GROUPS OF PROJECTIVITIES AND LEVI SUBGROUPS IN SPHERICAL BUILDINGS OF SIMPLY LACED TYPE

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Abstract. We determine the exact structure and action of Levi subgroups of parabolic subgroups of groups of Lie type related to thick, irreducible, spherical buildings of simply laced type. Therefore we introduce the special and general projectivity groups attached to simplices F. If the residue of F is irreducible, we determine the permutation group of the projectivity groups of F acting on the residue of F and show that this determines the precise action of the Levi subgroup of a parabolic subgroup on the corresponding residue. This reveals three special cases for the exceptional types $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$. Furthermore, we establish a general diagrammatic rule to decide when exactly the special and general projectivity groups of F coincide.

Contents

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1. Introduction

The theory of buildings evolved during the search for analogues of exceptional simple Lie groups over arbitrary fields; traditionally people only worked over the fields $\mathbb{C}$ and $\mathbb{R}$. This was of interest, since working over arbitrary fields would allow the field to be finite and with that, one could find new families of finite simple groups. In 1955 Chevalley managed to construct these analogues and the groups he found are now known as *Chevalley groups*. After Chevalley published his work, Jacques Tits developed the theory of buildings attaching geometric structures to these groups (see page 335-335 of [1]).

Chevalley groups defined over arbitrary fields are known to be *groups of Lie type* (as in [6]). Groups of Lie type have BN-pairs and are hence associated to buildings (see page 108, Proposition 8.2.1 of [6]). They can be described as groups of automorphisms of spherical buildings (i.e. buildings with finite Weyl groups, see section 6.2.6 BN-Pairs of [1]). Chevalley groups are always simple except in the cases $A_1(2)$, $A_1(3)$, $B_2(2)$, $G_2(2)$ (see page 172, Theorem 11.1.2 of [6]).

In this article we will focus on buildings of simply laced type and rank at least 3. Such buildings automatically admit so-called *root elations* (see [28]). Then we can define the Chevalley group attached to such a building Δ as the group of automorphisms generated by all root elations, which we will denote by $\mathsf{Aut}^+(\Delta)$. This agrees with what is known as the the *adjoint Chevalley group* (see page 198 of [6]), and is also called the *little projective group of* Δ. It is always simple in our cases, since we assume the rank to be at least 3 (compare with Main Theorem of [27]).

Parabolic subgroups of Chevalley groups have attracted much attention in the literature. They can be written as semidirect products of a *unipotent subgroup* and a *Levi subgroup* (see page 118 of [6]). So far, a lot of research focussed on the unipotent subgroups (see for example [11, 19]). In this article we aim to shed some light on the Levi subgroups.

Let Σ be an apartment of Δ and C a chamber in Σ that we will consider to be the fundamental chamber. Let F be a face of C. A Levi subgroup of the parabolic subgroup G_F of $\mathsf{Aut}^+(\Delta)$ is a subgroup L_F such that G_F is the semi-direct product of L_F and a unipotent subgroup. This matches with how it has been traditionally defined in the literature (see page 158, Definition 11.22 of [2]). The parabolic subgroups opposite G_F correspond bijectively to the Levi subgroups of G_F (see page 199, Proposition 14.21 of [2]). Hence a Levi subgroup fixes a simplex and a unique opposite simplex pointwise, and it acts as a group of automorphisms on the link (or residue) of each of these simplices. In the present paper we determines the precise action of the Levi subgroup on that link. To the best of our knowledge, this was not recorded before.

Our method is geometric and uses *special and general projectivity groups*. In Theorem A we show that the special projectivity group of F coincides with the faithful permutation group induced by the stabiliser $\mathsf{Aut}^+(\Delta)_F$ of F in $\mathsf{Aut}^+(\Delta)$ on the residue $\mathsf{Res}_\Delta(F)$ of F in Δ. This connects the special projectivity group of a simplex F to Levi subgroup of F. Since we determine all general and special groups of projectivites, this determines the precise action of the Levi subgroup of a parabolic subgroup on the corresponding residue.

In the course of our proof, we also develop some basic and general theory about the projectivity groups. In projective geometry, the *groups of projectivities*, or *projectivity groups* play an important role in many proofs. For instance, projectivities between lines in a projective plane can be used to define non-degenerate conics (Steiner's approach) and prove properties of them. In [21], Knarr defined groups of projectivities and groups of even projectivities for generalised polygons and determined them in the finite case. This was further generalised to large infinite classes in [30], where the group of projectivities was called the *general projectivity group* and the group of even projectivities the *special projectivity group* related to a point or line. A generalisation of the definitions to all spherical buildings is obvious and natural questions are, for instance,

- when does the general projectivity group coincide with the special projectivity group, and

- can one determine the various general and special projectivity groups, particularly in the case where the residues are irreducible?

In the present paper we answer these questions for irreducible spherical buildings Δ with a simply laced diagram (see Remark 8.23 for the other cases). It will turn out that for residues of rank 1, we always have $\mathsf{PGL}_2(\mathbb{K})$ in its natural permutation group action. This is Theorem D. For (irreducible) residues R of rank at least 2, in most cases we generically obtain the maximal linear (algebraic or projective) group, including possible dualities if opposition in the Coxeter diagram of the ambient building is trivial, and the one in the Coxeter diagram of R is not trivial. There are only these four classes of exceptions:

(i) If Δ has type D_n and the type of R contains the types $n-1$ and n (hence R is of type D_ℓ, for some $\ell < n$), then the projectivity groups are contained in $\mathsf{PGO}_{2\ell}(\mathbb{K})$. Here, $\mathbb{K}$ is the underlying field. (Hence there are no similitudes in the projectivity groups.)

(ii) If Δ has type E_6 and R has type A_5, then the special and general projectivity group consists of those members of $\mathsf{PGL}_6(\mathbb{K})$ which correspond to matrices for which the determinant is a third power in the field $\mathbb{K}$ of definition.

(iii) If Δ has type E_7 and R has type A_5 containing the type 2 (in Bourbaki labelling), then the special projectivity group consists of those members of $\mathsf{PGL}_6(\mathbb{K})$ which correspond to matrices for which the determinant is a square in the field $\mathbb{K}$ of definition. The general projectivity group extends this group with a duality, for instance a symplectic polarity, with corresponding matrix of square determinant.

(iv) If Δ has type E_7 and R has type D_6, then the special and general projectivity group are the simple group $\mathsf{P}\Omega_{12}(\mathbb{K})$ extended with a class of diagonal automorphisms.

This is Theorem E. A complete list in tabular form of all special and general projectivity groups acting on irreducible residues of buildings of type $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ and D_n (for $n \geq 4$) is included in Section 8. In our arguments, the so-called polar vertices of the diagram will play a crucial role, and our results will entail a new combinatorial characterisation of the polar type. Theorem B and C below show that these polar types are basically the only ones responsible for the special and general projectivity groups to coincide.

The exceptions (i) to (iv) show that the questions stated above are not trivial and that the answer is rather peculiar, with exactly three special cases for the exceptional groups.

We now get down to definitions and statements of our Main Results.

2. Preliminaries and statement of the Main Results

We will need some notions and notation related to spherical buildings, and of point-line geometries related to those. Excellent references for buildings are the books [1] and [28], since it will be convenient to consider buildings as simplicial complexes. Standard references for the point-line approach to (spherical) buildings are [4] and [23].

2.1. **Spherical buildings.** Let Δ be a spherical building. We will assume, as in [28], that Δ is a thick numbered simplicial chamber complex, and we will usually denote the type set by $I = \{1, 2, \ldots, r\}$, where r is the rank of Δ, and the set of chambers by $\mathscr{C}(\Delta)$. The type $\mathsf{typ}(F)$ of a simplex F is the set of types of its vertices. A *panel* is a simplex of size $r-1$. *Adjacent* chambers are chambers intersecting in a panel. This defines in a natural way the *chamber graph*. The *(gallery) distance* $\delta(C, C')$ between two chambers C and C' is the distance in the *chamber graph* of the vertices corresponding to C and C'.

One of the defining axioms of a spherical building is that every pair of simplices is contained in an *apartment*, which is a thin simplicial chamber subcomplex isomorphic to a finite Coxeter complex $\Sigma(W, S)$ with associated Coxeter system (W, S), where W is a Coxeter group with respect to the generating set S of involutions. If $S = \{s_1, \ldots, s_r\}$, then let $P_i = \langle s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_r \rangle$, $i \in I$, be the maximal parabolic subgroups. The vertices of $\Sigma(W, S)$ of type $i \in I$ are the right cosets of P_i. The chambers are the sets of cosets of maximal parabolic subgroups containing a given member w of W. For each pair (C, C') of adjacent chambers there exists exactly one *folding*, that is, a type preserving idempotent morphism of $\Sigma(W,S)$ mapping C' to C, and such that each chamber in the image has two chambers in its pre-image. The image α of a folding is called a *root*. The root associated to the *opposite folding*, namely, the folding mapping C to C' is called the *opposite* root, and is denoted by $-\alpha$. The intersection $\alpha \cap (-\alpha)$, called a *wall*, is denoted by $\partial\alpha$ (and hence also by $\partial(-\alpha)$), and is also referred to as the *boundary of* α. Every root contains a unique simplex that is fixed under each automorphism of $\Sigma(W,S)$ preserving α (and not necessarily type preserving). This simplex is called the *centre* of the root. If $\Sigma(W,S)$, or equivalently, Δ, is irreducible (see below), the type of such simplex is called a *polar type* of Δ. In the reducible case, the polar types of the connected components will be called *polar types* of the building.

For each vertex v of $\Sigma(W,S)$, there exists a unique other vertex v' of $\Sigma(W,S)$ with the property that every wall containing v also contains v' (and then automatically every wall containing v' contains v); then v and v' are called *opposite* vertices. *Opposite* simplices of $\Sigma(W,S)$ are two simplices A, B with the property that the vertex opposite to any vertex in A is contained in B, and vice versa. We denote $A \equiv B$. Opposition defines a permutation, also denoted by $\equiv$, of order at most 2 on the type set I. A subset $J \subseteq I$ is called *self-opposite* if $J^{\equiv} = J$. The permutation $\equiv$ acting on I induces in fact an automorphism of the corresponding Coxeter diagram. Recall that the vertices of the Coxeter diagram correspond to the types, that is, the elements of I, and two vertices i and j are connected by an edge of weight $m_{ij} - 2$, where m_{ij} is the order of s_is_j in W. Throughout, we use the Bourbaki labelling of connected spherical Coxeter diagrams [3]. The Coxeter diagram, and by extension the chamber complex $\Sigma(W,S)$ and the building Δ, are called *simply laced* if $m_{ij} \in \{2,3\}$, for all $i, j \in \{1, 2, \dots, r\}$, $i \neq j$. The building Δ is irreducible if the Coxeter diagram is connected. The polar type in the simply laced and irreducible case is unique. It is the set of nodes to which the additional generator is joined when constructing the affine diagram. Hence it is $\{1, r\}$ in case A_r, it is 2 in case of D_r, and $2, 1, 8$ for $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$, respectively.

