_4(F)$ the set of all self-projectivities $F \barwedge F_2 \barwedge F_3 \barwedge F_4 \barwedge F$ of F of length 4 with $F \sim F_3$, $F_2 \sim F_4$. Suppose that $\Pi_4(F)$ is geometric. Then $\Pi^+(F) = \langle \Pi_4(F) \rangle$.*

Remark 4.6. In Lemma 4.5 we may also assume that $F \cap F_3$ and $F_2 \cap F_4$ are opposite. Indeed, if not, then we can select in $\langle F, F_3 \rangle$ a subspace F_3' of dimension j disjoint from $\mathsf{proj}_F(F_2 \cap F_4)$. We then write

$$F \barwedge F_2 \barwedge F_3 \barwedge F_4 = (F \barwedge F_2 \barwedge F_3' \barwedge F_4 \barwedge F) \cdot (F \barwedge F_4) \cdot (F_4 \barwedge F_3' \barwedge F_2 \barwedge F_3 \barwedge F_4) \cdot F_4 \barwedge F.$$

Likewise, if $j \leq r-2$, we may also assume that $\langle F, F_3 \rangle$ and $\langle F_2, F_4 \rangle$ are opposite. Indeed, noting that a singular subspace U_1 through $F \cap F_3$ is opposite a singular subspace U_2 through $F_2 \cap F_4$ if, and only if, $U_1 \cap (F \cap F_3)^\perp \cap (F_2 \cap F_4)^\perp$ is opposite $U_2 \cap (F \cap F_3)^\perp \cap (F_2 \cap F_4)^\perp$ in the polar space $(F \cap F_3)^\perp \cap (F_2 \cap F_4)^\perp$ (where we may assume, due to the previous paragraph, that $F \cap F_3$ and $F_2 \cap F_4$ are opposite), it suffices to verify the claim for $i = 0$.

First suppose the line $L_1 := \langle F, F_3 \rangle$ intersects the line $L_2 := \langle F_2, F_4 \rangle$ in a point p. Select a plane π through L_1 not in a 3-space with L_2, and choose a point F_3' in π not on L_1 and not collinear to L_2. As above, we may substitute F_3 by F_3' in our sequence of perspectivities. Now none of $\langle F, F_3' \rangle$ or $\langle F_3, F_3' \rangle$ intersect L_2, and so we may assume now that L_1 does not intersect L_2. Then there is a unique point p_2 on L_2 collinear to all points of L_1, defining the plane $\pi := \langle p_2, L_1 \rangle$. We select a plane α through L_1 not in a 3-space with π. It follows that $\mathsf{proj}_\alpha L_2 \in L_1$. Thus we may consider any point F_3' of $\alpha \setminus L_1$ and we find that L_2 is opposite both $\langle F, F_3' \rangle$ and $\langle F_3, F_3' \rangle$. This completes the argument.

Lemma 4.7. *Let Δ be a metasymplectic space. Suppose that for each pair of points p, p', the graph $\Gamma(S)$, where S is the set of points opposite both p and p', is connected. Suppose also that there exists a point opposite any given triple of points. Let p be a given point. Denote by $\Pi_4(p)$ the set of all self-projectivities $p \barwedge p_2 \barwedge p_3 \barwedge p_4 \barwedge p$ of p of length 4 with $p \perp p_3$, $p_2 \perp p_4$ and pp_3 opposite p_2p_4. Suppose that $\Pi_4(F)$ is geometric. Then $\Pi^+(F) = \langle \Pi_4(F) \rangle$.*

Proof. The statement, without the condition that pp_3 is opposite p_2p_4, is [5, Lemma 8.17] specialised to buildings of type F_4 and vertices of type 1 or 4.

Now let $p \barwedge p_2 \barwedge p_3 \barwedge p_4 \barwedge p$ be a self-projectivity of p with $p \perp p_3$ and $p_2 \perp p_4$. Let L_3 be the projection of pp_3 onto the residue of p_2. Set $L_4 := p_2p_4$. In $\mathsf{Res}_\Delta(p_2)$, the elements L_3 and L_4 have distance $0, 1, 2$ or 3 from each other. If they have distance 3, then [22, Proposition 3.29] implies that pp_3 and p_2p_4 are opposite. Hence, we may assume that their distance d is 0, 1 or 2. Then there exists a line L_2 through p_2 at distance $d+1$ from L_3 in $\mathsf{Res}_\Delta(p)$ and at distance 1 from L_4 in that residue, that is, coplanar with L_4 in Δ. Let π_0 be the plane spanned by L_2 and L_4. Let L_1 be the projection of L_2 onto (the residue of) p, and let q_2 be a point on L_2 opposite

p. The projection of the line p_4q_2 from p_4 onto p is contained in the projection of π_0 onto p, is different from the projection of p_4p_2 onto p and hence, has distance $d+1$ to L_3. If we call the *width* of $p \barwedge p_2 \barwedge p_3 \barwedge p_4 \barwedge p$ the distance d in $\mathsf{Res}_\Delta(p)$ between the lines pp_3 and the projection of p_2p_4 onto p, then we can write

$$p \barwedge p_2 \barwedge p_3 \barwedge p_4 \barwedge p = (p \barwedge p_2 \barwedge p_3 \barwedge q_2 \barwedge p) \cdot (p \barwedge q_2 \barwedge p_3 \barwedge p_4 \barwedge p),$$

where the width of both self projectivities of p on the right hand side is $d+1$ (look at the inverse of the second to see this). An induction argument on d proves the assertion. □

Remark 4.8. One might wonder, why one would bother to reduce the computation of the special projectivity groups to the computation of self-projectivities of length 4, when we can reduce it to the computation of those of length 3 with Lemma 4.3. The reason is, firstly, that in general, a generic subspace opposite two given subspaces has an algebraically complicated form with many parameters (especially when the dimension of F is large). In the length 4 case (see Remark 4.6), there are essentially only two single parameters: one to fix F_3, and one to fix F_4, whereas $F, F_2, F \cap F_3$ and $F_2 \cap F_4$ can be chosen freely. Secondly, the conditions are different, and sometimes those of Lemma 4.5 are easier to meet than those of Lemma 4.3.

5. Preparations and auxiliary results

5.1. **Conditions for reduction for polar spaces.** In this subsection, we check the conditions of Lemma 4.3 and Lemma 4.5 in the cases where we shall apply them. We start with the polar spaces.

Regarding Lemma 4.3, we can be brief: the following is proved in [6].

Proposition 5.1. *Let Δ be a polar space of rank $r \geq 2$. Suppose first that each line contains at least four points and let $0 \leq j < r-1$. Then, if $j > 0$, for each quadruple of singular subspaces of dimension j containing at least one opposite pair, there exists a singular subspace of dimension j opposite all the members of the given quadruple. If $j = 0$, the same conclusion holds for each quadruple of points with the property that the pairwise intersections of the perps are not all the same. Secondly, suppose that each submaximal subspace is contained in at least four maximal singular subspaces, and $r \geq 3$, then each quadruple of maximal singular subspaces, containing at least one opposite pair, admits a common opposite singular subspace.*

Now we consider the main condition in Lemma 4.5. First we treat the rank 2 case, although we only consider the projectivity groups for rank at least 3 — we need the rank 2 case for induction purposes. Strictly speaking, we could assume that the rank 2 polar spaces we deal with are Moufang, but we prove the result for all generalised quadrangles (with at least four points per line and four lines through a point, respectively).

Proposition 5.2. *Let Δ be a thick polar space of rank 2. Let p_1 and p_2 be two points. Let S be the set of points opposite both p_1 and p_2. Then $\Gamma(S)$ is connected, if each line contains at least four points.*

Proof. Assume each line contains at least four points. Let q_1 and q_2 be two points in $\Delta \setminus (p_1^\perp \cup p_2^\perp)$. If q_1 and q_2 are collinear in Δ, then they are also collinear in $\Delta \setminus (p_1^\perp \cup p_2^\perp)$. Suppose q_1 and q_2 are opposite in Δ. Let L be a line through q_1 in Δ.

If $\mathsf{proj}_L(q_2)$ is not equal to $\mathsf{proj}_L(p_1)$ or $\mathsf{proj}_L(p_2)$, then $\mathsf{proj}_L(q_2)$ is a vertex of $\Delta \setminus (p_1^\perp \cup p_2^\perp)$ and the lines $q_2\mathsf{proj}_L(q_2)$ and $q_1\mathsf{proj}_L(q_2)$ give rise to edges in $\Gamma(\Delta \setminus (p_1^\perp \cup p_2^\perp))$ that form a path between the vertices corresponding to q_1 and q_2 via the vertex corresponding to $\mathsf{proj}_L(q_2)$.

Only if for every line L through q_1 it is the case that $\mathsf{proj}_L(q_2)$ is equal to either $\mathsf{proj}_L(p_1)$ or $\mathsf{proj}_L(p_2)$, we can not immediately find such a path. So assume this is the case. Then every point in $q_1^\perp \cap q_2^\perp$ is either collinear to p_1 or to p_2.

Suppose $p_1^\perp \cap p_2^\perp \cap q_1^\perp \cap q_2^\perp \neq \varnothing$ and let x be collinear to all of p_1, p_2, q_1, q_2. Let L be a line through q_1 not containing x. Without loss of generality, we may assume that the unique point

y on L collinear to q_2 is collinear to p_1. Let s_1 be a point on L not collinear to p_2, and distinct from q_1 and y. Let s_2 be the point on q_2x collinear to s_1. None of s_1 or s_2 is collinear to either p_1 or p_2 and we have the path $q_1 \sim s_1 \sim s_2 \sim q_2$.

Hence, we may suppose that each point of $q_1^\perp \cap q_2^\perp$ is collinear to either p_1, or p_2, but never to both. Without loss of generality, we may assume that p_1 is collinear to at least 2 points o_1, o_2 of $q_1^\perp \cap q_2^\perp$. If there are at least 5 points on a line, then there is a point r_1 on q_1o_1 distinct from all of $q_1, o_1, \mathsf{proj}_{q_1o_1}p_2$ and $\mathsf{proj}_{q_1o_1}\mathsf{proj}_{q_2o_2}p_2$. Then we have the path $q_1 \sim r_1 \sim \mathsf{proj}_{q_2o_2}r_1 \sim q_2$. Consequently, we may assume that there are exactly 4 points per line, and hence, in view of the main result of [2], there are a finite number of lines through each point, say n. By assumption, p_1 is collinear to $m \geq 2$ points $o_1, o_2, \ldots, o_m$ of $q_1^\perp \cap q_2^\perp$. Hence, there are $n - m$ lines through p_2 containing a point of $q_1^\perp \cap q_2^\perp$. The other m lines through p_2 each have to meet at least one of the lines q_10_i, q_2o_i, $1 \leq i \leq m$, and each such line must meet one of those m lines through p_2. It follows that there is a line L through p_2 meeting, without loss of generality, the line q_1o_1 and q_2o_2. Then the unique points r_1 and r_2 on q_1o_1 and q_2o_2, respectively, distinct from $q_1, q_2, o_1, o_2, q_1o_1 \cap L$ and $q_2o_2 \cap L$, are collinear and opposite both p_1 and p_2. Hence, we have the path $q_1 \sim r_1 \sim r_2 \sim q_2$. $\square$

Dually, we have:

Corollary 5.3. *Let Δ be a thick polar space of rank 2. Let L_1 and L_2 be two lines. Let S be the set of lines opposite both L_1 and L_2. Then $\Gamma(S)$ is connected, if each point is on at least four lines.*

Now we treat the higher rank cases.

Proposition 5.4. *Let Δ be a thick polar space of rank $n \geq 3$. Let U_1 and U_2 be two singular subspaces of dimension s. Let S_{U_1,U_2} be the set of all singular subspaces of dimension s opposite both U_1 and U_2. Then $\Gamma(S_{U_1,U_2})$ is connected for $s \leq n-2$, if each line has at least four points (this additional condition is not needed for $s = 0$) and for $s = n-1$, if either each submaximal singular subspace is contained in at least four maximal singular subspaces or each line contains at least four points.*

Proof. The proof goes with induction on the rank n, the base case being Proposition 5.2 and Corollary 5.3. However, for $s = 0$, we provide an independent proof, neglecting the additional conditions on the sizes of the lines.

Case 1: $s = 0$**.** Let p_1 and p_2 be two points. Let $\Delta \setminus (p_1^\perp \cup p_2^\perp)$ be the point-line geometry containing all points of Δ that are not collinear to either p_1 or p_2 and all lines of Δ between these points. These lines are exactly the lines L, which are not in a plane with either p_1 or p_2 and for which $L \setminus (\mathsf{proj}_L(p_1) \cup \mathsf{proj}_L(p_2))$ contains more than one point. We have to show that the point graph of $\Delta \setminus (p_1^\perp \cup p_2^\perp)$ is connected.

Let q_1 and q_2 be two arbitrary points of $\Delta \setminus (p_1^\perp \cup p_2^\perp)$. If q_1 and q_2 are collinear in Δ, then they are also collinear in $\Delta \setminus (p_1^\perp \cup p_2^\perp)$. Suppose q_1 and q_2 are opposite in Δ. As in the proof of Proposition 5.2, we may assume that every point in $q_1^\perp \cap q_2^\perp$ is either collinear to p_1 or to p_2. Since q_1 and q_2 are opposite, $q_1^\perp \cap q_2^\perp =: \Lambda$ defines a thick polar space of rank $n - 1 \geq 2$. Both $p_1^\perp \cap \Lambda$ and $p_2^\perp \cap \Lambda$ define geometric hyperplanes that we will denote by H_1 and H_2 and we have $\Lambda = H_1 \cup H_2$.

Suppose H_1 and H_2 are proper geometric hyperplanes. Then we obtain a contradiction, because a polar space can never be the union of two proper geometric hyperplanes (see Exercise 2.5 in [25]): Let x be a point of $H_1 \setminus H_2$, such that $x^\perp \neq H_1$. Then H_1 induces a proper geometric hyperplane in $\mathsf{Res}_\Lambda(x)$ and H_2 has to contain the complement. Let M be a line through x that is not contained in H_1. Then $M \setminus \{x\}$ has to be contained in H_2 and with that, x has to be contained in H_2, which is a contradiction.

So we may assume, without loss of generality, $p_1^\perp \cap \Lambda = \Lambda$. Since H_2 is a hyperplane, it contains two opposite points o_1 and o_2 and since $H_2 \subseteq H_1 = \Lambda$, the points o_1 and o_2 are both collinear to q_1, q_2, p_1 and p_2 and the lines q_1o_1 and q_2o_2 are opposite. Let x_1 be a point on $q_1o_1 \setminus \{q_1\}$ that is not in Λ. The projection of x_1 onto q_2o_2 is a point x_2 that is not in Λ and that does not coincide with q_2. Furthermore, x_1 and x_2 are not collinear to either p_1 or p_2 and thus contained in $\Delta \setminus (p_1^\perp \cup p_2^\perp)$. The lines q_1x_1, x_1x_2 and q_2x_2 give rise to edges in $\Gamma(S_{p_1,p_2})$ that form a path $q_1 \sim x_1 \sim x_2 \sim q_2$ between the vertices corresponding to q_1 and q_2.

This concludes the proof for the case $s = 1$. For cases 2 and 3 below we assume that each line has at least four points. But before moving to these other cases, we prove a common claim under our general assumptions.

Claim: *If $1 \leq s \leq n-1$, then different components of $\Gamma(S_{U_1,U_2})$ only contain vertices corresponding to disjoint subspaces.* Indeed, let V_1 and V_2 belong to $\Gamma(S_{U_1,U_2})$, with $x \in V_1 \cap V_2$. The projection of x onto U_i is a hyperplane H_i of U_i, $i \in \{1,2\}$. With [22, Proposition 3.29], it follows that every subspace X through x, that is opposite $\langle x, H_i \rangle$ in $\mathsf{Res}(x)$, is opposite U_i in Δ. Since $\mathsf{Res}(x)$ is a polar space of rank $n-1$, it follows from the induction hypothesis that we can find a path between the vertices corresponding to V_1 and V_2 in $\Gamma(S_{U_1,U_2})$. The claim is proved.

Case 2: $s = 1$**.** Remember the rank of Δ is at least 3, so Δ contains planes. Let L_1 and L_2 be two lines in Δ and let S_{L_1,L_2} be the set of all lines opposite both L_1 and L_2. Let J_1 and J_2 be two lines of S_{L_1,L_2}.

(I) If J_1 and J_2 intersect in a point, then by the above claim, we can find a path in $\Gamma(S_{L_1,L_2})$ connecting J_1 and J_2.

(II) Now suppose J_1 and J_2 do not intersect in a point, but there is some point $x \in J_1$ collinear to all points of J_2. Set $\pi := \langle x, J_2 \rangle$. Since J_j is opposite L_i, $i,j = 1,2$, the projection of L_i onto π is a point $u_i \in \pi \setminus (J_2 \cup \{x\})$. Select a line J_1' through x avoiding both u_1 and u_2, then J_1' is opposite both L_1 and L_2 and intersects both J_1 and J_2. By (I) we can connect J_1 with J_1' and J_1' with J_2.