Opposite simplices in Δ are simplices that are opposite in some apartment, and then the building axioms guarantee that they are opposite in every apartment in which they are both contained.

We say that a vertex v and a simplex F are *joinable* if $v \notin F$ and $F \cup \{v\}$ is a simplex; notation $v \sim F$. (Note that we denote simplices with capital letters such as F since the letter S already has a meaning. The letter F stands for "flag", which is a synonym of simplex in the language of geometries.) The simplicial complex induced on the vertices joinable to a given simplex F of a building Δ forms a building called the *residue of* F *in* Δ and denoted by $\mathsf{Res}_\Delta(F)$ (also sometimes called the *link*). It is well known that the Coxeter diagram of that residue is obtained from the Coxeter diagram of the building by deleting the vertices with type in $\mathsf{typ}(F)$. The opposition relation in $\mathsf{Res}_\Delta(F)$ will be denoted by $\equiv_F$ (also on the types), and two simplices of $\mathsf{Res}_\Delta(F)$ opposite in $\mathsf{Res}_\Delta(F)$ will be occasionally called *locally opposite at* F. The cotype $\mathsf{cotyp}(F)$ of a simplex F is $I \setminus \mathsf{typ}(F)$, and the *type* of the residue $\mathsf{Res}_\Delta(F)$ is the cotype of F.

Now let F and F' be two opposite simplices. Let $C \in \mathscr{C}(\Delta)$ be such that $F \subseteq C$. Then there exists a unique chamber $C' \supseteq F'$ at minimal gallery distance from C. The chamber C' is called the *projection of* C *from* F *onto* F' and denoted $\mathsf{proj}^F_{F'}(C)$. That mapping is a bijection from the set of chambers through F to the set of chambers through F' and preserves adjacency in both directions. It follows that it defines a unique isomorphism from $\mathsf{Res}(F)$ to $\mathsf{Res}(F')$, which we denote by $\mathsf{proj}^F_{F'}$ (as it is a special case of the projection operator, see 3.19 of [28]), see also Theorem 3.28 of [28]. When the context makes F clear, we sometimes remove the F from the notation for clarity and simply write $\mathsf{proj}_{F'}$. This projection has the following property.

Proposition 2.1 (Proposition 3.29 of [28])**.** *Let F and F' be opposite simplices of a spherical building Δ. Let v be a vertex of Δ adjacent to each vertex of F, and set $i := \mathsf{typ}(v) \in I$. Then the type i' of the vertex $\mathsf{proj}^F_{F'}(v)$ is the opposite in $\mathsf{Res}(F')$ of the opposite type of i in Δ, that is, $i' = (i^{\equiv})^{\equiv_{F'}}$. Also, vertices $v \sim F$ and $v' \sim F'$ are opposite in Δ if, and only if, $v' \equiv_{F'} \mathsf{proj}^F_{F'}(v)$.*

Now let α be a root of Δ. Let U_α be the group of automorphisms of Δ pointwise fixing every chamber that has a panel in $\alpha \setminus \partial\alpha$. The elements of U_α are called *(root) elations* and U_α itself is called a *root group*. If U_α acts transitively on the the set of apartments containing α, then we say that α is *Moufang*. If every root is Moufang, then we say that Δ is Moufang. The automorphism group of Δ is denoted by $\mathsf{Aut}\ \Delta$ and, if Δ is Moufang, then the subgroup generated by the root elations is denoted by $\mathsf{Aut}^+\ \Delta$ and called the *little projective group of* Δ.

Every spherical building Δ of rank $r \geq 3$ is Moufang. If Δ is simply laced, then the root group U_α associated to the root α only depends on the centre of α, that is, each member of U_α pointwise fixes each chamber of Δ having a panel in $\beta \setminus \partial\beta$, for β any root having the same centre as α.

2.2. **Groups of projectivity.** Let Δ be a spherical building and F, F' two simplices which are opposite, and which are not chambers. Then we call the isomorphism $\mathsf{proj}^F_{F'}$ a *perspectivity (between residues)* and denote $F \barwedge F'$. If $F_0, F_1, \ldots, F_\ell$ is a sequence of consecutively opposite simplices, then the isomorphism $\mathsf{Res}(F_0) \to \mathsf{Res}(F_\ell)$ given by $\mathsf{proj}^{F_{\ell-1}}_{F_\ell} \circ \cdots \mathsf{proj}^{F_1}_{F_2} \circ \mathsf{proj}^{F_0}_{F_1}$ is called a *projectivity (of length ℓ)*. If ℓ is even, it is called an *even* projectivity, and if $F_0 = F_\ell$, it is called a *self-projectivity*. The set of all self-projectivities of a simplex F is a group called the *general projectivity group of* F and denoted $\Pi(F)$. Likewise, the set of all even self-projectivities of a simplex F is a group called the *special projectivity group of* F and denoted $\Pi^+(F)$. Note that $\Pi(F) = \Pi^+(F)$ as soon as $(\mathsf{typ}(F))^\equiv \neq \mathsf{typ}(F)$.

Let $\Pi(F)$ be the general projectivity group of the simplex F of a spherical building Δ, with F not a chamber. Then, as an abstract permutation group, $\Pi(F)$ only depends on the type of F. Likewise, the special projectivity group $\Pi^+(F)$ only depends on the type of F. We have the natural inclusion $\Pi^+(F) \trianglelefteq \Pi(F)$ and $[\Pi(F) : \Pi^+(F)] \leq 2$. We denote the number $[\Pi(F) : \Pi^+(F)]$ by $n(J)$, where the type of F is J. We trivially have $n(J) = n(J^\equiv)$, because it is 1 if $J^\equiv \neq J$.

In the case that Δ has rank 2, that is, Δ is the building of a generalised polygon, F is necessarily a single vertex and can be thought of as either a point (type 1) or a line (type 2) of the generalised polygon. Knarr [21] shows that, if Δ is Moufang, then for every point or line x of Δ, the group $\Pi^+(x)$ coincides with the stabiliser of x in the little projective group of Δ, that is, the group generated by the root groups. We generalise this to arbitrary simplices in arbitrary Moufang spherical buildings of simply laced type. This is our first main result, Theorem A. The strategy of the proof is the same as for the rank 2 case. However, the proof requires that the unipotent radical of a parabolic subgroup in a Moufang spherical building pointwise stabilises the corresponding residue, and acts transitively on the simplices opposite the given residue. This follows from the Levi decomposition of parabolic subgroups in Chevalley groups. We provide a brief introduction.

2.3. **The Levi decomposition in Chevalley groups.** Let Δ be a building and F a simplex of type J. Suppose Δ is of simply laced type and has rank at least 3. Then, by the classification in [28], Δ admits an automorphism group G which is a Chevalley group, or, in case Δ corresponds to a projective space of dimension d defined over a noncommutative skew field $\mathbb{L}$, we can take for G the full linear group $\mathsf{PGL}_{d+1}(\mathbb{L})$. The stabiliser P_F of F is called a *parabolic subgroup* and, if G is a Chevalley group, admits a so-called *Levi decomposition* $P_F = U_F L_F$, see Section 8.5 of [6], where U_F is the so-called *unipotent radical* of P_F and L_F is called a *Levi subgroup*.

We provide an explicit description of P_F, U_F and L_F for $\mathsf{PGL}_{d+1}(\mathbb{L})$ in the case that we will need most in the present paper, namely when $\mathsf{Res}_\Delta(F)$ is irreducible. In that case one chooses the basis in such a way that each subspace of F of dimension i is generated by the first $i+1$ base points. Also, F consists of i-dimensional subspaces with $0 \leq i \leq d_1 - 1$ and $d - d_3 \leq i \leq d - 1$, where $|F| = d_1 + d_3$. Set $d_2 := d + 1 - d_1 - d_3$. Note that $J = \{1, \ldots, d_1, d - d_3 + 1, \ldots, d\}$.

Then a generic element of P_F looks like

$$\begin{pmatrix} T_{d_1} & M_{d_1\times d_2} & M_{d_1\times d_3} \\ O_{d_2\times d_1} & M_{d_2\times d_2} & M_{d_2\times d_3} \\ O_{d_3\times d_1} & O_{d_3\times d_2} & T_{d_3} \end{pmatrix},$$

where T_{d_i}, $i = 1, 3$, is an arbitrary invertible upper triangular matrix over F (needless to say that T_{d_1} and T_{d_3} are independent of each other; even if $d_1 = d_3$ they are considered different), $M_{d_i\times d_j}$ is an arbitrary $d_i \times d_j$ matrix, $i \in \{1, 2\}$ and $j \in \{2, 3\}$ (with similar remark as for the T_{d_i}), and $O_{d_i\times d_j}$ is the $d_i \times d_j$ zero matrix, $i \in \{2, 3\}$, $j \in \{1, 2\}$. With similar notation, and on top with U_{d_i}, $i \in \{1, 3\}$, an arbitrary unipotent upper triangular $d_i \times d_i$ matrix, D_{d_i}, $i \in \{1, 3\}$ an arbitrary invertible diagonal $d_i \times d_i$ matrix and I_{d_2} the $d_2 \times d_2$ identity matrix, generic elements of U_F and L_F look like (blanks replace zero matrices)

$$\begin{pmatrix} U_{d_1} & M_{d_1\times d_2} & M_{d_1\times d_3} \\ & I_{d_2} & M_{d_2\times d_3} \\ & & U_{d_3} \end{pmatrix} \text{ and } \begin{pmatrix} D_{d_1} & & \\ & M_{d_2\times d_2} & \\ & & D_{d_3} \end{pmatrix},$$

respectively. One indeed checks that $P_F = U_F L_F$ and $U_F \cap L_F = \{I_{d+1}\}$. Also, the following lemma is easily checked in this case, and hence we only provide a proof for the Chevalley groups.

Lemma 2.2. *Let Δ be a spherical Moufang building of simply laced type and let F be a simplex of Δ of type J. Let P_F be the stabiliser of F in $\mathsf{Aut}^+(\Delta)$. Then the unipotent radical $U_F \leq P_F$ acts sharply transitively on the set $F^{\equiv}$ of simplices opposite F, and pointwise fixes $\mathsf{Res}_\Delta(F)$.*

Proof. From the definition of U_F in Section 8.5 of [6], it is readily deduced that U_F pointwise fixes $\mathsf{Res}_\Delta(F)$, because it is generated by root groups whose corresponding roots contain F, but not in their boundary. Furthermore, a Levi subgroup L_F is just the stabiliser in G of F and an opposite simplex F' as described in the introduction. Since P_F acts transitively on $F^{\equiv}$ (by the BN-pair property of Chevalley groups), we find that U_F acts transitively on $F^{\equiv}$. Since $U_F \cap L_F$ is just the identity (see Theorem 8.5.2 of [6]), the lemma follows. □

We will be interested in the faithful permutation group induced by L_F on $\mathsf{Res}_\Delta(F)$.