(III) Finally, suppose that J_1 and J_2 are opposite. Let α_1 and β_1 be two arbitrary planes containing J_1 and let α_2 and β_2 be the planes through J_2 intersecting α_1 and β_1, respectively. Set $a := \alpha_1 \cap \alpha_2$ and $b := \beta_1 \cap \beta_2$. If either a or b is not collinear to either L_1 or L_2, say a, then we find a line J_1' through a in α_1 opposite both L_1 and L_2, and we are reduced to Cases (I) and (II). In the other case, say $a \perp L_1$ and $b \perp L_i$, $i \in \{1,2\}$, there is a unique point a' in α_1 collinear to L_2, and there is a unique point b' in β_2 collinear to L_j, $\{j\} = \{1,2\} \setminus \{i\}$. Since lines have at least four points, we can now find a line L_2' in β_2 distinct from both J_2 and $\mathsf{proj}_{\beta_2} a'$, and avoiding both b and b'. Then we can find a line J_1' in α_1 through $\mathsf{proj}_{\alpha_1} J_2'$ avoiding a and a'. Hence, both J_1' and J_2' are opposite both L_1 and L_2 and by (I), L_1 and L_1' are connected in $\Gamma(S_{L_1,L_2})$ and L_2 and L_2' are connected in $\Gamma(S_{L_1,L_2})$; by (II) also J_1' and J_2' are connected in $\Gamma(S_{L_1,L_2})$. Hence, L_1 and L_2 are connected in $\Gamma(S_{L_1,L_2})$.

Case 3: $2 \leq s \leq n-1$**.** Let U_1 and U_2 be two singular subspaces of dimension s and V_1 and V_2 both opposite both U_1 and U_2. On top of the induction on the rank of Δ, we also perform an induction on the dimension s of U_1 and U_2. The base cases here are Case 1 and Case 2.

(I) If V_1 and V_2 are not disjoint, then by the above claim, we can find a path in $\Gamma(S_{U_1,U_2})$ connecting V_1 and V_2.

(II) Secondly, assume $V_1 \cap V_2 = \varnothing$. Let A_1 be some $(s-1)$-dimensional subspace in V_1. Then A_1 is opposite some $(s-1)$-dimensional subspaces B_1 of U_1 and B_2 of U_2. Let A_2 be some $(s-1)$-dimensional subspace in V_2 that is opposite both B_1 and B_2. Then we can find a path between A_1 and A_2 in $\Gamma(S_{B_1,B_2})$. Let $A_1 = X_1, X_2, \ldots, X_r = A_2$ be the $(s-1)$-dimensional subspaces corresponding to all vertices of that path, such that X_j is adjacent to X_{j+1} for $j \in \{1,2,\ldots,r-1\}$. Then X_j is an $(s-1)$-dimensional subspace that intersects both X_{j-1} and X_{j+1} in $(s-2)$-dimensional subspaces and that is opposite both B_1 and B_2.

The projection of U_i onto X_j (for $j \in \{2,3,\ldots,r-1\}$ and $i \in \{1,2\}$) is a singular subspace W_{ij} of dimension s containing X_j. Let W_j be an s-dimensional subspace containing X_j opposite in $\mathsf{Res}(X_j)$ both W_{1j} and W_{2j}. Then, again using [22, Proposition 3.29], W_j is opposite both

U_1 and U_2. Set $W_1 := V_1$ and $W_r = V_2$. Since X_j intersects X_{j+1}, $j \in \{1, 2, \ldots, r-1\}$, and $X_j \subseteq W_j$, we find that W_j intersects W_{j+1}, for all $j \in \{1, 2 \ldots, r-1\}$. Now we can use (I) again the find paths between W_j and W_{j+1} in $\Gamma(S_{U_1,U_2})$, for $j \in \{1, 2 \ldots, r-1\}$, which, taken together, form one big path between $W_1 = V_1$ and $W_r = V_2$ in $\Gamma(S_{U_1,U_2})$. This concludes the proof of Case 3.

For the final case, we assume that each submaximal singular subspace is contained in at least four maximal singular subspaces.

Case 4: $s = n - 1$**.** Let M_1 and M_2 be two maximal singular subspaces and let N_1, N_2 be two maximal singular subspaces opposite both M_1 and M_2. As in the previous case, by the Claim above, we may assume that $N_1 \cap N_2 = \varnothing$, hence N_1 and N_2 are themselves opposite. We again proceed by induction on n, the case $n = 2$ being Corollary 5.3. So we assume $n > 2$. Select $x \in N_1$, and let N be the unique maximal singular subspace containing x and intersecting N_2 in a hyperplane of N_2. Then $M_i \cap N$ is at most a point, $i = 1, 2$, and we find a hyperplane H of N containing x and disjoint from both M_1 and M_2. Through H, there are at most two maximal singular subspaces not opposite either M_1 or M_2, and so there exists a maximal singular subspace N' through H opposite both M_1 and M_2. Applying the claim above to N_1 and N', and to N' and N_2 (which meet in $H \cap N_2 \neq \varnothing$), Case 4 is proved. □

5.2. **Conditions for reduction for metasymplectic polar spaces.** We now check the conditions of Lemma 4.7. We will only need the result for points.

First, we present two lemmas for dual polar spaces. We only need them in rank 3, but the proofs are the same for general rank.

Lemma 5.5. *A dual polar space Γ is not the union of two proper geometric hyperplanes.*

Proof. Let H and H' be two geometric hyperplanes of Γ with $H \cup H' = X$, where X is the point set of Γ. Recall that a *deep point* of H is a point of H such that $x^\perp \subseteq H$. Each point outside H belongs to H', as well as all points of H that are not deep. We now claim that a deep point x of H belongs to H' as soon as $x^\perp$ contains a point that is not deep. Indeed, let $y \perp x$ not be deep and let L be a line through y not contained in H. Let ξ be a symp containing x and L. Since x is deep, $H \cap \xi = x^\perp \cap \xi$. Hence, no point of $L \setminus \{x\}$ is deep. We conclude that $L \subseteq H'$, proving the claim. Hence, if H' is proper, then the set of deep points of H is closed under taking perps. Since Γ is connected, only X is a non-empty subset of points closed under taking perps. The lemma now follows. □

Lemma 5.6. *Let H and H' be two proper geometric hyperplanes of a dual polar space Γ. Then H contains a point x opposite some point $x' \in H'$.*

Proof. Let $y \in H$ be arbitrary. Suppose no point of H' is opposite y. Then $H' \subseteq y^{\not\equiv}$. Since no point at distance 2 can be removed from $y^{\not\equiv}$ without losing the property of being a hyperplane — and then also no point of $y^\perp$ can be deleted — we see that this implies $H' = y^{\not\equiv}$, Now let $x \in H$ be any other point of H (hence $x \neq y$). Lemma 3.5 yields a point x' not opposite y but opposite x. Then $x' \in H'$ and the assertion is proved. □

We are now ready to deal with the condition in Lemma 4.7 about the connectivity of $\Gamma(S)$.

Lemma 5.7. *Let Δ be a metasymplectic space in which either every line has at least four points or each plane is contained in at least 4 symps. For each pair of points p_0, p_1 the graph $\Gamma(S)$, where S is the set of points opposite both p_0 and p_1 is connected.*

Proof. (I) Let q_0 and q_1 be two points opposite both p_0 and p_1. If q_0 and q_1 are collinear in Δ, then they are connected in $\Gamma(S)$. Suppose q_0 and q_1 are symplectic in Δ and denote by $\xi(q_0, q_1)$ the symp containing both of them. Then no p_i, for $i \in \{0, 1\}$, can be contained in $\xi(q_0, q_1)$ and can also not be close to $\xi(q_0, q_1)$, since otherwise both q_0 and q_1 would not be opposite p_i. That means $p_i^{\perp\!\!\perp} \cap \xi(q_0, q_1)$ has to be a unique point r_i. We can find a path in the graph corresponding to $\xi(q_0, q_1) \setminus (r_0^\perp \cup r_1^\perp)$ between q_0 and q_1.

(II) Suppose q_0 and q_1 are special in Δ. Let q be the unique point collinear to both q_0 and q_1. Then we can not find a path in $\Gamma(S)$ between q_0 and q_1 immediately, if q is collinear to a point of $p_0^\perp$ or $p_1^\perp$. Let p be a point in $p_0^\perp \cup p_1^\perp$ collinear to q. We may assume $p \perp p_0$.

We have paths $p_0 \perp p \perp q \perp q_0$ and $p_0 \perp p \perp q \perp q_1$ and know that p_0 is opposite both q_0 and q_1. Therefore, we know that the pairs (p, q_0), (p, q_1) and (q, p_0) are special and if p_1 has distance 2 to q, then the pair (q, p_1) is also special. We first consider this case and define p' as the unique point of $p_1^\perp \cap q^\perp$.

In the residue of q we can see the lines pq, q_0q, q_1q and $p'q$ as points. Since $\mathsf{Res}(q)$ is a dual polar space, we can apply Proposition 5.4 and find a path between the points q_0q and q_1q in $\mathsf{Res}(q)$ all members of which are locally opposite pq and $p'q$. In Δ we can see that path as a chain of planes such that consecutive planes intersect in lines L locally opposite both pq and $p'q$. On each such line L, we choose a point distinct from both q and the projection of p_1 onto L. The thus obtained path in Δ connects q_0 with q_1 within $p_0^{\equiv} \cap p_1^{\equiv}$.

(III) Suppose q_0 and q_1 are opposite in Δ. Set $Z_0 = q_0^\perp \cap q_1^{\bowtie}$. If $q \in Z_0$ is opposite both p_0 and p_1, then we can find paths from q_0 to q and, by Case (II), from q to q_1 in $\Gamma(S)$ and the concatenation is a path between q_0 and q_1. Suppose such $q \in Z_o$ does not exist, in other words, $(p_0^{\not\equiv} \cup p_1^{\not\equiv}) \cap Z_0 = Z_0$. Now both $p_0^{\not\equiv}$ and $p_1^{\not\equiv}$ are geometric hyperplanes of Z_0. By Lemma 5.5, we may assume $Z_0 \subseteq p_0^{\not\equiv}$. Set $Y_0 = Z_0 \cap p_1^{\not\equiv}$. Likewise, defining $Z_1 = q_1^\perp \cap q_0^{\bowtie}$, one of $p_0^{\not\equiv}$ and $p_1^{\not\equiv}$ contains Z_1. The intersection of Z_1 with the other is denoted as Y_1. Let Y_1' be the projection of Y_1 onto Z_0. Then Lemma 5.6 yields a point $x_0 \in Y_0$ and a point $x_1' \in Y_1'$ with q_0x_0 locally opposite q_0x_1'. Projecting x_1' back onto Z_1, we obtain a point $x_1 \in Y_0$. Now pick $q_1' \in q_1x_1 \setminus \{x_1, q_1\}$ and note that, by this choice, $q_1' \in \Gamma(S)$. Then the projection q_0' of q_1' onto q_0x_0 is a point distinct from x_0 and hence, belongs to $\Gamma(S)$. Since x_0 is special to x_1, Case (II) implies that x_0 and x_1 are connected in $\Gamma(S)$. Hence, $q_0 \perp x_0$ is connected to $q_1 \perp x_1$ in $\Gamma(S)$ and the lemma is proved. □

Now we handle the condition in Lemma 4.7 about the existence of a point opposite three given points. It follows from Proposition 3.6 that this condition is automatically satisfied whenever the building has no residues isomorphic to the unique projective plane of order 2 (with 3 points per line). It follows from the main result in [7] that only triples of points that form a geometric line have no common opposite. But such a triple is determined by any pair of its elements. Now, it also follows from the proof of Lemma 4.7 in [5, Section 8] that the conclusion of Lemma 4.7, possibly without the claim of pp_3 being opposite p_2p_3, holds for all projectivities that can be written as a product

$$p_1 \,\overline{\wedge}\, p_2 \,\overline{\wedge}\, p_3 \,\overline{\wedge}\, \dots \,\overline{\wedge}\, p_{2\ell-1} \,\overline{\wedge}\, p_{2\ell} \,\overline{\wedge}\, p_1,$$

such that we can find a common opposite for the triples $\{p_1, p_{2k-1}, p_{2k+1}\}$, for all $k = 2, \dots, \ell-1$. Hence, if we were able to replace every subsequence $p_{2k-1} \,\overline{\wedge}\, p_{2k} \,\overline{\wedge}\, p_{2k+1}$, $k = 2, \dots, \ell-1$, for which $\{p_1, p_{2k-1}, p_{2k+1}\}$ is a geometric line, with a sequence $p_{2k-1} \,\overline{\wedge}\, p_{2k} \,\overline{\wedge}\, q_{2k} \,\overline{\wedge}\, p_{2k} \,\overline{\wedge}\, p_{2k+1}$, where $q_{2k} \equiv p_{2k}$ and none of $\{p_1, p_{2k-1}, q_{2k}\}$ or $\{p_1, q_{2k}, p_{2k+1}\}$ is a geometric line, then the conclusion of Lemma 4.5 would still hold. But this can be achieved since, by Lemma 3.5, there exists a point q_{2k} opposite p_{2k} and distinct from p_1. Since the proof of Lemma 4.5 in the present paper does not use the condition of the existence of a point opposite three given points, we conclude

Lemma 5.8. *Let Δ be a metasymplectic space. Let p be a given point. Denote by $\Pi_4(p)$ the set of all self-projectivities $p \,\overline{\wedge}\, p_2 \,\overline{\wedge}\, p_3 \,\overline{\wedge}\, p_4 \,\overline{\wedge}\, p$ of p of length 4 with $p \perp p_3$, $p_2 \perp p_4$ and pp_3 opposite p_2p_4. Suppose that $\Pi_4(F)$ is geometric. Then $\Pi^+(F) = \langle \Pi_4(F) \rangle$.*

5.3. **Collineations pointwise fixing a (sub)hyperpane.** We will see that self-projectivities of length 3 in $\Pi(x)$, for x a point in a polar space Δ, pointwise fix a hyperplane of $\mathsf{Res}_\Delta(x)$. Therefore, we prove some results about collineations of polar spaces pointwise fixing a hyperplane. We call a collineation of a polar space Δ that pointwise fixes a geometric hyperplane of Δ a *reflection* (In [12] it is called a *symmetry*).

Lemma 5.9. *Let Δ be a separable quadric of Witt index $r \geq 1$ in $\mathsf{PG}(V)$, for some vector space V over the field $\mathbb{K}$. We extend the perp-relation to all subspaces of $\mathsf{PG}(V)$ according to the non-degenerate polarity defined by the quadratic form defining Δ. Let θ be a central collineation of $\mathsf{PG}(V)$ with centre c and axis the hyperplane $H := c^\perp$. If θ maps some point x of Δ to a distinct point x^θ of Δ, then θ preserves Δ and consequently defines a collineation of Δ.*

Proof. If $\mathsf{char}\,\mathbb{K} \neq 2$, then this is a symmetry as defined by Dieudonné in [12, Section 10] and the result follows from that reference.

Now suppose $\mathsf{char}\,\mathbb{K} = 2$. This situation is less standard, especially when $\mathbb{K}$ is not perfect. We have to show that, if y is an arbitrary point of Δ, then θ maps y to a point of Δ. Considering the plane $\langle x, c, y\rangle$, this boils down to showing, given a conic $\mathscr{C}$ in a plane $\mathsf{PG}(2,\mathbb{K})$ and a line L through the nucleus n of $\mathscr{C}$, the existence of a non-trivial central elation with a given arbitrary centre c on L, with $c \neq n$. We can take the conic with equation $Y^2 = XZ$, and L has equation $X = kZ$, for some $k \in \mathbb{K}$. The centre c has coordinates $(\ell, 1, k\ell)$. Then the non-trivial elation pointwise fixing L with centre c and preserving $\mathscr{C}$ has matrix

$$\begin{pmatrix} (1+ab^2)^{-1} & 0 & b^2(1+ab^2)^{-1} \\ ab(1+ab^2)^{-1} & 1 & b(1+ab^2)^{-1} \\ a^2b^2(1+ab^2)^{-1} & 0 & (1+ab^2)^{-1} \end{pmatrix},$$

as one can check with elementary calculations. This concludes the proof. □

Lemma 5.10. *Let Δ be a separable quadric of Witt index $r \geq 1$ in $\mathsf{PG}(V)$, for some vector space V over the field $\mathbb{K}$. We extend the perp-relation to all subspaces of $\mathsf{PG}(V)$ according to the non-degenerate polarity defined by the quadratic form defining Δ. Let θ be a collineation pointwise fixing a subhyperplane G of $\mathsf{PG}(V)$. Set $L = G^\perp$. We assume that L is not a tangent, that is, that either $L \subseteq G^\perp$ with $L \cap \Delta = \varnothing$, or $L \cap G = \varnothing$, and the former only occurs in characteristic 2.*

Then θ is the product of two reflections.

Proof. Suppose first $L \cap G = \varnothing$. Then L is either disjoint from Δ, or intersects it in exactly two points. Let x be a point of Δ not contained in $G \cup L$. We also assume that, if L contains two points x_1, x_2 of Δ, then x is not contained in $x_1^\perp \cup x_2^\perp$. As θ fixes every subspace of G, it stabilises every subspace through L and hence, it stabilises the plane $\langle x, L\rangle$. So we can define $\{a\} = xx^\theta \cap L$. If $x \perp x^\theta$, then a belongs to Δ and so is one of x_1, x_2, contradicting our choice of x. Hence, a does not belong to Δ. Also, $a \notin x^\perp$ and $a \notin (x^\theta)^\perp$, since a is on a secant through x and x^θ. Since $L^\perp = G$, $a \in L$ implies $G \subseteq a^\perp$.

Let θ_1 be the unique central collineation with centre a that maps x to x^θ and fixes a^ρ pointwise. By Lemma 5.9, θ_1 defines a collineation of Δ. Define θ_2 to be $\theta\theta_1^{-1}$.

Let c be the point $\langle x, G\rangle \cap L$. Then c maps to $\langle x^\theta, G\rangle \cap L = \langle x^{\theta_1}, H\rangle \cap L$ under θ_1. Consequently $c^\theta = c^{\theta_1}$ and so c is fixed under $\theta\theta_1^{-1} = \theta_2$.