2.4. Main results.

Theorem A. *Let F be a simplex of a Moufang spherical building Δ. Let $\mathsf{Aut}^+(\Delta)$ be the automorphism group of Δ generated by the root groups. Then $\Pi^+(F)$ is permutation equivalent to the faithful permutation group induced by the stabiliser $\mathsf{Aut}^+(\Delta)_F$ of F in $\mathsf{Aut}^+(\Delta)$ on the residue $\mathsf{Res}_\Delta(F)$ of F in Δ.*

Going back to the case where Δ is a Moufang building of rank 2, the results in Chapter 8 of [30] show that $n(\{1\}) = n(\{2\}) = 1$ as soon as Δ is a so-called "Pappian polygon" (for a definition of the latter, see Section 3.5 of [30]). In any case, we always have $1 \in \{n(\{1\}), n(\{2\})\}$ due to Lemma 8.4.6 of [30]. One of the goals of the present paper is to generalise this to all spherical buildings. This will be achieved by proving a general sufficient condition in J for $n(J)$ being equal to 1. To state this, we say that the type J of a simplex is *polar closed* if we can order the elements of a partition of J into singletons and pairs, say $J_1, \ldots, J_k$ such that, for each $\ell \in \{1, \ldots, k\}$, the type J_ℓ is a polar type in the residue of $J_1 \cup \cdots \cup J_{\ell-1}$. We then have:

Theorem B. *Let Δ be a spherical building with type set I. If either $J \neq J^{\equiv}$ or $J \subseteq I$ is polar closed, then $n(J) = 1$.*

To see a partial converse of this statement, we restrict to the simply laced case (see also Remark 8.23).

Theorem C. *Let Δ be an irreducible spherical building of simply laced type with type set I. If $J \subseteq I$, $J^{\equiv} = J$ and $I \setminus J$ has at least one connected component K of size at least 2 such that $I \setminus K$ is not polar closed, then $n(J) = 2$.*

Note that, if J is polar closed, then for each connected component K of $I \setminus J$ the type set $I \setminus K$ is polar closed.

This implies the following combinatorial characterisation of the polar type in connected simply laced spherical diagrams. For $K \subseteq I$ we denote by $\overline{K}$ the union of all connected components of K of size at least 2.

Corollary 1. *The polar type of a connected simply laced spherical diagram D_I over the type set I is the unique smallest subset $J \subseteq I$ with the property that opposition in $D_{\overline{I \setminus J}}$ coincides with opposition in D_I.*

Corollary 1 does not hold in the non-simply laced case (since opposition does not determine the direction of the arrow in the Dynkin diagram). Indeed, for types B_n, C_n and F_4, there are each time two single types satisfying the given condition, reflecting the fact that, in characteristic 2, there are really two choices.

Finally, we consider the case left out in Theorem C above, where $I \setminus J$ has only connected components of rank 1. We reduce the action of $\Pi^+(F)$ on each panel to a case where $|I \setminus J| = 1$ and show:

Theorem D. *Let Δ be an irreducible spherical building of simply laced type with type set I. Let $J \subseteq I$ with $|I \setminus J| = 1$, and let P be a panel of type J. Then $\Pi^+(P)$ is permutation equivalent to the natural action of $\mathsf{PGL}_2(\mathbb{K})$ on the projective line $\mathsf{PG}(1, \mathbb{K})$, and equals $\Pi(P)$.*

In view of Theorem D, one could expect that the general and special projectivity groups of simplices whose residue is isomorphic to $\mathsf{A}_r(\mathbb{K})$ are isomorphic to $\mathsf{PGL}_{r+1}(\mathbb{K})$. This is indeed in most cases true, but not always. If it is not true, then necessarily the residue in question is not contained in a larger residue of type A_{r+1}. Our last main result determines the exact permutation representations of the special and general projectivity groups on the corresponding residues of the building.

Theorem E. *Let Δ be an irreducible spherical building of simply laced type with type set I. Let $I \neq J \subseteq I$ with $I \neq I \setminus J$ connected and let F be a simplex of type J. Then $\Pi^+(F)$ and $\Pi(F)$ are as in* Table 1 *and* Table 2 *for* $\mathsf{typ}(\Delta) \in \{\mathsf{D}_r, \mathsf{E}_m \mid r \geq 4, m = 6, 7, 8\}$*, and to* $\mathsf{PGL}_n(\mathbb{L})$ *in its natural action, if Δ has type A_r, $r \geq 2$, it is defined over the skew field $\mathbb{L}$, and $|I \setminus J| = n - 1$.*

The notation used in Tables 1 and 2 is explained in Section 8, where Theorem E is proved.

2.5. **Lie incidence geometries.** Some arguments — in particular those in Section 8 — will be more efficiently carried out in a specific point-line geometry related to the spherical building in question. We provide a brief introduction here. More details can be found in textbooks like [4] and [23].

2.5.1. *Point-line geometries, projective spaces, polar spaces and parapolar spaces.* Recall that a *point-line* geometry $\Gamma = (X, \mathscr{L})$ consists of a set X whose elements are called *points*, and a subset $\mathscr{L}$ of the full set of subsets of X, whose members are called *lines* (hence we disregard geometries with so-called repeating lines). The notion of *collinear points* will be used frequently. We denote collinearity of two points x and y with $x \perp y$, and $x^\perp$ has the usual meaning of the set of points collinear to x (including x). A *(proper) subspace* is a (proper) subset of the point set intersecting each line in either 0,1 or all of the points of the line. A *(proper) hyperplane* is a (proper) subspace intersecting each line non-trivially. The *point graph* of Γ is the graph with vertices the points, adjacent when collinear. A subspace is *convex* if its induced subgraph in the point graph is convex (all vertices on paths of minimal length between two vertices of the subspace are contained in the subspace). We will frequently regard a subspace as a subgeometry in the obvious way. A subspace is called *singular* if every pair of points in it is collinear. In our case singular subspaces will always be projective spaces. Lines and planes are short for 1- and 2-dimensional projective (sub)spaces, respectively.

The distance between points is the distance in the point graph and the diameter of the geometry is the diameter of the point graph.

We usually require that Γ is *thick*, that is, each line contains at least three points.

For example, the 1-spaces of any vector space V of dimension at least 3 over some skew field $\mathbb{L}$ form the point set of a thick geometry $\mathsf{PG}(V)$ the lines of which are the 1-spaces contained in a given 2-space. This geometry is a projective space. The hyperplanes correspond to the codimension 1 subspaces of V.

A *polar space* is a thick point-line geometry such that for each point x, the set $x^{\perp}$ is a hyperplane (which we require to be distinct from the whole point set).

A pair of points of Γ is called *special* if they are not collinear and there is a unique point of Γ collinear to both. Then Γ is called a *parapolar space* if every pair of points at distance at most 2 is contained in a convex subspace isomorphic to a polar space. Such convex subspaces are called *symplecta*, or *symps* for short. A pair p, q of non-collinear points of a symp is called *symplectic*; in symbols $p \perp\!\!\!\perp q$.

Given an irreducible spherical building Δ of rank r at least 2 of type X_r over the type set I. Let $J \subseteq I$ and define X as the set of all simplices of Δ of type J. The set $\mathscr{L}$ of lines consists of the sets of simplices of type J completing a given panel whose type does not contain J to a chamber. The geometry $(X, \mathscr{L})$ is usually referred to as the *Lie incidence geometry of type* $\mathsf{X}_{r,J}$ (where we replace J by its unique element if $|J| = 1$). The main observation here (see the above references), usually referred to as *Cooperstein's theory of symplecta* [9, 10] is that $(X, \mathscr{L})$ is either a projective space, a polar space, or a parapolar space.

In the present paper, we will only use projective spaces over arbitrary skew fields (they are related to buildings of type A_r), polar spaces (that are related to buildings of type D_r), and some specific parapolar spaces that are related to buildings of types E_6, E_7 and E_8 over a field $\mathbb{K}$). Polar spaces related to buildings of type D_r will usually be called *polar spaces of type* D_r, or *hyperbolic* polar spaces since in rank $r \geq 4$, they are in one-to-one correspondence to hyperbolic quadrics in projective spaces. Recall that a *hyperbolic quadric* is the projective null set of a quadratic form of maximal Witt index in a vector space V of even dimension. The standard form (using coordinates $x_{-r}, \ldots, x_{-1}, x_1, \ldots, x_r$) is given by

$$x_{-r}x_r + x_{-r+1}x_{r-1} + \cdots x_{-2}x_2 + x_{-1}x_1.$$

The automorphisms of Δ induced by elements of $\mathsf{PGL}(V)$ will be called *linear*. They conform to the elements of the corresponding (maximal) linear algebraic group. Note that hyperbolic quadrics contain two natural classes of maximal singular subspaces characterised by the fact that members of distinct classes intersect in subspaces of odd codimension (the *codimension* is the vector dimension of a complementary subspace).

Concerning types E_r, $r = 6, 7, 8$, we list here some basic properties of the Lie incidence geometries of types $\mathsf{E}_{6,1}$, $\mathsf{E}_{7,7}$ and $\mathsf{E}_{8,8}$ that we will make use of. Most of them can be read off the diagram, and others follow from considering an apartment of the building. They are called "facts" in papers like [14, 16]. For Lie incidence geometries of type $\mathsf{E}_{6,1}$, good references are [12] and [26], and for Lie incidence geometries of type $\mathsf{E}_{7,7}$ a good reference is [13]. In both papers the basic facts are explained in some more detail. On the other hand, Lie incidence geometries of type $\mathsf{E}_{8,8}$ are so-called *long root subgroup geometries* and therefore satisfy some well-known generic properties that can be found in [7].

2.5.2. *Lie incidence geometries of type* $\mathsf{E}_{6,1}$. These geometries have diameter 2 and contain no special pairs of points. Hence every pair of points is contained in a symp. Symps are polar spaces of type D_5. The basic properties which we shall use without notice are summarised in the following lemma.

Lemma 2.3. *Let Γ be a Lie incidence geometry of type $\mathsf{E}_{6,1}$ over a field $\mathbb{K}$. Then the following properties hold.*

(*i*) *Two distinct symps either meet in a unique point, or share a maximal singular subspace of either of them, referred to as a* 4*-space.*
(*ii*) *A point* p *and a symp* ξ*, with* $p \notin \xi$*, either satisfy* $p^\perp \cap \xi = \varnothing$*, or* $p^\perp \cap \xi$ *is a maximal singular subspace of* ξ*, referred to as a* $4'$*-space.*
(*iii*) *The* 4*-spaces in a given symp form one natural class of maximal singular subspaces of* ξ*; the* $4'$*-spaces form the other.*

In the building 4-spaces correspond to vertices of type 5, whereas $4'$-spaces correspond to simplices of type $\{2, 6\}$.

We now mention some other facts. The first one can be read off the diagram. It is also contained as Fact 4.14 in [14].

Lemma 2.4. *Let* Γ *be a Lie incidence geometry of type* $\mathsf{E}_{6,1}$ *over a field* $\mathbb{K}$*. Then the following hold.*

(*i*) *A* 4*-space and a* $4'$*-space that have a plane* π *in common intersect in a* 3*-space. Consequently, a* 4*-space and a* 5*-space that share a plane, share a* 3*-space.*
(*ii*) *Two distinct non-disjoint* 5*-spaces intersect in either a point or a plane. Consequently, a* $4'$*-space that shares a* 3*-space with a* 5*-space is contained in it.*
(*iii*) *Two disjoint* 5*-spaces that are not opposite contain respective planes contained in a common* 5*-space. Every point of each of the two* 5*-space is collinear to some point of the plane contained in the other* 5*-space.*

Lemma 2.5. *Let* Γ *be a Lie incidence geometry of type* $\mathsf{E}_{6,1}$ *over a field* $\mathbb{K}$*. Let* ξ *be a symp in* Γ *and let* π *be a plane in* Γ *intersecting* ξ *in a unique point* x*. Then there exists a unique plane* $\alpha \subseteq \xi$ *all points of which are collinear to all points of* π*.*

Proof. Let L be a line in π not intersecting ξ. The lemma now follows from Fact A.9 of [16]. □

2.5.3. *Lie incidence geometries of type* $\mathsf{E}_{7,7}$. These geometries have diameter 3 and contain no special pairs of points. Points at distance 3 correspond to opposite vertices of type 7 in the corresponding building. Hence every pair of non-opposite points is contained in a symp. Symps are polar spaces of type D_6. The basic properties which we shall use without notice are summarised in the following lemma.

Lemma 2.6. *Let* Γ *be a Lie incidence geometry of type* $\mathsf{E}_{7,7}$ *over a field* $\mathbb{K}$*. Let* x *be a point and* ξ *a symp. Then either*

(*i*) $x \in \xi$*, or*
(*ii*) $x \notin \xi$*,* x *is collinear to each point of a unique* $5'$*-space of* ξ *and symplectic to all other points of* ξ*, or*
(*iii*) $x \notin \xi$*,* x *is collinear to a unique point* x' *of* ξ*, symplectic to all points of* ξ *collinear to* x'*, and opposite each other point of* ξ*.*

In Case (*ii*) above, the point x is said *to be close to* ξ, whereas in Case (*iii*) it is said *to be far from* ξ.

Two distinct symps sharing at least a plane, share a 5-space. Again, the 5-spaces in a given symp form one natural class of maximal singular subspaces, whereas the $5'$-spaces form the other class.

Lemma 2.7. *Let* Γ *be a Lie incidence geometry of type* $\mathsf{E}_{7,7}$ *over a field* $\mathbb{K}$*. Let* M *be a maximal* 5*-space and let* ξ *and* ξ' *be two distinct symps containing* M*. Let* $p \in \xi \setminus M$ *and* $p' \in p^\perp \cap (\xi' \setminus M)$*. Then every point on the line* $\langle p, p' \rangle$ *is contained in a (unique) symp which contains* M*.*

2.5.4. *Lie incidence geometries of type* $\mathsf{E}_{8,8}$. These geometries have diameter 3 and contain special pairs of points. Points at distance 3 correspond to opposite vertices of type 8 in the corresponding building. Symps are polar spaces of type D_7. We have singular 6-spaces, occurring as the intersection of symps and corresponding to vertices of type 3 in the corresponding building, and singular $6'$-spaces, occurring as the the intersection of a symp with a singular 7-space and hence corresponding to simplices of type $\{1,2\}$ in the corresponding building.

The basic properties which we sometimes shall use without notice are summarised in the following lemmas. As already mentioned, these geometries are long root subgroup geometries. A property that has been used to characterise them, e.g. in[20], is the following.

Lemma 2.8. *Let* Γ *be a Lie incidence geometry of type* $\mathsf{E}_{8,8}$ *over a field* $\mathbb{K}$. *Let* p *be a point,* π *a plane and* $L \subseteq \pi$ *a line. Suppose* p *is collinear to some point* $q \in \pi \setminus L$ *and special to at least two points of* L. *Then it is symplectic to precisely one point of* L.

The following properties can be found in [7].

Lemma 2.9. *Let* Γ *be a Lie incidence geometry of type* $\mathsf{E}_{8,8}$ *over a field* $\mathbb{K}$. *Let* p, q *be two points and* ξ *a symp.*

- (i) *The points* p *and* q *are opposite if, and only if, there exist points* x, y *with* $p \perp x \perp y \perp q$ *such that* $\{p, y\}$ *and* $\{q, x\}$ *are special.*
- (ii) *Each line contains at least one point not opposite* p. *If it contains at least two such points, then no point on the line is opposite* p.
- (iii) *If* ξ *contains a point opposite* p, *then* ξ *contains a unique point* x *symplectic with* p; *all points of* $\xi \setminus \{x\}$ *collinear to* x *are special to* p *and all other points of* ξ *are opposite* p.

Also, being symplectic pointwise defines an isomorphism between opposite symps.

A point and a symp containing an opposite point will be called *far from each other*. The grading of the geometry as explained in [7] or, equivalently, the model of an apartment as given in [31], implies the following fact.

Lemma 2.10. *Let* Γ *be a Lie incidence geometry of type* $\mathsf{E}_{8,8}$ *over a field* $\mathbb{K}$. *Let* p *be a point and* ξ *a symp. Then exactly one of the following occurs.*

- (i) *The point* p *belongs to* ξ.
- (ii) *The point* p *is collinear to a* $6'$ *space contained in* ξ *and symplectic to all other points of* ξ.
- (iii) *The point* p *is collinear to a unique line* L *of* ξ, *symplectic to all points of* ξ *collinear with* L *and special to all other points of* ξ.
- (iv) *The point* p *is symplectic to all points of a unique* 6*-space of* ξ *and special to all other points of* ξ.
- (v) *The point* p *is far from* ξ.

The following can be easily read off the diagram.

Lemma 2.11. *Symps in parapolar spaces of type* E_8 *are polar spaces of type* D_7. *A* $6'$*-space is the intersection of a symp and a maximal singular subspace of type* A_7 *(hence a* 7*-dimensional singular subspace). The intersection of two* $6'$*-spaces in a symp has even codimension in both. Hence two distinct* 7*-spaces sharing a* 3*-space intersect precisely in a* 4*-space. Also, a symp sharing a* 5*-space with a* 7*-space intersects that* 7*-space in a* $6'$*-space.*

We also have the following properties.

Lemma 2.12. *Let* Γ *be a Lie incidence geometry of type* $\mathsf{E}_{8,8}$ *over a field* $\mathbb{K}$. *Let* ξ *and* ξ' *be two symps of* Γ *intersecting in a* 6*-space* U. *If* L *is a line intersecting* ξ *in a point* x *and* ξ' *in a point* x', *with* $x \neq x'$, *then the map* $\zeta \mapsto \zeta \cap L$ *defines a bijection between the set of symps containing* U *and the set of points on* L.

Proof. If there were a point u of U collinear to x but not to x', then the symp ξ' through x' and u would contain x, a contradiction. Hence there is a 7-space W containing L and a hyperplane H of U. A symp ζ containing U shares the 5-space H with W and hence, by the last assertion of Lemma 2.11, intersects W in a $6'$-space U_ζ. The latter has a unique point in common with L.

Conversely, let q be any point on L. Then the $6'$-space $\langle q, H\rangle$ is contained in a unique symp ζ which clearly intersects L in q. □

Lemma 2.13. *Let Γ be a Lie incidence geometry of type $\mathsf{E}_{8,8}$ over a field $\mathbb{K}$. Let U and U' be two opposite 6-spaces of Γ. Let $\xi \supseteq U$ and $\xi' \supseteq U'$ be two symps that are not opposite. Then there exist $6'$ spaces $V \subseteq \xi$ and $V' \subseteq \xi'$ such that no point of $V \cup V'$ is opposite any point of $\xi \cup \xi'$. The set of points of ξ' symplectic to a given point of V forms a 6-space intersecting V' in a 5-space. Likewise, the set of points of ξ symplectic to a given point of V' forms a 6-space intersecting V in a 5-space. Also, every point x of $\xi \setminus V$ is far from ξ' and the unique point of ξ' symplectic to x is contained in U'. Likewise, every point x' of $\xi' \setminus V'$ is far from ξ and the unique point of ξ symplectic to x' is contained in U.*

Proof. Since U is opposite U', the symp ξ is opposite some symp $\xi^* \supseteq U'$, and hence every point of U' is symplectic to a unique point of ξ. The set of points of ξ thus obtained is a $6'$-space V, since being symplectic defines an isomorphism between ξ and ξ^* by Lemma 2.9. Clearly, V is independent of ξ^* and consequently it is the unique set of points of ξ symplectic to some point of U'. Likewise, let V' be the set of points of ξ' symplectic to some point of U. Pick a point $v \in V$. Let u' be the unique point of U' symplectic to v. Let u be an arbitrary point of U not collinear to v. Then u and u' are opposite. Let $E(u, u')$ be the equator geometry as defined in [16]; then $E(u, u')$ consists of all points symplectic to both u and u' and with the induced line set it is a Lie incidence geometry of type $\mathsf{E}_{7,1}$. Note $v \in E(u, u')$. Let v' be the unique point of V' symplectic to u and note $v' \in E(u, u')$. The set of symps through U corresponds to a line $L \ni v$ in $E(u, u')$; likewise the set of symps through U' corresponds to a line $L' \ni v'$ in $E(u, u')$. Since U and U' are opposite, the lines L and L' are opposite. It follows that the points v and v' are special. Since u' is not collinear to v', Lemma 2.9(*iii*) implies that the point v is not opposite any point of ξ'. So, we have shown that no point of V is opposite any point of ξ'. Suppose now some point $x \in \xi \setminus V$ is not opposite every point of ξ'. Since ξ is hyperbolic, x is contained in a line K which intersects both U and V non-trivially. It then follows that also $K \cap U$ is not opposite every point of ξ', a contradiction. Hence V is precisely the set of points of ξ not opposite any point of ξ'. Likewise, V' is exactly the set of points of ξ' not opposite any point of ξ. It follows from Lemma 2.10 that each point of V is symplectic to each point of a 6-space U'_v of ξ'. We now show that $U'_v \cap V'$ is a 5-space.

To that aim, let $v \perp z \perp v'$ and note $z \in E(u, u')$. Let ζ be the symp through u and z, and let ζ' be the symp through u' and z. Let Z be the intersection of ξ and ζ, and let Z' be the intersection of ξ' and ζ'. Set $Z_0 = z^\perp \cap Z$ and $Z'_0 = z^\perp \cap Z'$. By Lemma 2.8 each point of Z_0 is symplectic to all points of at least a 4-space of Z'_0, and hence $Z_0 \subseteq V$. Likewise $Z'_0 \subseteq V'$.