With that, $\theta\theta_1^{-1}$ fixes G pointwise and also fixes x and c. Hence, it pointwise fixes $\langle x, H\rangle$, and hence, $\theta = (\theta\theta_1^{-1})\theta_1$ is the product of two central collineations.

Secondly, suppose $L \subseteq H$. Now L does not contain any point of Δ and we may pick x in Δ, but not in G, arbitrarily. As before, we define a as the intersection of $xx^\theta \cap L$. We also know that x is not collinear to x^θ.

Define θ_1 as the elation that fixes $a^\perp$ pointwise, has centre a and maps x to x^θ. Then, again by Lemma 5.9, θ_1 induces a collineation in Δ.

The line xx^θ already has two points in the quadric, so there are no more points and therefore x^θ maps back to x. That means θ is an involution (which we can of course also deduce from the fact that $\mathsf{char}\,\mathbb{K} = 2$).

The map $\theta\theta_1^{-1}$ fixes H pointwise and fixes both x and x^θ. Hence, it pointwise fixes $\langle H, x\rangle$. Consequently, $\theta = (\theta\theta_1^{-1})\theta_1$ is the product of two central collineations. □

6. Projectivity groups for upper residues in polar spaces

6.1. **Projectivity groups of points.**

6.1.1. *The generic case.*

Proposition 6.1. *Let Δ be a thick embeddable polar space of rank $n \geq 3$. Let p_1 and p_2 be two opposite points of Δ, let Γ be the polar space $p_1^\perp \cap p_2^\perp$ of rank $n-1$ inside Δ and let H be a hyperplane of Γ, obtained as intersection of a projective hyperplane of some projective space with some embedding of Γ in that projective space. Let k and k' be two distinct non-collinear points of Γ, both not contained in H. Suppose $k^\perp \cap H = k'^\perp \cap H$. Then there exists a point p_3 in Δ that is opposite both p_1 and p_2 with $p_3^\perp \cap \Gamma = H$, such that the odd projectivity $p_1 \barwedge p_2 \barwedge p_3 \barwedge p_1$ fixes all lines joining p_1 with a point of H, and moves p_1k to p_1k'.*

Proof. The fact that $p_1 \barwedge p_2 \barwedge p_3 \barwedge p_1$ fixes all lines through p_1 having a point in H implies that p_3 is collinear to all points of H. Also, the fact that $p_1 \barwedge p_2 \barwedge p_3 \barwedge p_1$ maps p_1k to p_1k' implies that p_3 is on a line L that intersects both p_1k' and p_2k. We select an arbitrary point k'' on the line p_1k' distinct from both p_1 and k', and we let L be the unique line through k'' intersecting p_2k. Set $K = k^\perp \cap H$. If H contains lines, then K is a hyperplane of H and so H is determined by K and some point $h \in H \setminus K$, meaning that, if a point is collinear to K and h, then it is collinear to H. If H only contains points, then the same follows from our assumption that H arises as the intersection of a projective hyperplane of some projective space with some embedding of Γ in that projective space.

We now define p_3 as the unique point of L collinear to h. Since all of p_1, p_2, k and k' are collinear to K, also k'' and all other points of L are collinear to K. In particular, p_3 is collinear to K, and since it is also collinear to h, it is collinear to H.

It follows that $p_1 \barwedge p_2 \barwedge p_3 \barwedge p_1$ fixes all lines joining p_1 with a point of H, and moves p_1k to p_1k'. $\square$

Corollary 6.2. *Let Δ be a thick polar space of rank $n \geq 3$ and let H be a hyperplane of Δ. Let k, k' be two points not contained in H such that $k^\perp \cap H = k'^\perp \cap H$. Then there exists a unique collineation θ of Δ pointwise fixing H and mapping k to k'.*

Proof. First we prove uniqueness,

Let K be the set $k^\perp \cap H = k'^\perp \cap H$. Let x be some point of K. Then the line kx maps to the line $k'x = k^\theta x$. Let a be some point on kx not equal to k or x. Then a^θ has to be on the line $k^\theta x$. Let y be some point in $(a^\perp \cap H) \setminus (k^\perp \cap H)$. Then a is the unique point of kx collinear to y, and with that, a^θ is uniquely determined as the unique point on $k^\theta x$ collinear to $y^\theta = y$. That means the images of all points in $k^\perp$ and H are uniquely determined. Playing the same game with all points of $k^\perp \setminus H$, and then again and again shows that θ is uniquely determined as the complement of a hyperplane in a polar space is always connected.

Next we prove existence. If Δ is non-embeddable, then by [8], the only hyperplanes are of the form $p^\perp$, for some point p, and then θ is a central elation (see [25, Chapter 5]). If Δ is embeddable, then we can view it as the intersection of $p_1^\perp$ and $p_2^\perp$, for two opposite points p_1, p_2 of a polar space of rank $n+1$. Then existence follows from Proposition 6.1. $\square$

Remark 6.3. It can happen that a (thick) polar space Δ of rank at least 3 possesses a hyperplane H, but that there is no nontrivial collineation of Δ pointwise fixing it. This is not in contradiction with Corollary 6.2, as for such hyperplanes H there do not exist two distinct points k, k' with $k^\perp \cap H = k'^\perp \cap H$. This situation occurs for instance in symplectic polar spaces over fields of characteristic 2. Indeed, Let Δ be a symplectic polar space of rank r at least 3 over a field $\mathbb{K}$ with $\mathsf{char}\,\mathbb{K} = 2$. The universal embedding of such a space corresponds with a (inseparable) quadric, and hence there are geometric hyperplanes of Δ which, as subsets of points of $\mathsf{PG}(2r-1, \mathbb{K})$ (in which Δ is naturally embedded), generate $\mathsf{PG}(2r-1, \mathbb{K})$. Let H be such a hyperplane. Select $2r-1$ points of H that generate a hyperplane J of $\mathsf{PG}(2r-1,\mathbb{K})$. Then $J=k^\perp$ for a unique point k of Δ. Hence there is no second point k' with $k^\perp\cap H=k'^\perp\cap H$.

If we combine our results of Proposition 4.1, Lemma 4.3, Proposition 5.1 and Proposition 6.1, we obtain the following theorem.

Theorem 6.4. *Let Δ be a polar space of rank at least 3 with at least 4 points per line, and let p be a point. Then $\Pi(x)$ is the subgroup of the automorphism group of $\mathsf{Res}(p)$ generated by all reflections of $\mathsf{Res}(p)$. If Δ is not a separable orthogonal polar space, then $\Pi^+(p)=\Pi(x)$. If Δ is a separable orthogonal polar space, then $\Pi^+(p)$ is the subgroup of $\mathsf{Aut}\,\mathsf{Res}(p)$ consisting of products of an even number of reflections.*

We have the following immediate consequence (taking into account that a polar space, which is not a separable quadric, automatically has at least 5 points per line, except, if it is a symplectic polar space over $\mathbb{F}_2$).

Corollary 6.5. *Let Δ be an embeddable, but not separable orthogonal or symplectic polar space of rank at least 2. Then the group of collineations generated by all reflections coincides with the group of collineation, which are the product of an even number of reflections.*

6.1.2. *Hermitian case and symplectic polar spaces.* Noting that, in the case that Δ is embeddable, in the proof of Proposition 6.1, the hyperplanes H arise as intersections of subspaces with Γ, we gather some immediate consequences of Theorem 6.4 in the following statements.

Corollary 6.6. (*i*) *If Δ is the polar space arising from a non-degenerate Hermitian form (including symmetric bilinear forms) in a finite dimensional vector space over a (commutative) field, then, for each point p, the group $\Pi(p)$ is the full linear group preserving the Hermitian form that defines $\mathsf{Res}(p)$.*

(*ii*) *If Δ is a symplectic polar space, and p is a point of Δ, then $\Pi(p)=\Pi^+(p)$ is the simple symplectic group corresponding to $\mathsf{Res}(p)$, except, if $\mathsf{Res}(p)$ has rank 2 and each line exactly has 3 points, in which case the group is isomorphic to the symmetric group on 6 letters.*

Proof. The bilinear case of (*i*) follows directly from Propositions 8 and 14 of [12]. Now let the form that defines Δ be Hermitian with nontrivial field automorphism σ. Since $\Pi^+(p)$ contains the little projective group of $\mathsf{Res}_\Delta(p)$, [12, Théorème 5] implies that it suffices to show that each field element x with $xx^\sigma=1$ can be obtained as a determinant of a reflection. It suffices to consider the 2-dimensional case (vector space dimension), where this is immediate: the mapping $(x,y)\mapsto(x,ay)$, with $aa^\sigma=1$, preserves the Hermitian form $xx^\sigma+kyy^\sigma$, has determinant a and fixes the hyperplane $(0,1)$ (and also the point $(1,0)$, which is the perp of $(0,1)$). Now (*i*) follows.

We now show (*ii*). By Remark 6.3, $\Pi^+(p)$ is generated by reflections for which the fixed hyperplanes are point perps. These are elations and hence $\Pi^+(p)$ is the simple symplectic group corresponding to $\mathsf{Res}(p)$, if lines have at least 4 points. By Proposition 4.1, $\Pi(p)=\Pi^+(p)$. The statement (*ii*) for polar spaces with exactly three points per line follows from Theorem 3.1, as the stabiliser of a point in the simple symplectic group is the symplectic group of the residue. (This argument can in fact also be used as an alternative for the larger symplectic polar spaces.) $\square$

There remain three things to be addressed in more detail: (1) If Δ is a separable orthogonal polar space, can we be more specific about when $\Pi^+(p)=\Pi(p)$ (in other words, are there generic situations in which this equality *always* or perhaps *never* holds true)? (2) If Δ is non-embeddable, then the description of $\Pi(p)$ above is not very transparent; can we provide a description using the bilinear form defining the dual (embeddable) generalised quadrangle? (3) The case when lines of Δ have exactly three points. We begin with (1).

6.1.3. *Separable orthogonal polar spaces.*

Proposition 6.7. *Let Δ be a separable, orthogonal polar space of rank n with at least 4 points per line, and anisotropic codimension r. Let p be a point. If r is even, then $\Pi^+(p)$ has index 2 in $\Pi(p)$. If r is odd, then $\Pi^+(p) = \Pi(p)$.*

Proof. If r is even, then each member of $\Pi^+(p)$ preserves the system of generators of the (imaginary) hyperbolic quadric over a splitting field of the quadric, whereas each single reflection interchanges these systems.

Now let r be odd. Since $\Pi(p)$ contains reflections with arbitrary spinor norm, in particular also non-trivial elements with trivial spinor norm, we find products of two reflections (which each are products of three perspectivities) with arbitrary spinor norm. The proposition follows. $\square$

6.1.4. *Polar spaces with* 3 *points per line.* Now, we address (3). For symplectic polar spaces (which are isomorphic to parabolic polar spaces), this is already contained in Corollary 6.6(*ii*). Hermitian polar spaces never contain exactly three points per line. There remain the so-called *elliptic* ones, having anisotropic dimension 2. For these, Proposition 6.1 implies that $\Pi(p)$ contains the special (projective) orthogonal group (which is generated by reflections by[12, Proposition 14]). Now, it is clear that Proposition 6.7 is also valid in the present case, and therefore $\Pi^+(p)$ is the simple orthogonal group (which has index 2 in the special orthogonal group).

6.1.5. *Non-embeddable polar spaces.* In order to deal with the thick non-embeddable polar spaces, we first take a look at residues isomorphic to the pseudo-quadratic polar spaces of rank 2 that are dual to separable orthogonal polar spaces of rank 2. We will need the notion of a similitude of a (quadratic or Hermitian) form.

A *similitude* of a form is a linear transformation preserving the form up to a non-zero constant, and in case of a quadratic form, it is called *direct* if the form is hyperbolic over a splitting field and the similitude preserves each natural system of maximal singular subspaces.

Let $\mathbb{K}$ be a field with an involution σ, and let $\mathbb{F}$ be the subfield pointwise fixed by σ. Let $\Delta(\mathbb{K},\mathbb{F})$ be the pseudo-quadratic polar space of rank 3 given by the pseudo-quadratic form

$$X_{-3}^{\sigma}X_3 + X_{-2}^{\sigma}X_2 + X_{-1}^{\sigma}X_1 \in \mathbb{F}.$$

In what follows, it is also allowed that $\mathbb{K}$ has characteristic 2 and is an inseparable quadratic extension of $\mathbb{F}$ (and then σ is the identity).

Let p_i be the base point corresponding to the x_i-coordinate, $i \in \{-3,-2,-1,1,2,3\}$. Let p be the point with coordinates $(1, a_{-2}, 0, 0, a_2, k - a_{-2}^{\sigma}a_2)$, with $a_{-2}, a_2 \in \mathbb{K}$ and $k \in \mathbb{F}$. Also, let $p(\ell_1,\ell_2)$ be the point with coordinates $(0,\ell_1,0,0,\ell_2,0)$, with $\ell_1,\ell_2 \in \mathbb{F}$. We assume that p is not collinear to either p_{-3} or p_3, that is, $k - a_{-2}^{\sigma}a_2 \neq 0$.

An elementary calculation shows that the projectivity $p_{-3} \barwedge p_3 \barwedge p \barwedge p_{-3}$ maps the plane $\langle p_{-3}, p_{-1}, p(\ell_1,\ell_2)\rangle$ to the plane

$$\langle p_{-3}, p_{-1}, p((a_{-2}^{\sigma}a_2 + a_2^{\sigma}a_{-2} - k)\ell_1 - a_{-2}^{\sigma}a_{-2}\ell_2, a_2a_2^{\sigma}\ell_1 - k\ell_2)\rangle.$$

It follows that, if Γ is the separable, orthogonal polar space of rank 2 dual to $p_{-3}^{\perp} \cap p_3^{\perp}$, and L is the line of Γ corresponding to the point p_{-1}, then the action on L by the above projectivity, is given in binary coordinates by

$$\begin{pmatrix}\ell_1\\ \ell_2\end{pmatrix} \mapsto \begin{pmatrix} a_{-2}^{\sigma}a_2 + a_2^{\sigma}a_{-2} - k & -a_{-2}^{\sigma}a_{-2} \\ a_2^{\sigma}a_2 & -k \end{pmatrix} \cdot \begin{pmatrix}\ell_1\\ \ell_2\end{pmatrix} =: A \cdot \begin{pmatrix}\ell_1\\ \ell_2\end{pmatrix}.$$

The determinant of the 2×2-matrix A in the above expression is

$$(k - a_{-2}^{\sigma}a_2)\cdot(k - a_{-2}^{\sigma}a_2)^{\sigma}.$$

Let Γ be embedded in $\mathsf{PG}(5,\mathbb{F})$. Note, that the fixed points of the projectivity above form an ovoid of $p_{-3}^{\perp}\cap p_3^{\perp}$ and hence, a spread of Γ. Taking six base points on three spread lines, amongst which we can choose L, we claim that the determinant of the matrix of the corresponding collineation is

$$[(k - a_{-2}^{\sigma}a_2) \cdot (k - a_{-2}^{\sigma}a_2)^{\sigma}]^3.$$

Indeed, let M be a second line with two base points. We can choose the basis such that the equation of $\Gamma \cap \langle L, M\rangle$ is $x_0x_2 + x_1x_3 = 0$, where $L = \langle(1,0,0,0),(0,1,0,0)\rangle$ and $M = \langle(0,0,1,0),(0,0,0,1)\rangle$. The projectivity now fixes every line of $\Gamma \cap \langle L, M\rangle$ disjoint from L (and also L). One calculates that the matrix of the collineation is the block matrix

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Doing this for the third line with base points again, the claim follows.

Hence, we conclude that special and general projectivity group of the upper residue of p_{-3}, viewed as collineation group of Γ, consist of all similitudes of the associated quadratic form, with factor a norm of the quadratic extension $\mathbb{K}/\mathbb{F}$ (note that each square of $\mathbb{F}$ is such a norm). Writing the equation (in some other basis) of Γ as $y_{-2}y_2 + y_{-1}y_1 = yy^{\sigma}$, where we view the underlying vector space as $\mathbb{F} \times \mathbb{F} \times \mathbb{K} \times \mathbb{F} \times \mathbb{F}$, with generic coordinates $(y_{-2}, y_{-1}, y, y_1, y_2)$, we see that the factor of each similitude is a norm.

We can now state:

Proposition 6.8. *Let Δ be the thick non-embeddable polar space associated with the Cayley division algebra $\mathbb{O}$ over the field $\mathbb{F}$ with standard involution $x \mapsto \overline{x}$. Let p be any point of Δ. Then $\Pi(p) = \Pi^+(p)$ is the group of all direct similitudes of the quadratic form*

$$q : \mathbb{F} \times \mathbb{F} \times \mathbb{O} \times \mathbb{F} \times \mathbb{F} \to \mathbb{F} : (x_{-2}, x_{-1}, x, x_1, x_2) \mapsto x_{-1}x_1 + x_{-2}x_2 + x\overline{x},$$

and the factor of such similitude is a norm of some element of $\mathbb{O}$.