We now also deduce that each point of V (for which we can take v again without loss of generality) is symplectic to all points of a 5-space W' of ξ', where $u' \in W'$ and $\dim(W' \cap V') = 4$. Let U'_v be the unique 6-space containing W', then it follows from Lemma 2.10(*iv*) that v is symplectic to all points of U'_v and moreover, by the properties of hyperbolic polar spaces, $V' \cap U'_v$ is 5-dimensional. Likewise, v' is symplectic to each point of a 6-space $U_{v'}$ of ξ which intersects V in a 5-space.

We have already noted that every point x of $\xi \setminus V$ is far from ξ'. By the properties of hyperbolic polar spaces, x is contained in a 6-space intersecting U in a point (for which we can take u without loss of generality) and V in a 5-space. Hence $x \in U_{v'}$ and the last assertions follow.

This completes the proof of the lemma. □

Lemma 2.14. *Let Γ be a Lie incidence geometry of type $\mathsf{E}_{8,8}$ over a field $\mathbb{K}$. Let U and U' be two opposite singular 4-spaces of Γ and let W and W' be two singular 7-spaces of Γ containing*

U and U', respectively, which are not mutual opposite. Then there exist unique planes $\alpha \subseteq W$ and $\alpha' \subseteq W'$ such that no point of α is opposite any point of W' and no point of α' is opposite any point of W.

Proof. Pick $x \in U$ and $x' \in U'$ opposite. Set $U^* = \mathsf{proj}_x^{x'}(U)$ and $W^* = \mathsf{proj}_x^{x'}(W)$ (cf. Section 2.1). We interpret U, W, U^* and W^* as 3- and 6-spaces in the residue at x, which is a geometry of type $\mathsf{E}_{7,7}$. Then, using Proposition 2.1, the assertion is equivalent with showing that there exists a plane in W no point of which is opposite any point of W^*. Choosing opposite points in $U \cup U^*$, the same argument reduces the assertion to showing in the geometry of type $\mathsf{E}_{6,6}$ over the field $\mathbb{K}$, for given disjoint but not opposite 5-spaces W and W^{**}, there exist planes $\alpha \subseteq W$ and $\alpha^{**} \subseteq W^{**}$ with the property that each point of W is collinear to any point of α^{**} and each point of W^{**} is collinear to some point of α. But this follows from Lemma 2.4(iii). □

3. General observations and proof of Theorem A

We start this section with a simple, though important observation, used in both [21] and Chapter 8 of [30], but not explicitly stated in either. We provide a proof for completeness.

Observation 3.1. *Let Δ be a spherical building over the type set I and let $J \subseteq I$ be self-opposite. Let F be a simplex of type J. Then $n(J) = 1$ if, and only if, the identity in $\Pi(F)$ can be written as the product of an odd number of perspectivities.*

Proof. If the identity in $\Pi(F)$ can be written as the product of an odd number of perspectivities, then, by composing this product with any even projectivity, we see that we can write any putative member of $\Pi(F) \setminus \Pi^+(F)$ as a product of an even number of perspectivities, that is, as a member of $\Pi^+(F)$, a contradiction. We conclude $\Pi^+(F) = \Pi(F)$ in this case.
Conversely, if $\Pi^+(F) = \Pi(F)$, then consider any odd projectivity θ. Our assumption implies that we can write θ^{-1} as an even projectivity. Composing those two products of perspectivities, we obtain the identity written as the product of an odd number of perspectivities. □

We can now prove Theorem A.

Proposition 3.2. *Let F be a simplex of a Moufang spherical building Δ of simply laced type. Let $\mathsf{Aut}^+(\Delta)$ be the automorphism group of Δ generated by the root groups. Then the special projectivity group of F coincides with the faithful permutation group induced by the stabiliser $\mathsf{Aut}^+(\Delta)_F$ of F in $\mathsf{Aut}^+(\Delta)$ on the residue $\mathsf{Res}_\Delta(F)$ of F in Δ.*

Proof. I) First we want to show that every even self-projectivity of $\mathsf{Res}(F)$ is induced by a product of elations that stabilises F. In fact, we are going to show that any even projectivity

$$\theta : \mathsf{Res}(F) \to \mathsf{Res}(T)$$

that maps F to a simplex T is induced by an elation. Since self-projectivities are products of projectivities, it then follows that every even self-projectivity is induced by a product of elations that stabilises F.
So let $\theta : \mathsf{Res}(F) \to \mathsf{Res}(T)$ be an even projectivity that maps F to a simplex T. It suffices to prove the assertion for the case that θ is a product of two perspectivities. Then there exists a simplex R opposite to both F and T, such that $\theta = \mathsf{proj}_T^R \circ \mathsf{proj}_R^F$. Since Δ is Moufang, it follows with Lemma 2.2 that there exists an elation g, which maps F to T and fixes R pointwise. For an element f in F, F^g is exactly the projection of $\mathsf{proj}_R(f)$ onto T, since elations preserve incidence. That means $g_{|\mathsf{Res}(F)} = \mathsf{proj}_T^R \circ \mathsf{proj}_R^F$.
II) Now let $g : \Delta \to \Delta$ be an elation. Let α be the corresponding root to g. Let T be a simplex containing the center of α. Then g fixes $\mathsf{Res}(T)$ pointwise and moves a simplex F of the same type to a simplex F^g. First we show that the restriction $g_{|\mathsf{Res}(F)}$ is an even projectivity from $\mathsf{Res}(F)$ to $\mathsf{Res}(F^g)$.

Since Δ is Moufang, Δ is thick and therefore there exists a simplex R in Δ opposite to both F and T. Since elations preserve incidence, the image R^g is opposite to both F^g and $T^g = T$.

Now for every $f \in F$ we have:

$$f^g = \mathsf{proj}_{F^g}^{R^g} \circ \mathsf{proj}_{R^g}^{T} \circ \mathsf{proj}_{T}^{R} \circ \mathsf{proj}_{R}^{F}(f)$$

For an element h of the little projective group that stabilises F, h is a product of elations and every elation can be written as an even projectivity like above. □

4. Projective spaces

In this section we completely settle the case of type A_r regarding the number $n(J)$. The proof will also contain a warm-up for a general statement we will prove later on, see Lemma 5.2. The main reason for treating this case separately is that we can provide an elementary proof only using projective geometry independent from building-theoretic notions (we do refer to Proposition 2.1, but this can easily be verified for projective spaces).

Theorem 4.1. *For buildings of type A_r with type set I and $J \subseteq I$, we have $n(J) = 1$ if, and only if, either $J^{\equiv} \neq J$, or $|J| = 2k$ for some $k \leq \frac{r-1}{2}$ and $J = \{1, 2, \ldots, k, r-k+1, r-k+2, \ldots, r\}$, that is, J is polar closed.*

Proof. Let F and F' be two opposite simplices of type J. First note that, for both the "if" and the "only if" parts, we may assume that J is self-opposite. First suppose $n(J) = 1$. Let $j \in I \setminus J$ be minimal with respect to the Bourbaki labelling of the diagram and let v be a vertex of type j incident to F. Since J is self-opposite, $j \leq \frac{r}{2}$. Then according to Proposition 2.1, the type j' of $\mathsf{proj}_{F'}^{F}(v)$ is the opposite type in $\mathsf{Res}_\Delta(F')$ of type $n+1-j$ (which belongs to $I \setminus J$ since $I \setminus J$ is self-opposite). If $n(J) = 1$, we should have $j = j'$. This is only possible if the integer interval $[j, r+1-j]$ belongs to $I \setminus J$. Putting $k = j-1$, we obtain the "only if" part of the statement.

Now we show the "if" part. We establish the identity projectivity as a product of three perspectivities. Let F be any simplex of type J. Suppose $F = \{U_i \mid i \in J, \dim U_i = i-1\}$. Note that, since F is a simplex, $U_i \leq U_j$ for $i \leq j$, with $i, j \in J$. Select a simplex F' opposite F and set $F' = \{U'_i \mid i \in J, \dim U'_i = i-1\}$. Choosing a basis $\{p_0, p_1, \ldots, p_r\}$ well, we may assume $U_i = \langle p_0, \ldots, p_{i-1}\rangle$ and $U'_i = \langle p_r, p_{r-1}, \ldots, p_{r-i+1}\rangle$. Let, for $0 \leq i \leq k-2$, the point q_i be an arbitrary point on the line $\langle p_i, p_{r-i}\rangle$ distinct from both p_i and p_{r-i}. Define $U''_i = \langle q_0, \ldots, q_{i-1}\rangle$, for $1 \leq i \leq k-1$, and $U''_i = \langle U_{r-i+1}, p_{r-i+1}, \ldots, p_{i-1}\rangle$. Then the simplex $F'' = \{U''_i \mid i \in J\}$ is easily checked to be opposite both F and F'. Let W be an arbitrary subspace of dimension k containing U_k and contained in U_{r-k+1}. Then W is generated by U_k and a point $p \in \langle p_k, \ldots, p_{r-k}\rangle$. The point p belongs to $U'_{r-k+1} \cap U''_{r-k+1}$. Consequently $\mathsf{proj}_{F'}^{F}(W) = \langle U'_k, p\rangle =: W'$, $\mathsf{proj}_{F''}^{F'}(W') = \langle U''_k, p\rangle =: W''$ and $\mathsf{proj}_{F}^{F''}(W'') = W$. This implies that $\mathsf{proj}_{F}^{F''} \circ \mathsf{proj}_{F''}^{F'} \circ \mathsf{proj}_{F'}^{F}$ is the identity and, by Observation 3.1, the assertion is proved. □

5. Proof of Theorem B

The following lemma is basically the gate property of buildings.

Lemma 5.1. *Let Δ be a spherical building over the type set I and let F_J be a simplex of type $J \subseteq I$. Let $K \subseteq J$ and let $F_K \subseteq F_J$ be a simplex of type K. Let F_J^* be opposite F_J and let $F_K^* \subseteq F_J^*$ be opposite F_K. Set $F'_J := \mathsf{proj}_{F_K}(F_J^*)$. Let $C \supseteq F_J$ be a chamber. Then*

$$\mathsf{proj}_{F_J^*}(C) = \mathsf{proj}_{F_J^*}(\mathsf{proj}_{F'_J}(C)).$$

Proof. This follows from the gate property of residues. Since $F'_J = \mathsf{proj}_{F_K}(F_J^*)$,

$$F'_J \subseteq \mathsf{proj}_{F_K}(\mathsf{proj}_{F_J^*}(C)).$$

The latter is on every minimal gallery joining $\mathsf{proj}_{F_J^*}(C)$ with C and hence equals $\mathsf{proj}_{F'_J}(C)$. The assertion follows. □

In the next lemma we use the following terminology. A triple of pairwise opposite simplices S_1, S_2, S_3 is called a *projective* 3-*cycle* if $\mathsf{proj}^{S_3}_{S_1} \circ \mathsf{proj}^{S_2}_{S_3} \circ \mathsf{proj}^{S_1}_{S_2} = \mathsf{id}$. Note that, if the triple S_1, S_2, S_3 is a projective 3-cycle, then so is the triple S_i, S_j, S_k, with (i, j, k) any permutation of $(1, 2, 3)$.

Lemma 5.2. *Let Δ be a spherical building over the type set I and let S_1, S_2, S_3 be a projective 3-cycle of type $J \subseteq I$. Let $K \subseteq I \setminus J$ be such that, for each pair of S_3-opposite simplices $T_3, T'_3 \in$ $\mathsf{Res}(S_3)$, there exists a simplex T''_3 such that T_3, T'_3, T''_3 is a projective 3-cycle in $\mathsf{Res}_\Delta(S_3)$. Then $n(J \cup K) = 1$. More exactly, if T_1 is a simplex of type K adjacent to S_1, then there exist simplices $T'_2 \sim S_2$ and $T''_3 \sim S_3$ of type K such that the triple $S_1 \cup T_1$, $S_2 \cup T'_2$, $S_3 \cup T''_3$ is a projective 3-cycle.*

Proof. Let T_1 be a simplex of type K adjacent to S_1. We want to write id in $\mathsf{Res}_\Delta(S_1 \cup T_1)$ as the product of three projections.

Since S_1, S_2, S_3 is a projective 3-cycle, $\mathsf{proj}^{S_1}_{S_2} T_1 = \mathsf{proj}^{S_3}_{S_2} T_3$, where $T_3 = \mathsf{proj}^{S_1}_{S_3} T_1$. Hence we have

$$\begin{aligned} T_2 &= \mathsf{proj}_{S_2}(T_1) = \mathsf{proj}_{S_2}(T_3), \\ T_3 &= \mathsf{proj}_{S_3}(T_2) = \mathsf{proj}_{S_3}(T_1), \\ T_1 &= \mathsf{proj}_{S_1}(T_3) = \mathsf{proj}_{S_1}(T_2). \end{aligned}$$

Let T'_2 be a simplex locally opposite T_2 at S_2. Then, by Proposition 2.1, the simplices T_1 and T'_2 are opposite in Δ. Set $T'_3 = \mathsf{proj}_{S_3}(T'_2)$. Then T'_3 is opposite T_2 in Δ (again by Proposition 2.1). Since $T_3 = \mathsf{proj}_{S_3}(T_2)$, this implies, again using Proposition 2.1, that T'_3 is locally opposite T_3 at S_2. Our assumption permits to choose a simplex $T''_3 \sim S_3$ of type K such that T_3, T'_3, T''_3 is a projective 3-cycle in $\mathsf{Res}_\Delta(S_3)$. Since, in particular, T''_3 is locally opposite both T_3 and T'_3 at S_3, we have similarly as before (using Proposition 2.1) the following opposite relations:

$$\begin{aligned} &T_1 \equiv T'_2 \equiv T''_3 \equiv T_1, \\ &T_3 \equiv_{S_3} T'_3 \equiv_{S_3} T''_3 \equiv_{S_3} T_3. \end{aligned}$$

Let v_1 be an arbitrary vertex adjacent to $S_1 \cup T_1$. We want to see that if we project v_1 first onto $S_2 \cup T'_2$, then onto $S_3 \cup T''_3$ and back to $S_1 \cup T_1$, then we get v_1 again. Define:

$$\begin{aligned} v_2 &:= \mathsf{proj}_{S_2 \cup T_2}(v_1), \\ v_3 &:= \mathsf{proj}_{S_3 \cup T_3}(v_1) = \mathsf{proj}_{S_3 \cup T_3}(v_2), \\ v'_2 &:= \mathsf{proj}_{S_2 \cup T'_2}(w_2); \text{ then } v'_2 = \mathsf{proj}_{S_2 \cup T'_2}(v_3), \\ v'_3 &:= \mathsf{proj}_{S_3}(v'_2); \text{ then } v'_3 \sim T'_3, \\ v''_3 &:= \mathsf{proj}_{S_3 \cup T''_3}(v'_2) \end{aligned}$$

We have that v_i is adjacent to T_i for $i \in \{1, 2, 3\}$, that v'_j is adjacent to T'_j for $j \in \{2, 3\}$ and that v''_3 is adjacent to T''_3, since incidences are preserved under projection.

By Lemma 5.1, the projection of v'_2 from $S_2 \cup T'_2$ onto $S_3 \cup T''_3$ is the same as the projection onto T''_3 of the projection of v'_2 from S_2 onto S_3 and this is the same as $\mathsf{proj}_{T''_3}(v'_3)$ (namely v''_3).

If we project v'_3 onto T_2, we get the vertex v'_2. If we project further onto T_2, we get the vertex v_2. The converse shows that v_3 maps to v'_3 under the projection locally at S_3 from T_3 to T'_3.

Now the projection of v_1 onto $S_3 \cup T_3''$ is obtained by first projecting onto S_3 (and this is v_3), and then projecting v_3 locally at S_3 onto T_3''. But since the triple T_3, T_3', T_3'' is a projective 3-cycle, we have locally at S_3:

$$\mathsf{proj}_{T_3''}(v_3) = \mathsf{proj}_{T_3''}(\mathsf{proj}_{T_3'}(v_3)) = \mathsf{proj}_{T_3''} v_3' = v_3'',$$

which shows that the triple $S_1 \cup T_1$, $S_2 \cup T_2'$, $S_3 \cup T_3''$ is a projective 3-cycle. This concludes the proof of the lemma. □

In view of Lemma 5.2, and in order to prove Theorem B, it suffices to show that, for any irreducible building Δ, there exists a triple of simplices of polar type which is a projective 3-cycle.

Proposition 5.3. *Let Δ be a spherical building. Let F and F' be two opposite simplices of polar type. Then F and F' are contained in a projective 3-cycle.*

Proof. Let C be a chamber containing F, let Σ be an apartment containing C and F', let α be the root in Σ with centre F (and so containing C) and let $C' = \mathsf{proj}_{F'}(C)$. Then F' is the centre of the opposite root $-\alpha$ of α in Σ. Let $\theta \in U_\alpha$ be a non-trivial root elation and set $F'' = F'^{\theta}$. Let $(C_0, C_1, \ldots, C_\ell)$ be a minimal path in the chamber graph of Δ connecting $C = C_0$ with $C' = C_\ell$. By symmetry, $\ell = 2k$ is even and $C_0, \ldots, C_k$ all belong to α, whereas $C_{k+1}, \ldots, C_{2k}$ belong to $-\alpha$. The root $(-\alpha)^\theta$ has centre F'' and contains $C_{k+1}^\theta, \ldots, C_{2k}^\theta =: C''$. Moreover, since θ fixes $\partial\alpha = \partial(-\alpha)$ pointwise, the union $(-\alpha) \cup (-\alpha)^\theta$ is an apartment and the chambers C_{k+1} and C_{k+1}^θ are adjacent. Hence F'' is opposite F' and $\delta(C, C') = \delta(C, C'') = \delta(C', C'')$. All this yields

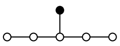
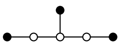
$$\mathsf{proj}_{F'}^{F''}(C'') = C'.$$

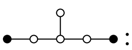
This shows that $\{F, F', F''\}$ is a projective 3-cycle. □

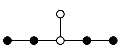
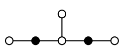
6. Proof of Theorem C

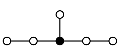
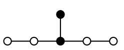
We prove Theorem C by verifying that, as soon as J is not polar closed and $I \setminus J$ contains a connected component of rank at least 2, then for some connected component of $I \setminus J$, Proposition 2.1 implies that a single perspectivity $\mathsf{proj}_{F'}^{F}$, with F, F' simplices of type J, does not preserve types. This implies that every member of $\Pi(F)$ which is the product of an odd number of perspectivities is a duality in $\mathsf{Res}_\Delta(F)$, and hence cannot be the identity. Observation 3.1 then yields $n(J) = 2$. We first treat the exceptional cases and then the infinite class of type D_n. The case A_n follows from Theorem 4.1.

6.1. **Type $\mathsf{E_6}$.** Out of the $2^6 - 2 = 62$ possible types of a nonempty non-maximal simplex, there are exactly $2^4 - 2 = 14$ self-opposite ones. Out of these 14, there are precisely seven for which $I \setminus J$ has a connected component of rank at least 2. We present the possibilities pictorially, colouring the vertices of types in J black. For the other seven $I \setminus J$ is the union of isolated vertices.

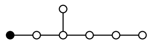
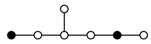
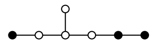
 and  are polar closed.

: According to Proposition 2.1, types 3 and 5 are interchanged by a perspectivity.

 and : According to Proposition 2.1, types 2 and 4 are interchanged by a perspectivity.

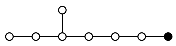
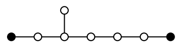
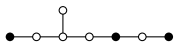
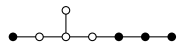
 and : According to Proposition 2.1, types 1 and 5 are interchanged, as are types 3 and 6, by a perspectivity.

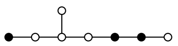
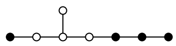
6.2. **Type E_7.** All of the $2^7 - 2 = 126$ possible types of nonempty non-maximal simplices are self-opposite, as opposition is trivial here. There are 18 polar closed types of which only three with a residue containing a connected component of rank at least 2. These components are of types D_4 and D_6; the three cases are

,  and .

Now, the only connected subdiagrams of size at least 2 admitting trivial opposition are precisely the ones of types D_4 and D_6. The above choices for J are the only ones for which $I \setminus J$ has a connected component of size at least 2 admitting trivial opposition. It follows from Proposition 2.1 that in all other cases where a connected component of $I \setminus J$ has rank at least 2, the corresponding group of projectivities contains a duality and hence $n(J) = 2$.

6.3. **Type E_8.** Here opposition is also trivial. There are 19 polar closed types of which only four with a residue containing a connected component of rank at least 2. These components are of types D_4, D_6 and E_7; the four cases are

, ,  and .

There is actually exactly one more type with a residue of rank 4 admitting trivial opposition: : Here the unique connected component $K = \{2,3,4,5\}$ of $I \setminus J = \{2,3,4,5,8\}$ has the property that $I \setminus K = \{1,6,7,8\} =$  is polar closed.
Since all other connected subdiagrams of size at least 2 are either of type $\mathsf{A}_2, \ldots, \mathsf{A}_7$, D_5, D_7 or E_6, we see that for all other types J such that $I \setminus J$ has a connected component of size at least 2, we have $n(J) = 2$.

6.4. **Type D_n, $n \geq 4$.** Obviously, the only connected subdiagrams of size at least 2 of a diagram of type D_n, $n \geq 4$, where opposition agrees with the opposition in D_n are of type D_{n-2k}, for $k \in \mathbb{N}$ such that $n - 2k \geq 3$. So a counterexample J to the assertion has $\max J = n - (n-2k) = 2k$ and the connected component K of size at least 2 of $I \setminus J$ is unique. Clearly, $I \setminus K$, which consists of the vertices of types $1, 2, \ldots, 2k$, is polar closed (indeed, consider the ordering $2,1;4,3;\ldots;2k,2k-1$).