Proof. Let p be a point of Δ. The rank 2 polar space $\mathsf{Res}_{\Delta}(p)$ is dual to a separable quadric of Witt index 2 in $\mathsf{PG}(11, \mathbb{F})$, given by the quadratic form mentioned in the statement. Clearly, $\Pi(p) = \Pi^+(p)$ is contained in the group of all direct similitudes of that quadratic form. So, it sufficed to prove that each such similitude can occur. But this follows from the discussion above taking into account that the pseudo-quadratic polar space $\Delta(\mathbb{K}, \mathbb{F})$, with $\mathbb{K}$ any 2-dimensional subfield of $\mathbb{O}$ containing $\mathbb{F}$, is a sub polar space of Δ, with the property that all planes of Δ through a line of $\Delta(\mathbb{K}, \mathbb{F})$ belong to $\Delta(\mathbb{K}, \mathbb{F})$. □

6.2. **Subspaces of dimension at least 1.**

Lemma 6.9. *Let Δ be a polar space of rank $n \geq 3$ embedded into $\mathsf{PG}(V)$ that is not a separable quadric. Let A_1, A_2, A_3 and A_4 be singular subspaces of dimension $m \leq n-2$, such that $A_1 \cap A_3 =: B$ and $A_2 \cap A_4 =: C$, with B and C opposite and of dimension $m-1$, such that A_i is opposite A_{i+1} for $i \in \mathbb{Z}/4\mathbb{Z}$ and such that A_1 and A_3 are not contained in a common subspace. Then $A_1 \barwedge A_2 \barwedge A_3 \barwedge A_4 \barwedge A_1$ is a product of two collineations in $\mathsf{Res}_{\Delta}(A_1)$ that each fix a hyperplane.*

Proof. We may assume that $\mathsf{PG}(V)$ is minimal, that is, the relation $\perp$ defines a non-degenerate polarity of $\mathsf{PG}(V)$.

(I) We will first assume that $m \leq n-3$. Then $A_1^{\perp} \cap A_2^{\perp}$ has rank $n-m-1$. Since $n-m-1 \geq 2$, $A_1^{\perp} \cap A_2^{\perp} \cap A_3^{\perp}$ is a hyperplane of $A_1^{\perp} \cap A_2^{\perp}$ that we will denote by H. If $A_4^{\perp} \cap A_1^{\perp} \cap A_2^{\perp} = H$, then H is fixed pointwise. So suppose $A_4^{\perp} \cap A_1^{\perp} \cap A_2^{\perp} \neq H$. Define:

$$x_1 := C^{\perp} \cap A_1, \quad x_2 := B^{\perp} \cap A_2,$$
$$x_3 := C^{\perp} \cap A_3, \quad x_4 := B^{\perp} \cap A_4$$

Since B and C are opposite, $x_1 \neq x_3$ and $x_2 \neq x_4$. In $\mathsf{PG}(V)$, there are planes $\langle x_1, x_2, x_3\rangle$ and $\langle x_1, x_3, x_4\rangle$ which share the line $\langle x_1, x_3\rangle$ in $\mathsf{PG}(V)$. On the line $\langle x_1, x_3\rangle$ in $\mathsf{PG}(V)$, there exists some other point that is contained in Δ, since Δ is not a separable quadric, and we denote it by

x_5. Since every point of C is collinear to x_1 and x_3, it follows that every point of C is collinear to x_5. The subspace $\langle C, x_5\rangle$ is a subspace in Δ and we denote it by A_5. Now $A_5^\perp \cap A_1^\perp \cap A_2^\perp = H$ and with that, we can write φ as the product of the projectivities

$$\varphi_1 \colon A_1 \barwedge A_2 \barwedge A_3 \barwedge A_5 \barwedge A_1 \text{ and } \varphi_2 \colon A_1 \barwedge A_5 \barwedge A_3 \barwedge A_4 \barwedge A_1.$$

Then φ_1 fixes H pointwise and, similarly, φ_2 fixes a hyperplane of $\mathsf{Res}_\Delta(A_1)$ pointwise. Therefore, $\varphi = \varphi_1\varphi_2$ is the product of two collineations in $\mathsf{Res}_\Delta(A_1)$ that each fix a hyperplane pointwise.

(II) Now suppose $m = n-2$. Then the intersection $A_1^\perp \cap A_2^\perp \cap A_3^\perp$ could be empty. In that case, we consider the subspace of $\mathsf{PG}(V)$ spanned by $A_3^\perp$ and intersect it with the subspace spanned by $A_1^\perp \cap A_2^\perp$. This intersection will be a hyperplane of the subspace spanned by $A_1^\perp \cap A_2^\perp$ and we take this as H. If the hyperplane spanned by $A_4^\perp$ intersects the space spanned by $A_1^\perp \cap A_2^\perp$ in H as well, we are done. Otherwise, we do the same construction as before and obtain a subspace A_5, such that the hyperplane spanned by $A_5^\perp$ intersects the space spanned by $A_1^\perp \cap A_2^\perp$ in H and $\varphi = \varphi_1\varphi_2$ as before with the same definitions for φ_1 and φ_2, and the same conclusion holds. $\square$

Lemma 6.10. *Let Δ be a polar space of rank $n \geq 3$ embedded into $\mathsf{PG}(V)$ that is a separable quadric. Let A_1, A_2, A_3 and A_4 be singular subspaces of dimension $m \leq n-2$, such that $A_1 \cap A_3 =: B$ and $A_2 \cap A_4 =: C$, with B and C opposite and of dimension $m-1$, such that A_i is opposite A_{i+1} for $i \in \mathbb{Z}/4\mathbb{Z}$ and such that A_1 and A_3 are contained in a common subspace and A_2 and A_4 are contained in an opposite subspace. Then $\theta \colon A_1 \barwedge A_2 \barwedge A_3 \barwedge A_4 \barwedge A_1$ is a product of two collineations in $\mathsf{Res}_\Delta(A_1)$ that each fix a hyperplane in the ambient projective space.*

Proof. Since Δ is a separable quadric, we can view the perp relation as a non-degenerate polarity. Let $\Gamma := A_1^\perp \cap A_2^\perp$. Then Γ has rank $n-m-1$. Since $m \leq n-2$, the rank of Γ is at least 1. Let H be the hyperplane $A_3^\perp \cap \langle\Gamma\rangle$ and G be the subhyperplane $A_4^\perp \cap H$. If $G = H$, we are done. So suppose G is a proper subset of H.

Define:

$$x_1 := C^\perp \cap A_1, \quad x_2 := B^\perp \cap A_2,$$
$$x_3 := C^\perp \cap A_3, \quad x_4 := B^\perp \cap A_4$$

Let c_1 be the point $A_2^\perp \cap \langle A_1, A_3\rangle$ and c_2 be the point $A_1^\perp \cap \langle A_2, A_4\rangle$. Note that both c_1 and c_2 belong to Γ. We have $A_2^\perp \cap \langle A_1, A_3\rangle \subseteq C^\perp \cap \langle A_1, A_3\rangle = x_1x_3$. That means c_1 is on the line x_1x_3 and analogously c_2 is on the line x_2x_4. Every point of G is collinear to x_1 and x_3 and with that also to c_1 and analogously to c_2. That means $G \subseteq c_1^\perp \cap c_2^\perp$, and since G is a subhyperplane of Γ, we have $G = \langle\Gamma\rangle \cap c_1^\perp \cap c_2^\perp$. We may view θ as a collineation in Γ; the fixed points correspond precisely with the points of G.

If $c_1 \perp c_2$, then the singular subspace generated by A_1 and A_3 is not opposite the one generated by A_2 and A_4, contradicting our hypothesis. So c_1 and c_2 are not collinear. Then we can apply Lemma 5.10 and the proof is complete. $\square$

Lemma 6.11. *Let Δ be a polar space of rank $r \geq 3$, which is a separable quadric in $\mathsf{PG}(V)$. Let $1 \leq d \leq r-2$. Let U_1 and U_2 be two opposite singular subspaces of dimension d. Let W_1 and W_2 be two singular subspaces of dimension $d+1$, containing U_1 and U_2, respectively, and intersecting in a point p. Let U_3 be a singular subspace of dimension d intersecting $\langle U_1, U_2\rangle$ in a $(d-1)$-dimensional subspace H. Suppose p is not collinear to all points of U_3. Then the image of W_2 under $U_2 \barwedge U_3 \barwedge U_1$ is W_1 if, and only if, d is odd.*

Proof. Set $u_i = \mathsf{proj}_{U_i} H$, $i = 1,2$. Since $U_1, U_2 \subseteq p^\perp$, we find that $H \subseteq p^\perp$. It follows that the line pu_i is the projection of H onto W_i, $i = 1,2$. Consequently, $\mathsf{proj}_{W_i} U_3 =: x_i \in pu_i$, $i = 1,2$. Note that by assumption, $p \notin \{x_1, x_2\}$. We conclude that the image of W_2 under $U_2 \barwedge U_3 \barwedge U_1$ is W_1 if, and only if, $x_1 \perp x_2$ (the "if"-part follows from the fact that the $(2d+1)$-dimensional subspace of $\mathsf{PG}(V)$ generated by U_1 and U_2 does not contain singular subspaces of dimension

$d+1$ as it intersects Δ in a non-degenerate hyperbolic quadraic). This is equivalent to $u_1 \perp u_2$, which, on its turn, is equivalent to $u_1 \in \langle H, u_2\rangle$. This happens if, and only if, U_1 and U_2 belong to the same natural system of maximal singular subspaces of the hyperbolic quadric obtained by restricting Δ to $\langle U_1, U_2\rangle$. Since hyperbolic quadrics contain opposite maximal singular subspaces of the same natural system if, and only if, their rank is even (and equivalently, the dimension of the maximal singular subspaces is odd), the assertion follows. $\square$

This has now the following consequences.

Theorem 6.12. *Let Δ be a polar space of rank r at least 3 with at least 4 points per line, and let U be a singular subspace of dimension at most $r-2$. Then $\Pi^+_{\geq}(U)$ is the subgroup of the automorphism group of $\mathsf{Res}(U)$ generated by products of two reflections of $\mathsf{Res}(U)$. If Δ is not a separable orthogonal polar space, then $\Pi^+_{\geq}(U) = \Pi_{\geq}(U)$ is also generated by all reflections. If Δ is a separable orthogonal polar space, then $\Pi_{\geq}(U) = \Pi^+_{\geq}(U)$, if $\dim U$ is odd, and $\Pi(U)$ is the subgroup of the automorphism group of $\mathsf{Res}(U)$ generated by all reflections, if $\dim U$ is even. In the latter case, $\Pi_{\geq}(U)$ is not equal to $\Pi^+_{\geq}(U)$, as soon as the anisotropic dimension of Δ (as a quadric), is even.*

Proof. Let G_1 be the group of collineations of $\mathsf{Res}(U)$ generated by the reflections, and let $G_2 \leq G_1$ be its subgroup consisting of those members, which are a product of an even number of reflections.

By Lemma 3.4 and Theorem 6.4, $G_2 \leq \Pi^+_{\geq}(U)$. By Lemma 6.9 and Lemma 6.10, we have $\Pi^+_{\geq}(U) \leq G_2$. This proves the first part of the theorem.

If Δ is not a separable orthogonal polar space, then $\Pi^+_{\geq}(U) = \Pi_{\geq}(U)$ by Proposition 4.1. Then the second assertion follows directly from Corollary 6.5.

Now let Δ be a separable orthogonal polar space. If $\dim U$ is odd, then Proposition 4.1 implies $\Pi_{\geq}(U) = \Pi^+_{\geq}(U)$. Now let $\dim U$ be even. Then Lemma 6.11 implies that there exists a non-trivial self-projectivity of U of length 3 that fixes a hyperplane of $\mathsf{Res}(U)$. It follows now that $\Pi_{\geq}(U)$ is generated by reflections. Finally, if the anisotropic dimension of Δ (as a quadric) is even, then this is also the case for $\mathsf{Res}(U)$. Then each member of $\Pi^+_{\geq}(U)$ preserves the system of generators of the (imaginary) hyperbolic quadric over a splitting field of the quadric, whereas each single reflection interchanges these systems. $\square$

The projectivity groups for upper residues in symplectic polar spaces follow directly from Theorem 3.1:

Proposition 6.13. *If Δ is a symplectic polar space of rank r, and U is a singular subspace of Δ of dimension $d \leq r-2$, then $\Pi_{\geq}(U) = \Pi^+_{\geq}(U)$ is the simple symplectic group corresponding to $\mathsf{Res}(U)$, except if $\mathsf{Res}(U)$ has rank 2 and each line has exactly 3 points, in which case the group is isomorphic to the symmetric group on 6 letters.*

Similarly as for points, we can also handle the case of elliptic polar space with three points per line.

Proposition 6.14. *Let Δ be a polar space of rank r with three points per line and distinct from a symplectic polar space. Let U be a singular subspace of dimension $d \leq r-2$. Then $\Pi^+_{\geq}(U)$ is the simple orthogonal group associated to $\mathsf{Res}(U)$. If d is odd, then $\Pi_{\geq}(U) = \Pi^+_{\geq}(U)$, whereas, if d is even, $\Pi_{\geq}(U)$ is the special orthogonal group associated to $\mathsf{Res}(U)$.*

A special case of Theorem 6.12 is worth mentioning.

Corollary 6.15. *Let Δ be a σ-Hermitian polar space over a commutative field $\mathbb{K}$ with anisotropic dimension 0, and let $\mathbb{F}$ be the fix field of σ. Let U be a submaximal singular subspace. Then $\Pi^+_{\geq}(U) = \Pi_{\geq}(U)$ is permutation equivalent to $\mathsf{PGL}_2(\mathbb{F})$.*

Since each non-embeddable polar space contains an ideal σ-Hermitian polar subspace (*ideal* means that every plane of the large space through a line of the subspace belongs entirely to the subspace), we immediately deduce from Corollary 6.15 the following result,

Corollary 6.16. *Let Δ be a non-embeddable polar space with planes isomorphic to projective planes over an octonion division algebra with centre $\mathbb{K}$. Let L be a line of Δ. Then $\Pi^+_{\geq}(L) = \Pi_{\geq}(L)$ is permutation equivalent to $\mathsf{PGL}_2(\mathbb{K})$.*

7. Projectivity groups for projective residues in polar spaces

7.1. Non-maximal projective residues.

7.1.1. *Embeddable polar spaces.* This case is straightforward, using Lemma 3.4. Indeed, the special projectivity group of a subspace D of dimension d of a Desarguesian projective space over the skew field $\mathbb{L}$ is the full linear group $\mathsf{PGL}(d, \mathbb{L})$, so it also coincides with the special projectivity group inherited from the polar space. For the general projectivity group, we have to add a duality, which is linear, if the polar space is orthogonal or symplectic, but which is semi-linear with companion skew field automorphism σ, if the polar space is associated to a σ-quadratic form. If $\dim D = 1$, then this duality is just a collineation of $\mathsf{PG}(1, \mathbb{L})$ (with $\mathbb{L}$ the underlying skew field) induced by σ. In the orthogonal case, the duality belongs to $\mathsf{PGL}_2(\mathbb{L})$ and so the general and special groups coincide; in the other cases the special group has index 2 in the general group.

7.1.2. *Non-embeddable polar spaces.* Here, the residues have type 1 or 2 in Bourbaki labelling. We look at the lower residue of a line, and at the planar line pencils. We first consider the lower residues of a line.

Proposition 7.1. *Let Δ be the non-embeddable polar space associated to the Cayley division algebra $\mathbb{O}$ over the field $\mathbb{K}$. Let L be a line of Δ. Then $\Pi^+_{\leq}(L)$ is permutation equivalent to the group of direct similitudes of the quadric of Witt index 1, defined by the anisotropic form given by the norm form of $\mathbb{O}$ over $\mathbb{K}$. Also, $\Pi_{\leq}(L)$ is isomorphic to the group generated by $\Pi^+_{\leq}(L)$ and a permutation induced by the standard involution of $\mathbb{O}$, when considering L as a projective line over $\mathbb{O}$.*

Proof. We begin by noting that the mentioned permutation group is equal to the projectivity group of a line in the Moufang plane $\mathsf{PG}(2, \mathbb{O})$, as proved by Grundhöfer [13, Satz]. So, in view of Lemma 3.4, it suffices to show that every (lower) even projectivity of L in Δ coincides with a projectivity inside some plane of Δ containing L. Thanks to Lemma 4.5, Remark 4.6, Proposition 5.1 and Proposition 5.4, it suffices to show this for lower projectivities of the form $L \barwedge L_2 \barwedge L_3 \barwedge L_4 \barwedge L$, where L and L_3 intersect in a point p_1 and generate a plane π_1, where L_2 and L_4 intersect in a point p_2 and generate a plane π_2, and where p_1 and p_2 are opposite, and π_1 and π_2 are also opposite. Set $c_i = \mathsf{proj}_{\pi_1} L_i$, $i = 2, 4$. Then one observes that, for each $x \in L$, $x_3 := \mathsf{proj}_{L_3}\mathsf{proj}_{L_2} x$ is contained in the line xc_2, and $x' := \mathsf{proj}_L \mathsf{proj}_{L_4} x_3$ lies on the line x_3c_4. Hence, $L \barwedge L_2 \barwedge L_3 \barwedge L_4 \barwedge L$ coincides with the projectivity $L \barwedge c_2 \barwedge L_3 \barwedge c_4 \barwedge L$ inside π_1. This completes the proof of the first part of the proposition.

The second part follows from Section 7.1.1 by noting that Δ contains (many) polar spaces over quaternion subfields. $\square$

Proposition 7.2. *Let Δ be the non-embeddable polar space associated to the Cayley division algebra $\mathbb{O}$ over the field $\mathbb{K}$. Let P be a planar line pencil of Δ. Then $\Pi^+(P)$ is permutation equivalent to the group of direct similitudes of the quadratic form of Witt index 1 defined by the norm form of $\mathbb{O}$ over $\mathbb{K}$. Also, $\Pi(P)$ is isomorphic to the group generated by $\Pi^+(L)$ and the permutation induced by the standard involution of $\mathbb{O}$, when considering P as a projective line over $\mathbb{O}$.*

Proof. We identify a line pencil with its point-plane pair. As in the previous proof, it suffices to show that, for line pencils P_2, P_3, P_4, with $P \sim P_3$ and $P_2 \sim P_4$ (see the next paragraph for more details on the adjacency $\sim$) such that both P and P_3 are opposite both P_2 and P_4, the projectivity $P \barwedge P_2 \barwedge P_3 \barwedge P_4 \barwedge P$ is a projectivity of P inside the plane it defines.