7. Projectivity Groups of panels—Proof of Theorem D

7.1. **A basic lemma.** The next lemma will enable us to pin down the special and general projectivity groups for residues which have the full linear group as respective projectivity group in a residue.

Lemma 7.1. *Let Δ be a spherical building over the type set I and let F_K be a simplex of type $K \subseteq I$. Let $K \subseteq J \subset I$ and let F_J be a simplex of type J containing F_K. Let $\Pi^+_K(F_J)$ be the special projectivity group of $F_J \setminus F_K$ in $\mathsf{Res}_\Delta(F_K)$. Then $\Pi^+_K(F_J) \leq \Pi^+(F_J)$.*

Proof. Let F'_J and F''_J be two simplices containing F_K such that $F'_J \setminus F_K$ is opposite both $F_J \setminus F_K$ and $F''_J \setminus F_K$ in $\mathsf{Res}_\Delta(F_K)$. We have to show that the product of the two perspectivities in $\mathsf{Res}_\Delta(F_K)$ from $\mathsf{Res}_\Delta(F_J)$ to $\mathsf{Res}_\Delta(F'_J)$, subsequently to $\mathsf{Res}_\Delta(F''_J)$ coincides with the product of two perspectivities in Δ. To that aim, let F^*_K be a simplex in Δ opposite F_K, and let F^*_J be the projection of F'_J onto F^*_K (hence $F^*_J = \mathsf{proj}^{F'_K}_{F^*_K}(F'_J)$).
Let C be any chamber containing F_J. Set

$$
\begin{aligned}
C' &= \mathsf{proj}_{F'_J}(C),\\
C'' &= \mathsf{proj}_{F''_J}(C') = \mathsf{proj}_{F''_J}(\mathsf{proj}_{F'_J}(C)),\\
C^* &= \mathsf{proj}_{F^*_J}(C').
\end{aligned}
$$

Then, according to Lemma 5.1, we have

$$C^* = \mathsf{proj}_{F_J^*}(C) \text{ and } C'' = \mathsf{proj}_{F_J''}(C^*),$$

which implies that C'' is indeed equal to the image of C under the product of two perspectivities in Δ. □

Recall that an automorphism of a spherical building Δ of simply laced type is called *linear*, if it belongs to $\mathsf{PGL}_{r+1}(\mathbb{L})$ in case Δ corresponds to $\mathsf{PG}_r(\mathbb{L})$, or if it belongs to the linear algebraic group corresponding to the building if Δ has type D_r, $r \geq 4$, or $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$. The next result is an immediate consequence of Lemma 7.1.

Corollary 7.2. *Let Δ be a spherical building over the type set I and let F_K be a simplex of type $K \subseteq I$. Let $K \subseteq J \subset I$ and let F_J be a simplex of type J containing F_K. Let $\Pi_K^+(F_J)$ be the special projectivity group of $F_J \setminus F_K$ in $\mathsf{Res}_\Delta(F_K)$. Suppose that $\Pi_K^+(F_J)$ is the full linear type preserving automorphism group of $\mathsf{Res}_\Delta(F_J)$. Then $\Pi^+(F_J)$ also coincides with the full linear type preserving automorphism group of $\mathsf{Res}_\Delta(F_J)$.*

7.2. **End of the proof.** Now Theorem D follows from Corollary 7.2 because every vertex of the Coxeter diagram of a simply laced irreducible spherical building of rank at least 3 is contained in a residue isomorphic to the building of a projective plane over some skew field $\mathbb{L}$, and in a projective plane the special projectivity group of a line is $\mathsf{PGL}_2(\mathbb{L})$ acting naturally on $\mathsf{PG}(1, \mathbb{L})$.

8. General and special projectivity groups of irreducible residues of rank at least 2

In this section we determine the exact projectivity groups for irreducible residues. We begin with some general results.

8.1. **General considerations.** The *fix set* of an automorphism ρ of a building Δ is the set of simplices fixed under ρ. The *structure of a fix set* describes how many simplices of which type are fixed and what exact relation they have to each other, disregarding the simplices themselves. E.g.: If two automorphisms of a building Δ both exactly fix every vertex of two opposite chambers, then they have the same fix structure (where the two opposite chambers do not have to be the same chambers in Δ, since the structure only captures the types and relations, but not the exact simplices). We say that a set Π of automorphisms of a building Δ is *geometric* if its members are characterised by their fix structure. Formally, this means that an automorphism belongs to Π if, and only if, its fix set is a member of a certain given set of subsets of the simplices of Δ, closed under the action of the full automorphism group of Δ.

Lemma 8.1. *Let Δ be a spherical building over the type set I and let $J \subseteq I$ be a self-opposite type. Suppose that for each quadruplet of simplices of type J, there exists a simplex of type J opposite all the given simplices. Let F, F', F'' be three pairwise opposite simplices of type J and denote by θ_0 the projectivity $F \barwedge F' \barwedge F'' \barwedge F$. Denote with $\Pi_3(F)$ the set of all self-projectivities of F of length 3 and suppose that $\Pi_3(F)$ is geometric. Then $\Pi(F) = \langle \Pi_3(F) \rangle$ and $\Pi^+(F) = \langle \theta_0^{-1}\theta \mid \theta \in \Pi_3(F) \rangle$.*

Proof. It is clear that the said groups are subgroups of the respective projectivity groups. Now we claim that every self-projectivity of F of length ℓ is the product of $\ell \mod 2\mathbb{Z}$ members of $\Pi_3(F)$. First note that, if F^* is a simplex of type J and $\theta : \mathsf{Res}(F) \to \mathsf{Res}(F^*)$ is an isomorphism, then $\theta\Pi_3(F^*)\theta^{-1} = \Pi_3(F)$, by the fact that $\Pi_3(F)$ is geometric.

Now let $F = F_0 \barwedge F_1 \barwedge \cdots \barwedge F_\ell = F$ be a self-projectivity of length ℓ. Suppose $\ell \geq 5$. Let G be opposite all of F_0, F_1, F_2 and F_3. Denote $\theta_i : F_0 \barwedge G \barwedge F_i$ and $\rho_i = F_i \barwedge G \barwedge F_{i-1} \barwedge F_i$, $i = 1, 2, 3$. Then we have

$$F_0 \barwedge F_1 \barwedge F_2 \barwedge F_3 = \theta_1\rho_1\theta_1^{-1} \cdot \theta_2\rho_2\theta_2^{-1} \cdot \theta_3\rho_3\theta_3^{-1} \cdot \theta_3.$$

Hence we can replace $F_0 \barwedge F_1 \barwedge F_2 \barwedge F_3$ by the product of three members of $\Pi_3(F)$ and the projectivity $\theta_3 = F \barwedge G \barwedge F_3$ of length 2. So, the claim will follow inductively, if we show it for $\ell = 4$, that is, in the above we have the additional perspectivity $F_3 \barwedge F_0$. Hence we have, with the same notation, and denoting additionally $\rho_4 = F_0 \barwedge G \barwedge F_3 \barwedge F_0$, which belongs to $\Pi_3(F)$,

$$F_0 \barwedge F_1 \barwedge F_2 \barwedge F_3 \barwedge F_0 = \theta_1\rho_1\theta_1^{-1} \cdot \theta_2\rho_2\theta_2^{-1} \cdot \theta_3\rho_3\theta_3^{-1} \cdot \rho_4,$$

which is a product of four, hence an even number of, elements of $\Pi_3(F)$. Now the assertions are clear, noting that every product $\theta_1\theta_2$ of members of $\Pi_3(F)$ can be written as the product $(\theta_0\theta_1^{-1})^{-1} \cdot (\theta_0\theta_2)$ of two automorphisms of the form $\theta_0\theta$, where $\theta \in \Pi_3(F)$. □

We will usually apply this lemma to the case where all members of $\Pi_3(F)$ are type-interchanging involutions, and so $\Pi^+(F)$ will also be the intersection of $\Pi(F)$ with the group of type preserving collineations.

In Lemma 8.1, there is the condition that we find a simplex opposite four given simplices. It is well-known that one can find a chamber opposite two given chambers, see Proposition 3.30 in [28]. We can generalise this so that the condition in Lemma 8.1 becomes automatic for buildings with thickness at least 5; for the simply laced case this just means that the building is not defined over the fields $\mathbb{F}_2$ or $\mathbb{F}_3$.

We say that a building *has thickness at least t* if every panel is contained in at least t chambers. The following generalises Proposition 3.30 of [28]. The proof is also a rather obvious generalisation.

Proposition 8.2. *If a spherical building has thickness at least $t+1$, then there exists a chamber opposite t arbitrarily given chambers. In particular, there exists a vertex opposite t arbitrarily given vertices of the same self-opposite type.*

Proof. We will prove the claim by induction. First consider the case that $t = 2$. Then the condition that every panel is contained in at least $t + 1 = 3$ chambers is equivalent to Δ being a thick building and the assertion follows with Proposition 3.30 of [28]. Now suppose $t > 2$. Suppose we know we can find a chamber opposite $t - 1$ given chambers. Let $C_1, \ldots, C_{t-1}$ be $t - 1$ different chambers in Δ and let C_t be another chamber in Δ. Among all the chambers in Δ opposite to each C_i, $i \in \{1, \ldots, t - 1\}$, let E be a chamber with maximal distance to C_t. Assume that C_t and E are not opposite. Then $\mathsf{dist}(C_t, E) \neq \mathsf{diam}\,\Delta$. Let Σ be an apartment containing both C_t and E. With Proposition 2.41 of [28], it follows that there exists a face A of codimension 1 of E, such that $E = \mathsf{proj}_A(C_t)$.

Since every panel is contained in $t + 1$ chambers, we can find a chamber E' having A as a face that is not equal to E and not equal to $\mathsf{proj}_A(C_i)$ for $i \in \{1, \ldots, t - 1\}$.

With Proposition 3.19.7 and Lemma 2.30.7 of [28], it follows that

$$\begin{cases} \mathsf{dist}(C_i, E') = \mathsf{dist}(\mathsf{proj}_A(C_i), C_i) + 1 = \mathsf{dist}(C_i, E) = \mathsf{diam}(\Delta), \text{ for } i \in \{1, \ldots, t-1\}, \\ \mathsf{dist}(C_t, E') = \mathsf{dist}(C_t, E) + 1. \end{cases}$$

That means E' is opposite to each C_i for $i \in \{1, \ldots, t-1\}$ and has a strictly greater distance to C_t than E. That contradicts the fact that E has maximal distance to C_t among the chambers opposite each C_i for $i \in \{1, \ldots, t - 1\}$. It follows that C_t and E are opposite. □

This proposition takes care of all situations where the field has order at least 4. Over $\mathbb{F}_2$, the projectivity groups will always be determined already by Proposition 3.2 (or Theorem A). So there remains to deal with $\mathbb{F}_3$. In this case, we will prove in the situations we need and more generally, that, if the simply laced spherical building is defined over the finite field $\mathbb{F}_q$, then we can find a simplex opposite $q+1$ given simplices of certain given types (see the next paragraphs).