Assume that $P = \{p, \pi\}$, with p a point in the plane π, and likewise $P_i = \{p_i, \pi_i\}$, $i = 2, 3, 4$. Then $P \sim P_3$ means that either $\pi = \pi_3$ and $p \neq p_3$, or $p = p_3$ and $\pi \cap \pi_3$ is a line. Suppose first that $\pi = \pi_3$. Let L be the line pp_3 and set $M := \mathsf{proj}_\pi p_2$. Note $M \cap L \notin \{p, p_3\}$. Let K be an arbitrary line of P. Then its image under $P \barwedge P_2$ is the line K_2 through p_2 and $\mathsf{proj}_{\pi_2} K$. It follows that $\mathsf{proj}_\pi K_2 = K \cap M$. But then the image K_3 of K_2 under $P_2 \barwedge P_3$ is the line joining p_3 with $K \cap M$. It follows that $P \barwedge P_2 \barwedge P_3$ coincides with the projectivity $p \barwedge M \barwedge p_3$ in the projective plane π.

Now assume $p = p_3$ and $\pi \cap \pi_3$ is a line $L \in P \cap P_3$. Set $M = \mathsf{proj}_\pi p_2$, $M_3 = \mathsf{proj}_{\pi_3} p_2$ and $M_2 = \mathsf{proj}_{\pi_2} p$. Let K again be an arbitrary line of P and set $x = K \cap M$, $x_2 = \mathsf{proj}_{\pi_2} K = \mathsf{proj}_{M_2} x$, $K_2 = p_2 x_2$, $x_3 = \mathsf{proj}_{\pi_3} K_2 = \mathsf{proj}_{M_3} x_2$, and $K_3 = px_3$. Then we find that K_3 is the image of K under $P \barwedge P_2 \barwedge P_3$, and x_3 is the image of x under $M \barwedge M_2 \barwedge M_3$ (lower perspectivities). Hence, $P \barwedge P_2 \barwedge P_3$ can be written as the product of $p \barwedge M$ in π, with $M \barwedge M_2 \barwedge M_3$ in Δ, and finally $M_3 \barwedge p$ in π_3. Since a similar decomposition holds for $P_3 \barwedge P_4 \barwedge P$, we conclude, using Proposition 7.1, that $P \barwedge P_2 \barwedge P_3 \barwedge P_4 \barwedge P$ is contained in the projectivity group of p inside π. The first part of the proposition is proved.

The second part follows from the second part of Proposition 7.1 by noting that, if $Q = \{q, \alpha\}$ is an arbitrary line pencil opposite P, the perspectivity $P \barwedge Q$ is the product of the perspectivity $p \barwedge \mathsf{proj}_\pi q$ inside π, the lower perspectivity $\mathsf{proj}_\pi q \barwedge \mathsf{proj}_\alpha p$ and the perspectivity $\mathsf{proj}_\alpha p \barwedge q$ inside the plane α. $\square$

Remark 7.3. The standard involution of a split octonion algebra interchanges the two systems of maximal singular subspaces, as follows immediately from the fact that it interchanges two of the eight standard basis vectors and maps the others to their additive inverse (in the representation of Zorn). Hence, the general projectivity groups in Proposition 7.1 and Proposition 7.2 are the groups of all similitudes of the said quadratic forms. This also holds for the quadratic associative division algebras over the field $\mathbb{K}$ with non-trivial standard involution. For further use, given a quadratic alternative division algebra $\mathbb{A}$ over $\mathbb{K}$ with non-trivial standard involution, we denote the group of all permutations of the projective line $\mathsf{PG}(1, \mathbb{A})$ corresponding to direct similitudes by $\mathsf{PGL}_2^+(\mathbb{A})$, and the group of all permutations corresponding to all similitudes by $\mathsf{PGL}_2(\mathbb{A})$, except, if $\mathbb{A}$ is commutative (and this notation would be ambiguous), then we use $\mathsf{PGL}_2^+(\mathbb{A}/\mathbb{K})$ and $\mathsf{PGL}_2(\mathbb{A}/\mathbb{K})$, respectively.

7.2. **Maximal singular subspaces.**

7.2.1. *Reduction to the composition of four perspectivities.*

Proposition 7.4. *Let Δ be a polar space of rank $r \geq 3$. Let M be a maximal singular subspace. of Δ. Suppose the set of projectivities $M \barwedge M_2 \barwedge M_3 \barwedge M_4 \barwedge M$, with $M \equiv M_2 \equiv M_3 \equiv M_4 \equiv M$ and $U_0 := M \cap M_2$ and $U_1 := M_1 \cap M_3$ opposite submaximal singular subspaces, is geometric. Then $\Pi^+(M)$ is generated by all such projectivities.*

Proof. This follows from Remark 4.6 and Proposition 5.4. $\square$

As we will see below, the projectivities $M \barwedge M_2 \barwedge M_3 \barwedge M_4 \barwedge M$, mentioned in the previous proposition, are homologies. So proving that the said set is geometric is equivalent to showing that the set of factors of these homologies are closed under conjugation with any element of the underlying skew field.

7.2.2. *Computation of the special projectivity group in the general (non-symplectic embeddable) case.* We use the standard form, see [25, Chapter 4]. Let Δ be an embeddable polar space with pseudo-quadratic description in $\mathsf{PG}(V)$, for some right vector space V of some skew field $\mathbb{L}$ and let σ be an involution of $\mathbb{L}$. Let V_0 be a subspace of V of codimension $2n$ and let

$$\{e_i \mid i \in \{-n, -n+1, \ldots, -1, 1, \ldots, n\}\}$$

be a (suitably chosen) basis of a complementary subspace, such that

$$\{\langle e_i \rangle \mid i \in \{-n, -n+1, \ldots, -1, 1, \ldots, n\}\}$$

is a polar frame. For general vectors $v, w \in V$ we write $v = v_0 + \sum_{i \in I} e_i x_i$ and $w = w_0 + \sum_{i \in I} e_i y_i$ (where $v_0, w_0 \in V_0$ and $x_i, y_i \in \mathbb{L}$ for all $i \in I$). Then there exists an anisotropic σ-quadratic form $\mathfrak{q}_0$ on V_0, where $\mathfrak{q}_0(v_0) = g_0(v_0, v_0) + \mathbb{L}_\sigma$, for the $(\sigma, 1)$-linear form $g_0 : V_0 \times V_0 \to \mathbb{L}$, with associated σ-Hermitian form f_0, such that

$$\begin{aligned} \mathfrak{q}(v) &= x_{-n}^\sigma x_n + \cdots + x_{-1}^\sigma x_1 + \mathfrak{q}_0(v_0), \\ f(v, w) &= x_{-n}^\sigma y_n + x_n^\sigma y_{-n} + \cdots + x_{-1}^\sigma y_1 + x_1^\sigma y_{-1} + f_0(v_0, w_0) \end{aligned}$$

define the points and collinearity, respectively, of the polar space Δ.

Let $v_0, w_0 \in V_0$, $t, u \in \mathbb{L}$ be such that

$$D := D(v_0, w_0, t, u) := (g_0(w_0, w_0) + u - u^\sigma)(g_0(v_0, v_0) + t - t^\sigma) + f_0(w_0, v_0) + 1 \neq 0.$$

We abbreviate $g_1 := g_0(v_0, v_0) + t - t^\sigma$ and $g_2 := g_0(w_0, w_0) + u - u^\sigma$ and define the following four maximal singular subspaces.

$$\begin{cases} M_1 = \langle e_{-1}, e_{-2} \ldots, e_{-n} \rangle, \\ M_2 = \langle e_1, e_2, \ldots, e_n \rangle, \\ M_3 = \langle g_1 e_1 + v_0 + e_{-1}, e_{-2}, \ldots, e_{-n} \rangle, \\ M_4 = \langle g_2^\sigma e_{-1} + w_0 + e_1, e_2, \ldots, e_n \rangle. \end{cases}$$

Then, by our condition above, we have $M_1 \equiv M_2 \equiv M_3 \equiv M_4 \equiv M_1$ and $M_1 \sim M_3$ and $M_2 \sim M_4$ (cf. Definition 4.4). Set $\theta := M_1 \barwedge M_2 \barwedge M_3 \barwedge M_4 \barwedge M_1$. Then θ pointwise fixes the hyperplane $\langle e_{-2}, \ldots, e_{-n} \rangle$ of M_1 and also the point $\langle e_{-1} \rangle$. Hence, θ is a homology, and we determine its factor. We can work in the subspace W generated by e_{-2}, e_{-1}, e_1, e_2 and V_0. We write a generic element of that subspace as $(x_{-2}, x_{-1}, x_0, x_1, x_2)$, where all elements belong to $\mathbb{L}$, except $x_0 \in V_0$.

We consider a generic point $(1, x, 0, 0, 0)$ on $\langle e_{-2}, e_{-1} \rangle$ distinct from e_{-2}, so we may assume $x \neq 0$. Its image in $M_2 \cap W$ under the perspectivity $M_1 \barwedge M_2$ is the point $(0, 0, 0, 1, -x^\sigma)$. The image of that point in $M_3 \cap W$ under the perspectivity $M_2 \barwedge M_3$ is the point $(x^{-1}, 1, v_0, g_1, 0)$. Likewise, the image of the latter in $M_4 \cap W$, under the perspectivity $M_3 \barwedge M_4$, is the point $(0, g_2^\sigma, w_0, 1, -x^\sigma D^\sigma)$. Finally, the image of that point in $M_1 \cap W$, back again under the perspectivity $M_4 \barwedge M_1$, is the point $(1, Dx, 0, 0, 0)$.

We can now formulate the main result of this subsection. The set

$$\{D(v_0, w_0, t, u) \mid v_0, w_0 \in V_0, t, u \in \mathbb{L}\}$$

will be referred to as the *norm set*.

Proposition 7.5. *The special projectivity group of a maximal singular subspace of an embeddable polar space with pseudo-quadratic description in standard form as above, is generated by homologies with factors in the norm set.*

Proof. In view of the preceding computations and Proposition 7.4, it only remains to show that the norm set is closed under conjugation with an arbitrary element $r \in \mathbb{L}^\times$. This follows straight from the identities $r^{-1} f_0(w_0, v_0) r = f_0(w_0 r^{-\sigma}, v_0 r)$, $r^\sigma (t - t^\sigma) r = (r^\sigma t r) - (r^\sigma t r)^\sigma$. $\square$

It strongly depends on g_0, what exactly the special projectivity group of a maximal singular subspace is. We present a few special cases.

For a parabolic polar space of rank r, $V_0 = \mathbb{L}$, $\sigma = \mathsf{id}$ and $g_0(x_0, y_0) = x_0 y_0$. It follows that $f_0(x_0, y_0) = 2x_0y_0$, and consequently, $D(x_0, y_0, t, u) = x_0^2 y_0^2 + 2x_0y_0 + 1 = (x_0y_0 + 1)^2$. Hence, the special projectivity group here is $\mathsf{PGL}_r(\mathbb{L})$, if r is odd, and is the linear subgroup of $\mathsf{PGL}_r(\mathbb{L})$, consisting of the matrices with a square determinant, if r is even.

Now, suppose $V_0 = \mathbb{L}\oplus\mathbb{L}$, $\mathsf{char}\,\mathbb{L} \neq 2$ and $\sigma = \mathsf{id}$. Let g_0 be given by $g_0((x, y), (x'y')) = xx' + \ell yy'$. Then $D((x, y), (x', y'), t, u) = (1+xx'+yy')^2 + \ell(xy'-x'y)^2$ and so the special projectivity group is generated by homologies with factor a norm of the quadratic extension of $\mathbb{L}$ corresponding to g_0. In the finite case, every element of $\mathbb{L}$ is a norm, and then we simply have $\mathsf{PGL}_r(q)$, $q = |\mathbb{L}|$.

For Hermitian polar spaces over commutative fields with non-trivial involution, see the next paragraph.

7.2.3. *Special projectivity groups for symplectic polar spaces and some Hermitian ones.* The previous subsection does not cover the symplectic polar spaces (it would, if we considered skew-Hermitian forms rather than Hermitian forms). However, in this case, the simple group (generated by the root groups) is easy to describe and hence, we can directly use Theorem 3.1. The same method can be applied to Hermitian polar spaces over commutative fields with non-trivial field involution.

Proposition 7.6. *Let Δ be a symplectic polar space of rank r, defined over the field $\mathbb{K}$. Let U be a maximal singular subspace. Then $\Pi^+(U) \cong \mathsf{PGL}_r(\mathbb{K})$ in its standard permutation representation. Also, $\Pi(U) \cong \mathsf{PGL}_r(\mathbb{K}) \rtimes \langle\rho\rangle$, with ρ any linear polarity.*

Proof. Let Δ be described by the alternating form

$$(x_{-r}y_r + x_{-r+1}y_{r-1} + \cdots x_{-1}y_1) - (y_{-r}x_r + y_{-r+1}x_{r-1} + \cdots y_{-1}x_1).$$

Using obvious notation, the subspace U, spanned by $e_{-1}, \dots, e_{-r}$, is a maximal singular subspace, and the subspace W generated by $e_1, \dots, e_r$ is an opposite maximal singular subspace. Set $X_- = (x_{-1}, \dots, x_{-r})$ and $X_+ = (x_1, \dots, x_r)$, which we also read as row matrices, then, for every non-singular $r \times r$ matrix M, the linear transformation

$$X_- \mapsto X_- \cdot M^t, \qquad X_+ \mapsto X_+ \cdot M^{-1}$$

induces a linear collineation in $\mathsf{PG}(2r-1, \mathbb{K})$, preserving the above alternating form. Since such a collineation of Δ belongs to the automorphism group generated by the root elations, as follows from [12, Théorème 1], we find, using Theorem 3.1, that $\Pi^+(U) \cong \mathsf{PGL}_r(\mathbb{K})$. Now a single perspectivity induces a linear duality (over $\mathbb{K}$) from one maximal singular subspace to another, concluding the proof of the proposition. $\square$

A similar argument can be given to obtain a slightly more concrete version of Proposition 7.5 in case of Hermitian polar spaces over a commutative field with non-trivial involution. For a given field $\mathbb{K}$ and subfield $\mathbb{F}$, let $\mathsf{SL}_r(\mathbb{K}; \mathbb{F})$ be the multiplicative group of $r \times r$ matrices with a determinant in $\mathbb{F}^\times$, and let $\mathsf{PSL}_r(\mathbb{K}; \mathbb{F})$ be the corresponding projective group, that is, the quotient group with its centre.

Proposition 7.7. *Let Δ be a Hermitian polar space of rank r, defined over the field $\mathbb{K}$, with non-trivial involution σ and fixed field $\mathbb{F}$. Let U be a maximal singular subspace.*

(i) If the anisotropic part is trivial, then $\Pi^+(U) \cong \mathsf{PSL}_r(\mathbb{K}; \mathbb{F})$ in its standard permutation representation. Also, $\Pi(U) \cong \mathsf{PSL}_r(\mathbb{K}; \mathbb{F}) \rtimes \langle\rho\rangle$, with ρ any polarity with companion involution σ.

(ii) If the anisotropic part is non-trivial, then $\Pi^+(U) \cong \mathsf{PGL}_r(\mathbb{K})$ in its standard permutation representation. Also, $\Pi(U) \cong \mathsf{PGL}_2(\mathbb{K}) \rtimes \langle\rho\rangle$, with ρ any polarity with companion involution σ.

Proof. First, we note that a single perspectivity induces a duality with companion field automorphism σ (as follows from the computations preceding Proposition 7.5). Hence, the special projectivity group will be a linear group. Let θ be any (linear) collineation of Δ in the little projective group $G^\dagger$ of Δ stabilising U. Since the automorphism group of Δ acts transitively on opposite (ordered) pairs of maximal singular subspaces, we may assume that θ stabilises an opposite maximal singular subspace. We consider the standard Hermitian form

$$(x_{-1}^\sigma x_1 + \cdots + x_{-r}^\sigma x_r) + (x_1^\sigma x_{-1} + \cdots + x_r^\sigma x_{-r}) + f_0(x_0, x_0),$$

where f_0 is an anisotropic Hermitian form on some vector space V_0 with shorthand coordinates x_0. We take for U the subspace generated by $e_{-r}, \ldots, e_{-1}$. By the above, we may assume that θ also stabilises the maximal singular subspace W, generated by $e_1, \ldots, e_r$. Let θ be given by

$$\begin{pmatrix} x_{-1} & x_{-2} & \ldots & x_{-r} \end{pmatrix} \mapsto \begin{pmatrix} x_{-1} & x_{-2} & \ldots & x_{-r} \end{pmatrix} \cdot M^t,$$

with, for now, M any non-singular $r \times r$ matrix over $\mathbb{K}$ (the transpose is taken for notational reasons) with determinant $k \in \mathbb{K}^\times$. It is clear that the action of θ on W is described by

$$\begin{pmatrix} x_1 & x_2 & \ldots & x_r \end{pmatrix} \mapsto \begin{pmatrix} x_1 & x_2 & \ldots & x_r \end{pmatrix} \cdot M^{-\theta}.$$

(*i*) If V_0 is trivial, then the determinant of θ as a linear map in $\mathsf{PG}(2r-1, \mathbb{K})$ is equal to $\ell := kk^{-\theta}$. Then, according to [12, Théorème 5], θ belongs to $G^\dagger$ if, and only if, $\ell = 1$, that is, $k = k^\theta$. Now the statement of (*i*) follows.

(*ii*) Suppose now that V_0 is not trivial. Since $\mathsf{PGL}_r(\mathbb{K})$ is the full group of linear transformations, it suffices to show the assertion in the case of $\dim V_0 = 1$. Then we may assume $f_0(x_0, x_0) = a_0 x_0^\sigma x_0$. We define the action of θ on the coordinate x_0 as $x_0 \mapsto \ell^{-1} x_0$. We find that $f_0(\ell x_0, \ell x_0) = a_0 \ell^\sigma \ell x_0^\sigma x_0 = a_0 x_0^\sigma x_0$, since $\ell^\sigma \ell = 1$, as one computes. With that, the determinant of θ as a linear map in $\mathsf{PG}(2r, \mathbb{K})$ is equal to 1, and hence, by [12, Théorème 5] again, θ belongs to $G^\dagger$. This proves the assertions in (*ii*). □

7.2.4. *Non-embeddable polar spaces.* In this case, we could do a computation similar to the one performed in the previous subsection. However, using a result from [17], it is more efficient to use Lemma 4.3. Let $\mathbb{O}$ be a Cayley division algebra over the field $\mathbb{K}$ with standard involution $x \mapsto \overline{x}$, and let $\mathsf{PG}(2, \mathbb{O})$ be the associated projective plane, coordinatised as in Section 2.5.4. Then, for all $k, \ell \in \mathbb{K}$, the mapping with action on the points (x, y), with $y \neq 0$, given by

$$(x, y) \mapsto [k\overline{y}^{-1}\overline{x}, \ell\overline{y}^{-1}]$$

induces, by [17, Section 6.2], a polarity $\rho(k, \ell)$ of $\mathsf{PG}(2, \mathbb{O})$, which we call a *standard polarity.*

Remark 7.8. Replacing $\mathbb{O}$ with any quadratic associative division algebra $\mathbb{A}$, the above expression remains a polarity in the corresponding projective plane, and we also call such a polarity a *standard polarity* (with respect to the given quadratic structure, that is, relative to the field $\mathbb{K}$, which might not be determined by $\mathbb{A}$, if $\mathbb{A}$ is commutative). If the standard involution of $\mathbb{A}$ is non-trivial, then the polarity is Hermitian.

The following result is now an immediate consequence of the computations in [17, Section 6.2].

Lemma 7.9. *Let Δ be a thick non-embeddable polar space of rank* 3*. Then every perspectivity $\pi_1 \,\overline{\wedge}\, \pi_2 \,\overline{\wedge}\, \pi_3 \,\overline{\wedge}\, \pi_1$, with π_1, π_2, π_3 pairwise opposite planes, is a standard polarity.*

Noting that the class of standard polarities is geometric, we immediately conclude that $\Pi(\pi)$, for π a plane of a non-embeddable thick polar space, is the group generated by all standard polarities. We can be slightly more specific. Recall, that a projective collineation is a collineation, which induces a projectivity on at least one line (and hence on all lines). The set of such collineations forms a group, called the *full projective group*, which we denote by $\mathsf{PGL}_3(\mathbb{O})$, with slight abuse of notation (because there does not exist a 3-dimensional vector space over $\mathbb{O}$).

Proposition 7.10. *Let Δ be the thick non-embeddable polar space associated with the Cayley division algebra $\mathbb{O}$ over the field $\mathbb{K}$. Let π be any plane of Δ. Then $\Pi^+(\pi)$ is the full projective group of $\mathsf{PG}(2, \mathbb{O})$, whereas $\Pi(\pi)$ is the full projective group extended with a standard polarity.*

Proof. By [17, Section 6.2], the action of $\rho(k,\ell)$ on lines $[m,k]$, with $k \neq 0$, is given by

$$[m,k] \mapsto (k^{-1}\ell\overline{m}\overline{k}^{-1}, \ell\overline{k}^{-1}).$$

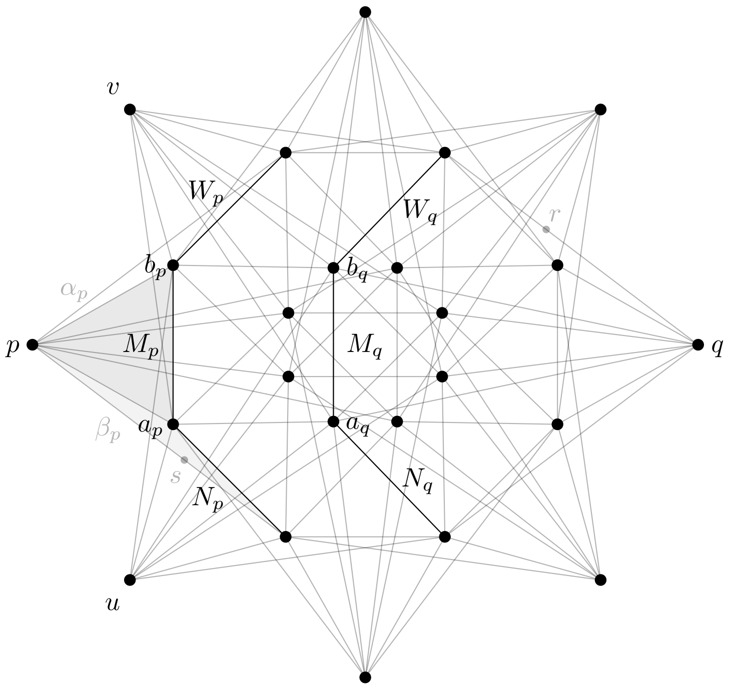
We can calculate the action of $\rho(k,\ell)\rho(1,1)$ on the points (x,y), $y \neq 0$ and obtain

$$(x,y) \mapsto (k\ell^{-1}x, \ell^{-1}y).$$

Setting $\ell = 1$, we see that we obtain all homologies with axis $[0]$ and centre (0). Then the assertion follows from [19, Satz 3]. $\square$

8. Projectivity groups of points of metasymplectic spaces

For the first proposition of this section, one might benefit from having a clear visualisation of an apartment of a building of type F_4. We provide one below. It is a graph, for which its vertices correspond to the vertices of type 1 (with the usual Bourbaki labelling) of the said apartment, the edges are those of type 2 , the 3-cliques are those of type 3, and the (skeletons of the) octahedra are the vertices of type 4.



As in [5], we call a collineation of a polar space a *homology*, if it pointwise fixes two opposite planes, which we call the *axes* of the homology.

Proposition 8.1. *Let p be a point of a metasymplectic space Γ. Let q be any opposite point, and let $s \perp p$ and $r \perp q$ be further points such that $ps \equiv qr$, $p \equiv r$ and $q \equiv s$. Then the projectivity $\rho_{r,s} : p \overline{\wedge} q \overline{\wedge} s \overline{\wedge} r \overline{\wedge} p$ is a homology of the polar space Δ_p, corresponding to the residue at p, which pointwise fixed planes α_s and α_r, the planes corresponding to the lines sp and the projection of rq onto p, respectively. Moreover, if L is any line of Δ_p intersecting α_r and α_s non-trivially, then r and s can be chosen, such that the projectivity $\rho_{r,s}$ maps a given plane of Δ through L, not adjacent to either α_r or α_s, to any other such plane.*

Proof. The line L of Δ_p corresponds to a plane α through p in Γ. Let Σ be an apartment in Γ containing the opposite lines ps and qr, containing the points p and q, and containing the plane β (this will only be crucial in the last part of the proof; hence, in the first part, we will not use this information). Let α_p and β_p be two planes in Σ through p that intersect in a line, and let $s \in \beta_p$. Also, for the last part of the proof (Claim 3), we choose these planes such that $\alpha_p = \alpha$. Let M_p and N_p be the lines in α_p and β_p, respectively, that are closest to q (meaning the points of these lines have distance two to q and are special to q). We denote by a_p the intersection point of M_p and N_p. Let M_q and N_q be the lines in $q^\perp \cap \Sigma$, such that each point of M_q (N_q, respectively) is collinear to a unique point of M_p (N_p). Let u be the point of Σ that is collinear to all points of M_p, N_p, M_q and N_q. Let W_p be the line of Σ that intersects α_p in exactly one point, whose points are all collinear to p and at distance 2 to q and which does not contain a point collinear to N_p. Let b_p be the intersection point of M_p and W_p. Again, let W_q be the line in $q^\perp \cap \Sigma$, such that each point of W_q is collinear to a unique point of W_p. Consider the plane $\langle M_p, u\rangle$ and let $S \subseteq \langle M_p, u\rangle$ be the line through a_p collinear to s (this line exists considering the symp containing p and u). We note that the plane $\langle S, s\rangle$ intersects α_p exactly in the point a_p, is not contained in $p^\perp$ and since s has distance 3 to q, we have $u \notin S$.

Let ξ be the octahedron of Σ containing M_p, M_q and u. Let v be the point in ξ that is opposite u and collinear to every point of M_p and M_q. Note that v is also collinear to W_p and W_q. Hence, v and q are contained in a common symp.

Let a_q be the point $M_q \cap N_q$; that is the unique point of M_q and N_q that is collinear to a_p. Then $\langle a_p, a_q, v\rangle$ spans a plane of Σ. We denote the point $M_q \cap W_q$ by b_q.

Claim 1. *We claim that the projectivity $p \barwedge q \barwedge s \barwedge r \barwedge p$ stabilises the planes $\langle p, N_p\rangle$ and $\langle p, W_p\rangle$ and fixes the lines pa_p and pb_p.*

Indeed, the plane $\langle p, N_p\rangle$ maps to $\langle q, N_q\rangle$ under the first projection and back to $\langle p, N_p\rangle$ under the second projection. It then maps to some plane through r, but since that plane is the image of $\langle p, N_p\rangle$, it maps back to $\langle p, N_p\rangle$ under the fourth projection. Similarly, $\langle p, W_p\rangle$ projects to $\langle q, W_q\rangle$, to some plane through s and then back to $\langle q, W_q\rangle$ and $\langle p, W_p\rangle$.

The line pa_p maps to qa_q, sa_p, the line between r and $R \cap va_q$ and back to pa_p. The line pb_p maps to the line between q and $W_q \cap M_q$, maps to the line between s and $S \cap ub_p$, maps to the line between r and $W_q \cap M_q$ and back to pb_p.

Claim 2. *We claim that the projectivity $p \barwedge q \barwedge s \barwedge r \barwedge p$ stabilises all planes through ps.*

This follows from Step 1 by varying Σ through $\{p, ps\}$ and $\{q, qr\}$.

Claim 3. *We claim that we can always define s and r in a way, such that the projectivity $p \barwedge q \barwedge s \barwedge r \barwedge p$ moves a line px with x on M_p to a line px' with x' on M_p, $x \neq x'$ and $x, x' \notin \{a_p, b_p\}$.*

Let x and x' be two distinct points on M_p not equal to a_p or b_p. Let x_1 be the unique point of M_q collinear to x. Then px moves to qx_1 under the first projection.

Considering the symp spanned by ξ, we can see that x_1 must be collinear to a unique point x_2 of S. With that, qx_1 has to move to sx_2 under the second projection. The points x_2 and x' are both each collinear to a line of $\langle v, a_q, b_q\rangle$. These lines are distinct as x_1 belongs to one of them but not the other. We define x_3 as the intersection point of these two lines. Let R be the connection line between b_q and x_3. Every point of $R \setminus \{b_q\}$ is collinear to the same line of $\langle q, W_q\rangle$ through b_q that we will denote by R'. In Σ we see that W_q and qb_q form a triangle and we denote the third point of that triangle in Σ by c_q. We define r to be the intersection point of R' and qc_q. Then sx_2 moves to rx_3 under the third projection and to px' under the last projection as desired. $\square$

We now first handle the case of type 1 vertices in $\mathsf{F}_4(\mathbb{K}, \mathbb{A})$, with $\mathbb{A}$ a quadratic alternative division algebra over $\mathbb{K}$. We may assume that $|\mathbb{A}| > 2$ as otherwise the groups of projectivity coincide with the full group of collineations $\mathsf{Sp}_6(2)$ by Theorem 3.1, since that group is also generated by the elations.

We denote by $\mathsf{PSp}_{2\ell}(\mathbb{K})$ the group of all collineations of $\mathsf{C}_{\ell,1}(\mathbb{K}, \mathbb{K})$ preserving the associated alternating form, $\ell \geq 2$. It is a simple group. We denote by $\overline{\mathsf{PSp}}_{2\ell}(\mathbb{K})$ the group generated by $\mathsf{PSp}_{2\ell}(\mathbb{K})$ and the *diagonal* collineations, that is, the linear collineations of the underlying vector space mapping the associated alternating form to a nonzero scalar multiple and represented by a diagonal matrix.

If $\mathsf{char}\,\mathbb{K}=2$, and $\mathbb{K}'$ is an overfield of $\mathbb{K}$ all of whose squares are contained in $\mathbb{K}$, then the polar space $\mathsf{C}_{\ell,1}(\mathbb{K}',\mathbb{K})$ is a polar subspace of $\mathsf{C}_{\ell,1}(\mathbb{K}',\mathbb{K}')$, and we denote the restriction of $\overline{\mathsf{PS}}\mathsf{p}_{2\ell}(\mathbb{K}')$ to $\mathsf{C}_{\ell,1}(\mathbb{K}',\mathbb{K})$ by $\overline{\mathsf{PS}}\mathsf{p}_{2\ell}(\mathbb{K}',\mathbb{K})$.

Likewise, for $\mathbb{A}$ a separable quadratic extension of $\mathbb{K}$, when $\mathsf{U}_6(\mathbb{A}/\mathbb{K})$ is the (simple) unitary group preserving a Hermitian form and whose elements correspond to matrices of determinant 1, then we denote by $\overline{\mathsf{U}}_6(\mathbb{A}/\mathbb{K})$ the group generated by $\mathsf{U}_6(\mathbb{A}/\mathbb{K})$ and all diagonal automorphisms with diagonal elements in $\mathbb{K}$ (with respect to the standard form).

Now let $\mathbb{A}$ be a quaternion division algebra over $\mathbb{K}$. Let $\mathsf{U}_6(\mathbb{A})$ denote the (simple) collineation group of $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ generated by the elations, then the group generated by all elations and diagonal automorphisms (with factors in $\mathbb{K}$ and with respect to the standard form) is denoted by $\overline{\mathsf{U}}_6(\mathbb{A})$. Note that we do not need to mention $\mathbb{K}$ in the notation, since it is unique as the centre of $\mathbb{A}$.

Finally, let $\mathbb{A}$ be a Cayley division algebra over $\mathbb{K}$. Here, there is no form of the corresponding polar space available (since it is non-embeddable) and hence we cannot consider diagonal automorphisms as in the previous paragraphs. However, we can either consider the group generated by all elations and homologies (which for the previous cases would have boiled down to the same groups), or all elations and the diagonal automorphisms of the universal embedding of the corresponding dual polar space, see the proof of Lemma 8.2. We denote these automorphism groups by $\mathsf{E}_{7,3}^{(28)}(\mathbb{A})$ and $\overline{\mathsf{E}}_{7,3}^{(28)}(\mathbb{A})$, respectively.

Lemma 8.2. *Let Δ be a polar space of rank 3 isomorphic to $\mathsf{C}_{3,3}(\mathbb{A},\mathbb{K})$, with $\mathbb{A}$ a quadratic alternative division algebra over $\mathbb{K}$, and let α and β be two disjoint planes. Let L be a line intersecting both α and β non-trivially. Then there exists a unique homology with axes α and β and mapping a given plane π though L, not adjacent to either α or β, to an arbitrary other given plane like that. In particular, there is a unique set of homologies with axes α and β, acting transitively on the set of planes through L, not adjacent with either α or β, and hence, such a set is geometric.*

Proof. This is best seen through the (universal) representation of the corresponding dual polar space in a projective space over $\mathbb{K}$, as established uniformly in [10]. We take the slightly more symmetric algebraic description from [11, Definition 10.1]. Let V be the vector space $\mathbb{K}^4\oplus\mathbb{A}^3\oplus\mathbb{K}^3\oplus\mathbb{A}^3\oplus\mathbb{K}$ (note that we view $\mathbb{A}$ as a vector space over $\mathbb{K}$ in the natural way, that is, coming from the algebra over $\mathbb{K}$). Then the projective (or Zariski) closure of the point set given by the following parameter form, with the induced line set, forms the dual polar space $\mathsf{C}_{3,3}(\mathbb{A},\mathbb{K})$. We denote by $x\mapsto\overline{x}$ the standard involution in $\mathbb{A}$.

$$(1,\ell_1,\ell_2,\ell_3,x_1,x_2.x_3,x_1\overline{x}_1-\ell_2\ell_3,x_1\overline{x}_1-\ell_2\ell_3,x_1\overline{x}_1-\ell_2\ell_3,$$
$$\ell_1\overline{x}_1-x_2x_3,\ell_1\overline{x}_1-x_2x_3,\ell_1\overline{x}_1-x_2x_3,\ell_1x\overline{x}_1+\ell_2x_2\overline{x}_2+\ell_3x_3\overline{x}_3-x_1(x_2x_3)-\overline{x}_3(\overline{x}_2\overline{x}_1)-\ell_1\ell_2\ell_3),$$

where $\ell_1,\ell_2,\ell_3\in\mathbb{K}$ and $x_1,x_2,x_3\in\mathbb{A}$. We let α and β correspond to the points $a=(1,0,0,\ldots,0)$ and $b=(0,0,\ldots,0,1)$, respectively, We write coordinates of points of $\mathsf{PG}(V)$ as 14-tuples with entries in $\mathbb{K}\cup\mathbb{A}$ in the natural way, according to the definition of V above. Then one sees that the set of points $a^\perp\cap b^{\not\equiv}$ spans the subspace $(0,*,*,*,*,*,*,0,\ldots,0)$ and $a^{\not\equiv}\cap b^\perp$ spans the subspace $(0,\ldots,0,*,*,*,*,*,*,0)$. Each homology with axes α and β has to fix all points of these subspaces, and one can see that such a homology stems from the following linear map, where $t\in\mathbb{K}^\times$:

$$\begin{aligned}&V\to V\\&(k,\ell_1,\ell_2,\ell_3,x_1,x_2,x_3,m_1,m_2,m_3,y_1,y_2,y_3,k')\\&\mapsto(k,t\ell_1,t\ell_2,t\ell_3,tx_1,tx_2,tx_3,t^2m_1,t^2m_2,t^2m_3,t^2y_1,t^2y_2,t^2y_3,t^3k')\end{aligned}$$

According to [10], the line generated by $a'=(0,0,1,0,0,\ldots,0)$ and $b'=(0,\ldots,0,1,0,0,0,0,0,0)$ corresponds to a line joining a point of α with one of β (in the coordinates of [10] these points are $(0,0,0,0,0)$ and $(0,0)$, and the line has type **(I)**). The point a' is collinear to a and b' to

b. The above linear mappings, for varying $t \in \mathbb{K}^\times$, act (sharply) transitively on the points of $a'b' \setminus \{a', b'\}$.

This proves the lemma. □

Corollary 8.3. *Let v be a vertex of type 1 in $\mathsf{F_4}(\mathbb{K}, \mathbb{A})$, with $\mathbb{A}$ a quadratic alternative division algebra over $\mathbb{K}$, $|\mathbb{A}| > 2$. Then $\Pi^+(\{p\})$ is generated by all homologies of the corresponding polar space.*

Proof. This follows from Lemma 5.8, Lemma 8.2 and Proposition 8.1. □

In view of Proposition 4.2, this proves the case (C3) of Table 1

We now consider vertices of type 4 in $\mathsf{F_4}(\mathbb{K}, \mathbb{A})$. Since $\mathsf{F_{4,1}}(\mathbb{K}, \mathbb{K}') \cong \mathsf{F_{4,4}}(\mathbb{K}'^2, \mathbb{K})$, if $\mathsf{char}\,\mathbb{K} = 2$ and $\mathbb{K}'$ is an inseparable extension of $\mathbb{K}$, we may restrict ourselves to the case, where $\mathbb{A}$ is separable; that is, either equal to $\mathbb{K}$ with $\mathsf{char}\,\mathbb{K} \neq 2$, or a separable quadratic extension of $\mathbb{K}$ (in any characteristic), or a quaternion or octonion division algebra over $\mathbb{K}$.

First we consider the case $\mathbb{A} = \mathbb{K}$, with $\mathsf{char}\,\mathbb{K} \neq 2$. Let p be a point of $\mathsf{F_{4,4}}(\mathbb{K}, \mathbb{K})$. We have the following analogue of Lemma 8.2.

Lemma 8.4. *Let Δ be a parabolic polar space of rank 3 over the field $\mathbb{K}$ with $\mathsf{char}\,\mathbb{K} \neq 2$, and let α and β be two disjoint planes. Let L be a line intersecting both α and β non-trivially. Then there exists a unique homology with axes α and β mapping a given plane π though L, not adjacent to either α or β, to an arbitrary other given plane like that. In particular, there is a unique set of homologies with axes α and β acting transitively on the set of planes through L, not adjacent with either α or β, and hence such set is geometric. Also, there is a unique homology with axes α and β that is a reflection.*

Proof. We consider the standard equation of a parabolic quadric in $\mathsf{PG}(6, \mathbb{K})$, that is,

$$X_{-3}X_3 + X_{-2}X_2 + X_{-1}X_1 = X_0^2.$$

We write the coordinates as $(X_{-3}, X_{-2}, X_{-1}, X_0, X_1, X_2, X_3)$. We can take, with self-explaining notation,

$$\begin{cases} \alpha = (*, *, *, 0, 0, 0, 0), \\ \beta = (0, 0, 0, 0, *, *, *), \\ L = \langle (1, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 1, 0) \rangle. \end{cases}$$

Then $\pi_k = \langle L, (0, 0, k^{-1}, 1, k, 0, 0) \rangle$, $k \in \mathbb{K}^\times$, is a generic plane through L in Δ not adjacent to either α or β. With that, the linear map

$$(x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3) \mapsto (x_{-3}, x_{-2}, x_{-1}, kx_0, k^2x_1, k^2x_2, k^2x_3)$$

maps π_1 to π_k. This is a reflection if, and only if, it fixes each point with $x_0 = 0$, that is, if, and only if, $k^2 = 1$, or $k \in \{1, -1\}$.

Now suppose a homology φ with axes α and β also stabilises the plane π_1. Then φ stabilises each line of π_1 through $\pi_1 \cap \alpha$ (as such a line is the projection onto π_1 of some fixed point of α), and also each line of π_1 through $\pi_1 \cap \beta$ (for the analogous reason). Hence, π_1 is fixed pointwise, and this readily implies that $\mathsf{PG}(6, \mathbb{K})$ is fixed pointwise.

All assertions are proved. □

We denote by $\mathsf{O}_7(\mathbb{K})$ the group of linear collineations of a 7-dimensional vector space V over $\mathbb{K}$, preserving the bilinear form associated to the standard equation of a parabolic quadric in $\mathsf{PG}(6, \mathbb{K})$, and by $\mathsf{PO}_7(\mathbb{K})$ the quotient with its centre. Note that the former coincides with $\mathsf{O}_7^+(\mathbb{K})$ with the notation of [12], as $\dim V = 7$ is odd. We then have:

Lemma 8.5. *Let p be a point of $\mathsf{F_{4,4}}(\mathbb{K}, \mathbb{K})$. Then $\Pi^+(p) = \Pi(p) \cong \mathsf{PO}_7(\mathbb{K})$ in its standard action.*

Proof. By Lemma 5.8, Proposition 8.1 and Lemma 8.4, $\Pi^+(p)$ is generated by all homologies, hence, it is contained in $\mathsf{PO}_7(\mathbb{K})$. Now let q be opposite p and consider the extended equator geometry $\widehat{E}(p,q)$. If we now restrict to perspectivities $x \barwedge y$ with x and y points of $\widehat{E}(p,q)$, then we see that the special projectivity group of p in $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{K}) \cong \widehat{E}(p,q)$ is a subgroup of the special projectivity group $\Pi^+(p)$ inside Δ. Hence Corollary 6.6 yields $\mathsf{PO}_7(\mathbb{K}) \leq \Pi^+(p)$. Also, using Proposition 6.1, we find that there exists a self-projectivity of Δ of length 3, exhibiting a reflection of the residue in p. Hence, $\Pi^+(p) = \Pi(p) = \mathsf{PO}_7(\mathbb{K})$ and the proof is complete. $\square$

This takes care of the first line of (B3) of Table 1 (and note that for $\mathsf{char}\,\mathbb{K} = 2$, the groups $\mathsf{PO}_7(\mathbb{K})$ and $\overline{\mathsf{PS}}\mathsf{p}_6(\mathbb{K})$ coincide, so we do not have to require that $\mathsf{char}\,\mathbb{K} \neq 2$ on that line in the table).

Now let $\mathbb{A}$ be a separable quadratic alternative division algebra over the field $\mathbb{K}$, with $d := \dim_{\mathbb{K}} \mathbb{A} \in \{2,4,8\}$. Let Δ be the polar space of rank 3, obtained from the quadric in $\mathsf{PG}(5+d,\mathbb{K})$ with standard equation

$$X_{-3}X_3 + X_{-2}X_2 + X_{-1}X_1 = \mathsf{norm}(X_0),$$

where the coordinate X_0 belongs to $\mathbb{A}$, viewed as vector space over $\mathbb{K}$ in the natural way, and where $\mathsf{norm}(X_0) = X_0\overline{X}_0$, with $X_0 \mapsto \overline{X}_0$ the standard involution in $\mathbb{A}$ with respect to the quadratic algebra structure. Let V be the vector space $K^3 \oplus \mathbb{A} \oplus \mathbb{K}^3$. Then we denote by $\mathsf{O}_{d+6}(\mathbb{K},\mathbb{A})$ or $\overline{\mathsf{O}}_{d+6}(\mathbb{K},\mathbb{A})$ the group of all linear transformations of V preserving the quadratic form associated with the above equation, or mapping it to a scalar multiple (the similitudes; the corresponding scalar is called the *factor* of the similitude), respectively, and we denote with $\mathsf{PO}_{d+6}(\mathbb{K},\mathbb{A})$ and $\overline{\mathsf{PO}}_{d+6}(\mathbb{K},\mathbb{A})$ the respective quotients with the centre. Let $\mathbb{F}$ be a splitting field of $\mathbb{A}$, that is, a quadratic extension of $\mathbb{K}$ over which $\mathbb{A}$ splits as an algebra. Then there are two natural systems of maximal singular subspaces (corresponding to those of the associated hyperbolic quadric) of the corresponding quadratic form over $\mathbb{F}$. The subgroups of $\mathsf{O}_{d+6}(\mathbb{K},\mathbb{A})$ and $\overline{\mathsf{O}}_{d+6}(\mathbb{K},\mathbb{A})$, preserving each of these systems, will be denoted by $\mathsf{O}^+_{d+6}(\mathbb{K},\mathbb{A})$ and $\overline{\mathsf{O}}^+_{d+6}(\mathbb{K},\mathbb{A})$, respectively, partially following Dieudonné [12] (instead of the bilinear form, we included the algebra in the notation). The corresponding projective groups are then $\mathsf{PO}^+_{d+6}(\mathbb{K},\mathbb{A})$ and $\overline{\mathsf{PO}}^+_{d+6}(\mathbb{K},\mathbb{A})$.

Furthermore, let α and β be two opposite planes of Δ. Then we say that the homology group with axes α and β *acts transitively*, or *is a transitive homology group* if for some line L intersecting both α and β non-trivially (and then for each such line), and each pair of planes π,π' through L but not intersecting either α or β in a line, there exists a homology with axes α and β mapping π to π'.

Note that the factors of the similitudes are precisely the non-zero norms of $\mathbb{A}$.

We now prepare for the determination of the projectivity groups of a point in $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$, with $\mathbb{A}$ separable and of dimension $2,4$ or 8 over $\mathbb{K}$.

Lemma 8.6. *Let G be a group of collineations of $\mathsf{B}_{3,1}(\mathbb{A},\mathbb{K})$, with $\mathbb{A}$ a separable quadratic alternative division algebra over the field $\mathbb{K}$, with $d = \dim_{\mathbb{K}} \mathbb{A} \in \{2,4,8\}$, containing $\mathsf{PO}^+_{d+6}(\mathbb{K},\mathbb{A})$ and containing a transitive homology group. Then G contains $\overline{\mathsf{PO}}^+_{d+6}(\mathbb{K},\mathbb{A})$.*

Proof. It suffices to show that, for each norm $r \in \mathbb{K}$, there exists a member of G mapping a given quadratic form describing $\mathsf{B}_{3,1}(\mathbb{A},\mathbb{K})$ to an r-multiple. We consider, with previous notation, the quadratic form

$$X_{-3}X_3 + X_{-2}X_2 + X_{-1}X_1 - \mathsf{norm}(X_0).$$

As in the proof of Lemma 8.4, we set

$$\begin{cases} \alpha = (*,*,*,0,0,0,0), \\ \beta = (0,0,0,0,*,*,*), \\ L = \langle (1,0,0,0,0,0,0),(0,0,0,0,0,1,0)\rangle. \end{cases}$$

Since r is a norm, there exists some $a \in \mathbb{A}$ with $\mathsf{norm}(a) = r$. Consider the plane π_a spanned by L and the point $p_a = (0, 0, 1, a, r, 0, 0)$. Let π_1 be the plane spanned by L and $p_1 = (0, 0, 1, 1, 1, 0, 0)$. Then there exists some homology h with axes α and β mapping π_1 to π_a. Now p_1 and p_a are the unique points of π_1 and π_a, respectively, collinear to the fixed points $(0, 1, 0, 0, 0, 0, 0)$ and $(0, 0, 0, 0, 0, 0, 1)$. Hence, the point p_1 is mapped by h to p_a. Since all points of $\alpha \cup \beta$ are fixed, the matrix of h is diagonal, and hence, this diagonal is, up to a scalar multiple, equal to $(1, 1, 1, M, r, r, r)$, where M is a $d \times d$ matrix with as first column a (in its coordinates over $\mathbb{K}$; the action is on the right). It follows that h is a similitude with factor r and the lemma is proved. □

Lemma 8.7. *Let p be a point of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$, with $\mathbb{A}$ separable and of dimension $2, 4$ or 8 over $\mathbb{K}$. Then $\Pi^+(p) \cong \overline{\mathsf{PO}}^+_{d+6}(\mathbb{K}, \mathbb{A})$ and $\Pi(p) \cong \overline{\mathsf{PO}}_{d+6}(\mathbb{K}, \mathbb{A})$.*

Proof. This time, the set of homologies obtained from Proposition 8.1 is not (necessarily) geometric. However, Lemma 5.8 still shows that $\Pi^+(p)$ is generated by a set of homologies, which contains a transitive group of homologies. It already shows that $\Pi^+(p) \leq \overline{\mathsf{PO}}_{d+6}(\mathbb{K}, \mathbb{A})$, such that $\Pi^+(p)$ contains a similitude with an arbitrary norm as factor.

However, since splitting the forms over $\mathbb{F}$ (see above) produces the exceptional buildings of types $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ for $d = 2, 4, 8$, respectively, and in these buildings, an odd projectivity *always* switches the natural classes of maximal singular subspaces of the hyperbolic quadratic associated to the splitting of the symps (as can be read off of Table 2 of [5] in the lines labelled (D4), (D5) and (D7)), we deduce that $\Pi^+(p) \leq \overline{\mathsf{PO}}^+_{d+6}(\mathbb{K}, \mathbb{A})$.

Now we select a point q opposite p and restrict the projectivity group $\Pi(p)$ to self-projectivities using only perspectivities between points of the extended equator geometry $\widehat{E}(p, q)$. Then Proposition 6.1 implies that every reflection is contained in $\Pi(p)$, seen as a polar space. Hence, $\mathsf{PO}_{d+6}(\mathbb{K}, \mathbb{A}) \leq \Pi(p)$ and the previous paragraphs and Lemma 8.6 imply now that $\Pi(p)$ contains $\overline{\mathsf{PO}}_{d+6}(\mathbb{K}, \mathbb{A})$. But since $\Pi^+(p) \leq \overline{\mathsf{PO}}^+_{d+6}(\mathbb{K}, \mathbb{A})$, because $\Pi^+(p)$ has index at most 2 in $\Pi(p)$ and $\overline{\mathsf{PO}}^+_{d+6}(\mathbb{K}, \mathbb{A})$ has exactly index 2 in $\overline{\mathsf{PO}}_{d+6}(\mathbb{K}, \mathbb{A})$, all projectivity groups now follow. □

9. Projectivity groups of non-maximal residues in metasymplectic spaces

We start with the residues isomorphic to generalised quadrangles, that is, buildings of type B_2 (or C_2). The groups $\mathsf{PO}_5(\mathbb{K})$ and $\overline{\mathsf{PO}}^+_{d+4}(\mathbb{K}, \mathbb{A})$, with $\mathbb{A}$ a separable quadratic alternative division algebra over $\mathbb{K}$ of dimension $2, 4$ or 8, are similarly defined as their higher dimensional analogues above. We then have the following lemma.

Lemma 9.1. *Let F be a simplex of type $\{1, 4\}$ of a building $\Delta := \mathsf{F}_4(\mathbb{K}, \mathbb{A})$. Then $\Pi^+(F) = \Pi(F)$. If $\mathbb{A} = \mathbb{K}'$ is an inseparable extension of $\mathbb{K}$, when $\mathsf{char}\,\mathbb{K} = 2$, then $\Pi(F) \cong \overline{\mathsf{PS}}\mathsf{p}_4(\mathbb{K}', \mathbb{K})$; this includes $\mathbb{K} = \mathbb{K}'$ for $\mathsf{char}\,\mathbb{K} = 2$. If $\mathbb{K} = \mathbb{A}$, then $\Pi(F) \cong \mathsf{PO}_5(\mathbb{K}) \cong \overline{\mathsf{PS}}\mathsf{p}(\mathbb{K})$. If $\mathbb{A}$ is separable with $\dim_{\mathbb{K}} \mathbb{A} \in \{2, 4, 8\}$, then $\Pi(F) \cong \overline{\mathsf{PO}}^+_{d+4}(\mathbb{K}, \mathbb{A})$.*

Proof. The fact that $\Pi^+(F) = \Pi(F)$ follows from [5, Lemma 5.2]. More concretely, let $F_1 = \{x_1, \xi_1\}$ be an incident point-symp pair of $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$, and let $F_2 = \{x_2, \xi_2\}$ be an opposite point-symp pair. Let $x_3 \notin \{x_1, x_2\}$ be a point on the imaginary line defined by x_1 and x_2, that is, $x_3 \in \{x_1, x_2\}^{\perp\!\!\!\perp\perp\!\!\!\perp} \setminus \{x_1, x_2\}$. Finally, let ξ_3 be a symp through x_3, intersecting $E(x_1, x_2)$ in a point of the hyperbolic line of $E(x_1, x_2)$, defined by $E(x_1, x_2) \cap \xi_i$, $i = 1, 2$. Since $E(x_1, x_2)$ is either a symplectic, a mixed, a unitary, or a thick non-embeddable polar space, we can choose ξ_3 disjoint from $\xi_1 \cup \xi_2$, and it follows that ξ_3 is opposite both ξ_1 and ξ_2. Now one can check that $F_1 \barwedge F_2 \barwedge F_3 \barwedge F_1$ is the identity.

Set $F = \{x, \xi\}$, with x of type 1 and ξ of type 4. By Theorem 3.1, $\Pi^*(F)$ is the group induced on $\mathsf{Res}_\Delta(F)$ by the little projective group $G^\dagger$ of Δ. Then also $\mathsf{Res}_\Delta(\xi)$ is stabilised and it follows that $\Pi^+(F)$ is the restriction of the stabiliser of x in $\Pi^+(\xi)$ to $\mathsf{Res}_\Delta(F)$. All assertions now follow. □

We note the following isomorphisms (for appropriate definitions of the unitary groups, similarly to their higher dimensional analogues):

$$\mathsf{PO}_5(\mathbb{K}) \cong \overline{\mathsf{PSp}}_4(\mathbb{K}); \quad \overline{\mathsf{PO}}_6^+(\mathbb{K},\mathbb{L}) \cong \overline{\mathsf{U}}_4(\mathbb{L}/\mathbb{K}); \quad \overline{\mathsf{PO}}_8^+(\mathbb{K},\mathbb{H}) \cong \overline{\mathsf{U}}_4(\mathbb{H}),$$

where $\mathbb{L}$ is a separable quadratic extension of $\mathbb{K}$, and $\mathbb{H}$ is a quaternion division algebra over $\mathbb{K}$.

We now turn to the planes and the rank 1 residues. Recall the notion of a standard polarity in planes $\mathsf{PG}(2,\mathbb{A})$, with $\mathbb{A}$ a quadratic alternative division algebra over $\mathbb{K}$ (see Remark 7.8).

Lemma 9.2. *If F is a simplex of a building $\Delta := \mathsf{F}_4(\mathbb{K},\mathbb{A})$, whose residue is isomorphic to $\mathsf{PG}(2,\mathbb{B})$, with $\mathbb{B} \in \{\mathbb{K},\mathbb{A}\}$, then $\Pi^+(F) \cong \mathsf{PGL}_3(\mathbb{B})$ and $\Pi(F) \cong \mathsf{PGL}_3(\mathbb{B}) \rtimes 2$, where the extension is with a standard polarity if $\mathbb{B} = \mathbb{A}$, and with an ordinary orthogonal polarity if $\mathbb{B} = \mathbb{K}$. If $\mathsf{Res}_\Delta(F)$ is the rank 1 building over $\mathbb{B}$, then $\Pi^+(F) = \Pi(F) \cong \mathsf{PGL}_2(\mathbb{B})$, if $\mathbb{B} = \mathbb{K}$ or, if the standard involution of $\mathbb{B}$ is trivial; otherwise $\Pi^+(F) = \mathsf{PGL}_2^+(\mathbb{B})$ and $\Pi(F) \cong \mathsf{PGL}_2(\mathbb{B})$.*

Proof. Let α be a plane of $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$, let ξ be a symplecton containing α and set $F = \{\alpha, \xi\}$ (then F is a simplex of type $\{3,4\}$ of the corresponding building). By Theorem 3.1, $\Pi^+(F)$ is induced by the little projective group $G^\dagger$ of the building. Since also $\Pi^+(\{\xi\})$ is induced by $G^\dagger$, we find that $\Pi^+(F)$ contains the stabiliser of α in the group $\Pi^+(\{\xi\})$. The latter is determined in Corollary 8.3 (for the inseparable case), Lemma 8.5 and Lemma 8.7 (for the other cases). These stabilisers are the full linear groups, hence, $\Pi^+(F) = \mathsf{PGL}_3(\mathbb{K})$. Similarly, if F is a simplex of type $\{1,2\}$, then $\Pi^+(F) = \mathsf{PGL}_3(\mathbb{A})$. Now we determine the general groups. So we exhibit a self-projectivity of length 3. Let $F = \{x, L\}$, with x a point incident with some line of $\mathsf{F}_{4,i}(\mathbb{K},\mathbb{A})$, with $i \in \{1,4\}$, and let $F' = \{x', L'\}$ be an opposite simplex. The residue of F induces a plane π in $E(x,x')$, viewed as polar space (the set of points symplectic with both x and x' via a symp through L). Similarly, the residue of F' also induces a plane π' in $E(x,x')$. Now, let ζ be a symp through L, intersecting $E(x,x')$ in a point $w \in \pi$. The plane β' through L' nearest to ζ is characterised by the property that every symp ζ' through β' is special to ζ. It follows that such symps ζ' intersect $E(x,x')$ in a point symplectic to $E(x,x') \cap \zeta$. This implies that these intersections form the projection of w to π'. Hence, we can consider a self-projectivity of length 3 in a symp of $\mathsf{F}_{4,5-i}(\mathbb{K},\mathbb{A})$. If $i = 4$, then it suffices to consider a self-projectivity in a parabolic polar space, and a direct computation shows that this is always a linear duality. If $i = 1$, then, according to Lemma 7.9 (which is valid for arbitrary $\mathbb{A}$ and not only for $\mathbb{A} = \mathbb{O}$), this is a standard polarity.

Now suppose that F has size 3. Then the residue of F is also a residue in a residue that is a projective plane, and hence, $\Pi^+(F)$ is the same as the special projectivity group in that projective plane. Completely similarly as in the previous paragraph, a self-projectivity of length 3 can be equivalently seen as a self-projectivity of length 3 of either (the lower residue of) a line or a planar line pencil. This reduces the determination of $\Pi(F)$ to the polar spaces $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ and $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$. For the former, the results follow from Proposition 7.1, Proposition 7.2 and Remark 7.3; for the latter, it follows from Section 7.1.1. □

We note the following isomorphisms:

$$\mathsf{PO}_3(\mathbb{K}) \cong \mathsf{PGL}_2(\mathbb{K}); \ \overline{\mathsf{PO}}_4^+(\mathbb{K},\mathbb{L}) \cong \mathsf{PGL}_2^+(\mathbb{L}/\mathbb{K});$$
$$\overline{\mathsf{PO}}_6^+(\mathbb{K},\mathbb{H}) \cong \mathsf{PGL}_2^+(\mathbb{H}), \overline{\mathsf{PO}}_{10}^+(\mathbb{K},\mathbb{O}) \cong \mathsf{PGL}_2^+(\mathbb{O}),$$

where $\mathbb{L}$ is a separable quadratic extension of $\mathbb{K}$, $\mathbb{H}$ is a quaternion division algebra over $\mathbb{K}$, and $\mathbb{O}$ an octonion division algebra over $\mathbb{K}$.

10. Conclusion for buildings of type F_4

We summarise the results obtained in the previous two sections for metasymplectic spaces in the following concluding theorem, including a tabular form.

Theorem 10.1. *Let $\mathsf{F}_4(\mathbb{K},\mathbb{A})$ be a building of type F_4, with $\mathbb{K}$ a field, and $\mathbb{A}$ a quadratic alternative division algebra over $\mathbb{K}$. Let F be a simplex with irreducible residue. Then $\Pi^+(F)$ and $\Pi(F)$ are as given in* Table 1, *where*

* $\mathbb{L}$ *denotes a separable quadratic extension of* $\mathbb{K}$*,*
* $\mathbb{H}$ *denotes a quaternion division algebra over* $\mathbb{K}$*,*
* $\mathbb{O}$ *denotes a Cayley (octonion) division algebra over* $\mathbb{K}$*,*
* $\mathbb{K}'$ *denotes an inseparable extension of* $\mathbb{K}$*, with* $\mathsf{char}\,\mathbb{K} = 2$*,*
* $\mathbb{A}'$ *denotes a separable quadratic alternative division algebra over* $\mathbb{K}$ *with* $\dim_{\mathbb{K}} \mathbb{A}' = d \in \{2,4,8\}$*.*

Reference	Δ	$\mathsf{Res}_\Delta(F)$	$\mathsf{cotyp}(F)$	$\Pi^+(F)$	$\Pi(F)$	
(A1)	$\mathsf{F}_4(\mathbb{K},\mathbb{A})$	$\mathsf{A}_1(\mathbb{K})$	$\{1\},\{2\}$	$\mathsf{PGL}_2(\mathbb{K})$	$\mathsf{PGL}_2(\mathbb{K})$	✓
	$\mathsf{F}_4(\mathbb{K},\mathbb{K})$	$\mathsf{A}_1(\mathbb{K})$	$\{3\},\{4\}$	$\mathsf{PGL}_2(\mathbb{K})$	$\mathsf{PGL}_2(\mathbb{K})$	
	$\mathsf{F}_4(\mathbb{K},\mathbb{A}')$	$\mathsf{A}_1(\mathbb{A}')$	$\{3\},\{4\}$	$\overline{\mathsf{PO}}^+_{d+2}(\mathbb{K},\mathbb{A}')$	$\overline{\mathsf{PO}}_{d+2}(\mathbb{K},\mathbb{A}')$	
(A2)	$\mathsf{F}_4(\mathbb{K},\mathbb{A})$	$\mathsf{A}_2(\mathbb{K})$	$\{1,2\}$	$\mathsf{PGL}_3(\mathbb{K})$	$\mathsf{PGL}_3(\mathbb{K}) \rtimes 2$	
	$\mathsf{F}_4(\mathbb{K},\mathbb{A})$	$\mathsf{A}_2(\mathbb{A})$	$\{3,4\}$	$\mathsf{PGL}_3(\mathbb{A})$	$\mathsf{PGL}_3(\mathbb{A}) \rtimes 2$	
	$\mathsf{F}_4(\mathbb{K},\mathbb{K}')$	$\mathsf{A}_2(\mathbb{K})$	$\{1,2\}$	$\mathsf{PGL}_3(\mathbb{K})$	$\mathsf{PGL}_3(\mathbb{K}) \rtimes 2$	
(B2)	$\mathsf{F}_4(\mathbb{K},\mathbb{K})$	$\mathsf{B}_2(\mathbb{K},\mathbb{K})$	$\{2,3\}$	$\mathsf{PO}_5(\mathbb{K})$	$\mathsf{PO}_5(\mathbb{K})$	✓
	$\mathsf{F}_4(\mathbb{K},\mathbb{A}')$	$\mathsf{B}_2(\mathbb{K},\mathbb{A}')$		$\overline{\mathsf{PO}}^+_{d+4}(\mathbb{K},\mathbb{A}')$	$\overline{\mathsf{PO}}^+_{d+4}(\mathbb{K},\mathbb{A}')$	✓
(C2)	$\mathsf{F}_4(\mathbb{K},\mathbb{K}')$	$\mathsf{B}_2(\mathbb{K},\mathbb{K}')$	$\{2,3\}$	$\overline{\mathsf{PSp}}_4(\mathbb{K}',\mathbb{K})$	$\overline{\mathsf{PSp}}_4(\mathbb{K}',\mathbb{K})$	✓
(B3)	$\mathsf{F}_4(\mathbb{K},\mathbb{K})$	$\mathsf{B}_3(\mathbb{K},\mathbb{K})$	$\{1,2,3\}$	$\mathsf{PO}_7(\mathbb{K})$	$\mathsf{PO}_7(\mathbb{K})$	
	$\mathsf{F}_4(\mathbb{K},\mathbb{A}')$	$\mathsf{B}_3(\mathbb{K},\mathbb{A}')$		$\overline{\mathsf{PO}}^+_{d+6}(\mathbb{K},\mathbb{A}')$	$\overline{\mathsf{PO}}_{d+6}(\mathbb{K},\mathbb{A}')$	
(C3)	$\mathsf{F}_4(\mathbb{K},\mathbb{K})$	$\mathsf{C}_3(\mathbb{K},\mathbb{K})$	$\{2,3,4\}$	$\overline{\mathsf{PSp}}_6(\mathbb{K})$	$\overline{\mathsf{PSp}}_6(\mathbb{K})$	✓
	$\mathsf{F}_4(\mathbb{K},\mathbb{L})$	$\mathsf{C}_3(\mathbb{L},\mathbb{K})$		$\overline{\mathsf{U}}_6(\mathbb{L}/\mathbb{K})$	$\overline{\mathsf{U}}_6(\mathbb{L}/\mathbb{K})$	✓
	$\mathsf{F}_4(\mathbb{K},\mathbb{H})$	$\mathsf{C}_3(\mathbb{H},\mathbb{K})$		$\overline{\mathsf{U}}_6(\mathbb{H})$	$\overline{\mathsf{U}}_6(\mathbb{H})$	✓
	$\mathsf{F}_4(\mathbb{K},\mathbb{O})$	$\mathsf{C}_3(\mathbb{O},\mathbb{K})$		$\overline{\mathsf{E}}^{28}_{7,3}(\mathbb{O})$	$\overline{\mathsf{E}}^{28}_{7,3}(\mathbb{O})$	✓
	$\mathsf{F}_4(\mathbb{K},\mathbb{K}')$	$\mathsf{C}_3(\mathbb{K}',\mathbb{K})$		$\overline{\mathsf{PSp}}_6(\mathbb{K}',\mathbb{K})$	$\overline{\mathsf{PSp}}_6(\mathbb{K}',\mathbb{K})$	✓

TABLE 1. Projectivity groups in the exceptional case F_4

The last column of Table 1 contains a checkmark (✓), if [5, Lemma 5.2] automatically yields $\Pi^+(F) = \Pi(F)$; see also Section 4.2. Note that there are other cases, for which $\Pi^+(F) = \Pi(F)$ generically holds, in contrast to the simply laced case, see [5].

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REFERENCES

[1] P. Abramenko & K. S. Brown, *Buildings. Theory and applications*, Graduate Texts in Math. **248**, Springer, New York, 2008.

[2] A. E. Brouwer, A non-degenerate generalized quadrangle with lines of size four is finite, **in** *Adv. Finite Geom. and Designs*, Proceedings Third Isle of Thorn Conference on Finite Geometries and Designs, Brighton 1990 (ed. J. W. P. Hirschfeld *et al.*), Oxford University Press, Oxford (1991), 47–49.

[3] F. Buekenhout & A. M. Cohen, *Diagram Geometry, Related to Classical Groups and Buildings*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics **57**, Springer Berlin, Heidelberg, 2013

[4] F. Buekenhout & E. E. Shult, On the foundations of polar geometry, *Geom. Dedicata* **3** (1974), 155–170.

[5] S. Busch, J. Schillewaert & H. Van Maldeghem, Groups of projectivities and Levi subgroups in spherical buildings of simply laced type, submitted.
[6] S. Busch & H. Van Maldeghem, A characterisation of lines in finite Lie incidence geometries of classical type, submitted.
[7] S. Busch & H. Van Maldeghem, Lines and opposition in Lie incidence geometries of exceptional type, in preparation.
[8] A. M. Cohen & E. E. Shult, Affine polar spaces, *Geom. Dedicata* **35** (1990), 43–76.
[9] B. De Bruyn & H. Van Maldeghem, Non-embeddable polar spaces. *Münster J. Math.* **7** (2014), 557–588.
[10] B. De Bruyn & H. Van Maldeghem, Dual polar spaces of rank 3 defined over quadratic alternative division algebras, *J. Reine Angew. Math.* **715** (2016), 39–74.
[11] A. De Schepper, J. Schillewaert, H. Van Maldeghem & M. Victoor, Construction and characterization of the varieties of the third row of the Freudenthal-Tits magic square, *Geom. Dedicata* **218** (2024), Paper No. 20, 57pp.
[12] J. Dieudonné, *Sur les groupes classiques*, Actualité scientifique et industrielles **1040**, *Publ. Inst. Math. Univ. Strasbourg VI*, Hermann, Paris, Nouvelle édition, 1973.
[13] T. Grundhöfer, Über Projectivitätengruppen affiner und projektiver Ebenen unter besonderer Berücksichtigung van Moufangebenen, *Geom. Dedicata* **13** (1983), 435–458.
[14] N. Knarr, Projectivities of generalized polygons, *Ars Combin.* **25B** (1988), 265–275.
[15] L. Lambrecht & H. Van Maldeghem, Automorphisms and opposition in spherical buildings of exceptional type, III. Metasymplectic spaces, submitted.
[16] B. Mühlherr, H. Petersson & R. M. Weiss, *Descent in Buildings*, Annals of Mathematics Studies **190**, Princeton University Press, 2015.
[17] A. Pasini & H. Van Maldeghem, An essay on Freudenthal-Tits polar spaces, *J. Algebra* **656** (2024), 367–393.
[18] S. Petit & H. Van Maldeghem, Generalized hexagons embedded in metasymplectic spaces, *J. Korean Math. Soc.* **60** (2023), 907–929.
[19] G. Pickert, Bemerkungen über die projektive Gruppe einer Moufang-Ebene, *Illinois J. Math.* **3** (1959), 169–173.
[20] E. E. Shult, On characterizing the long-root geometries, *Adv. Geom.* **10** (2010), 353–370.
[21] E. E. Shult, *Points and Lines: Characterizing the Classical Geometries*, Universitext, Springer-Verlag, Berlin Heidelberg, 2011.
[22] J. Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Math. **386**, Springer-Verlag, Berlin, 1974 (2nd printing 1986).
[23] J. Tits & R. Weiss, *Moufang Polygons*, Springer Monographs in Mathematics, Springer, 2002.
[24] H. Van Maldeghem, *Generalized Polgons*, Monographs in Mathematics **93**, Birkhaeuser, 1998.
[25] H. Van Maldeghem, *Polar Spaces*, Münster Lectures in Mathematics, Europ. Math. Soc. Press, 2024.

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