Notation 8.3. Let

$$\mathsf{PSL}_n(\mathbb{K}, a)) := \{M \in \mathsf{GL}_n(\mathbb{K}) \mid \det M = k^a, k \in \mathbb{K}\}.\mathsf{Sc}_n(\mathbb{K})/\mathsf{Sc}_n(\mathbb{K}),$$

where $\mathsf{Sc}_n(\mathbb{K})$ is the group of all scalar matrices over $\mathbb{K}$. We get $\mathsf{PGL}_n(\mathbb{K})$ by putting $a = 1$ and $\mathsf{PSL}_n(\mathbb{K})$ by putting $a = n$.

8.2. **Projective spaces.** Here, Δ is a projective space over a skew field $\mathbb{L}$. We will show that the special projectivity groups of any irreducible residue of rank ℓ is isomorphic to $\mathsf{PGL}_{\ell+1}(\mathbb{L})$. The general group always coincides with the special group, either because the type of the simplex is not self-opposite, or the type is polar closed.

Theorem 8.4. *Let Δ be a building of type A_r, defined over the skew field $\mathbb{L}$. Let F be any simplex such that $I \setminus \mathsf{typ}(F)$ is connected in the Coxeter diagram (say of type A_ℓ). Then both $\Pi^+(F)$ and $\Pi(F)$ are permutation isomorphic to $\mathsf{PGL}_{\ell+1}(\mathbb{L})$.*

Proof. Applying Proposition 3.2 and Corollary 7.2, it suffices to show that the stabiliser G of a hyperplane H of $\mathsf{PG}(r, \mathbb{L})$ in $\mathsf{PSL}_{r+1}(\mathbb{L})$ acts on H as $\mathsf{PGL}_r(\mathbb{L})$. Let g be an arbitrary element of $\mathsf{PGL}_r(\mathbb{L})$ acting on H. Then we can represent g with respect to an arbitrarily chosen basis B in H with an $r \times r$ matrix M. We have to find a member $g^* \in \mathsf{PSL}_{r+1}(\mathbb{L})$ inducing g in H. We can extend B to a basis B^* of $\mathsf{PG}(r, \mathbb{L})$ by adding one point $p_0 \notin H$ and a suitable unit point. Let d belong to the coset of the (multiplicative) commutator subgroup C of $\mathbb{L}^\times$ given by the Dieudonné determinant of M (see [17]). Then the block matrix $M^* := \begin{pmatrix} d^{-1} & 0 \\ 0 & M \end{pmatrix}$ represents a member g^* of $\mathsf{PGL}_{r+1}(\mathbb{L})$ fixing p_0, stabilising H and inducing g in H. Moreover, by the properties of the Dieudonné determinant, in particular those established in the proof of [17, Theorem 1], the determinant of M^* is equal to the product of the coset $d^{-1}C$ and the coset $\det M$. By the definition of d, this product is exactly C, and so $g^* \in \mathsf{PSL}_{r+1}(\mathbb{L})$. The proof is complete. $\square$

8.3. **Polar spaces of rank at least 3.** We prove various claims we will use in the proof of Lemma 8.18, but which are also interesting in their own right.

Notation 8.5. For a spherical building Δ over I and a type set $J \subseteq I$, we denote by Γ_J the graph with vertices the simplices of type J, adjacent when contained in adjacent chambers. Adjacent vertices F, F' in Γ_J are denoted $F \sim F'$. Let v and v' be vertices of type J. We denote the subgraph of Γ_J induced on the vertices opposite both v and v' by Γ'_J.

Claim 1: *Let Γ be a polar space of rank at least 3 such that each line contains at least 4 points. Then Γ is not the union of three (proper) geometric hyperplanes H_1, H_2, H_3.* Indeed, there exists a point $p_1 \in H_1 \setminus H_2$ (clearly $H_1 \neq H_2$) such that $H_1 \neq p_1^\perp$. Then H_1 induces a proper geometric hyperplane in $\mathsf{Res}_\Gamma(p_1)$ and clearly, H_3 has to contain the complement, which is ridiculous. The claim is proved.

Claim 2: *Let Γ be a polar space of rank at least 4 such that each line contains at least 4 points. If Γ is the union of 4 proper geometric hyperplanes H_1, H_2, H_3, H_4, then $H_i \cap H_j = H_k \cap H_\ell$, for $\{i, j, k, \ell\} \subseteq \{1, 2, 3, 4\}$, $i \neq j$ and $k \neq \ell$.* As before, we can find a point $p_1 \in H_1 \setminus H_2$ such that $H_1 \neq p_1^\perp$. We may also assume $H_3 \neq p_1^\perp \neq H_4$, as this only excludes at most two more points. If $p_1 \in H_3$, then in order to cover $p_1^\perp$, we need to cover $\mathsf{Res}_\Gamma(p_1)$ with the three proper hyperplanes H_1, H_2, H_3, contradicting Claim 1. Hence $p_1 \notin H_3 \cup H_4$. If $H_3 = p_3^\perp$, with $p_3 \in H_1 \setminus H_2$, then an arbitrary line L in H_1 through p_3 contains a point p of $H_3 \cap H_1$ that does not belong to H_2 and such that $p^\perp \neq H_4$, as L contains at least 4 points. This contradicts what we just argued. Likewise, there is no point $p_4 \in H_1 \setminus H_2$ such that $p_4^\perp = H_3$. We conclude that no point of $H_1 \setminus H_2$ belongs to $H_3 \cup H_4$. But since $H_1 \cap H_2$ is a geometric hyperplane of H_1, this implies that $H_1 \cap H_2 \subseteq H_3 \cap H_4$ and $H_1 \cap H_2 = H_1 \cap H_3 = H_1 \cap H_4$. The claim now follows.

Claim 3: *Let Γ be a polar space of rank at least 4 such that each line contains at least 4 points. Then the complement C of the union of two geometric hyperplanes H_1, H_2 of Γ is a connected geometry.* Indeed, let x and y be two non-collinear points of C. If $x^\perp \cap y^\perp$ is neither contained in H_1 nor in H_2, then by Claim 1 it is not contained in $H_1 \cup H_2$ and so there exists a point in $x^\perp \cap y^\perp \cap C$. So, we may assume that $x^\perp \cap y^\perp \subseteq H_1$. Since H_1 contains lines, it contains at least one point $p_2 \in H_2$. Let L be a line through y opposite the line xp_2. On L we can find a point z distinct from y, and not belonging to $H_1 \cup H_3$ (since L contains at least 4 points). On the line xp_2 only the point p_2 belongs to $H_1 \cup H_2$, and p_2 is not collinear to z. Hence the unique point z' on xp_2 collinear to z belongs to C. We have $x \perp z' \perp z \perp y$ all in C and the claim is proved.

Also here, we first prove some things about polar spaces. We continue our claims. Note that, to avoid technical issues, we do not always assume the most general conditions, but content ourselves with situations that are applicable to our case.

Claim 4: *Let Γ be a polar space of rank at least 4. Then given four maximal singular subspaces M_1, M_2, M_3, M_4, there exists a line disjoint from $M_1 \cup M_2 \cup M_3 \cup M_4$.* Indeed, it is easy to see that Γ is not the union of M_1 up to M_4 (this is obvious for infinite polar spaces; for finite ones the ovoid number is never less than 5, where the ovoid number indicates the number of maximal singular subspaces needed to partition the point set of Γ). Let p be a point outside $M_1 \cup M_2 \cup M_3 \cup M_4$. Let M_i' be the maximal singular subspace containing p and intersecting M_i in a hyperplane, $i = 1, 2, 3, 4$. The same argument (which would lead to technicalities if Γ had rank 3) as above applied in the residue $\mathsf{Res}_\Gamma(p)$ yields the wanted line L through p.

Claim 5: *Let Γ be a polar space of rank at least 4. Then given two maximal singular subspaces M_1, M_2, of the same kind if Γ is hyperbolic, and a line L disjoint from $M_1 \cup M_2$, there exists a maximal singular subspace through L disjoint from $M_1 \cup M_2$.* Looking in the residue $\mathsf{Res}_\Gamma(L)$, it suffices to show that in every polar space of rank at least 2 we can find a maximal singular subspace disjoint from the union of two given maximal singular subspaces, of the same kind in the hyperbolic case, which we can call M_1, M_2. By taking subsequently the residue in a point outside these maximal singular subspaces, we may assume that the rank of the residue is equal to 2. Then M_1 and M_2 are lines. Consider a line L intersecting both and distinct from M_i, $i = 1, 2$ (such a line exists since either every point is contained in at least three lines, or we are dealing with a grid in case M_1 and M_2 are disjoint). Then, since lines have size at least 3, there is a point $x \in L$ not in $M_1 \cup M_2$. Any line through x distinct from L is disjoint from $M_1 \cup M_2$. The claim is proved.

Claim 6: *Let Γ be a hyperbolic polar space of rank d at least 4, and let M_1, M_2 be two maximal singular subspaces of the same kind. Then, given two maximal singular subspaces W, W' disjoint from both M_1 and M_2, there exists a sequence of maximal singular subspaces $W = W_0 \sim W_1 \sim \cdots \sim W_k = W'$, where $U \sim U'$ means that $U \cap U'$ has codimension 2 in both U and U' (this is adjacency in the graph $\Gamma_{\{d\}}$, with above notation), and k is some natural number.* Let p be a point of $W \setminus W'$. Let M be the maximal singular subspace through p intersecting W' in a hyperplane. Then $M \cap M_i$ is a point p_i, $i = 1, 2$. It is easy to see that there is a point q in $W \setminus (W' \cup p_1^\perp \cup p_2^\perp)$, since no projective space is the union of three proper subspaces among which at least one not a hyperplane. Since q is not collinear to p_i, $i = 1, 2$, the unique maximal singular subspace W'' through pq intersecting W' in a hyperplane is disjoint from $M_1 \cup M_2$ and is closer to W with respect to the relation $\sim$ than W'. An obvious induction argument now concludes the proof of the claim.

Claim 7: *Let Γ be a polar space of rank d at least 3. Let L, L' be two lines, viewed as vertices of the graph $\Gamma_{\{2\}}$. Then the corresponding graph $\Gamma'_{\{2\}}$ is connected.* The proof of this claim is completely similar to the first part of this proof, since the line Grassmannian of a polar space is a so-called *hexagonal geometry*, that is, is shares the properties with the geometries of type $\mathsf{E}_{6,2}$ and $\mathsf{E}_{7,1}$ used above, If the polar space is hyperbolic (and we will only use Claim 7 in that case), then it actually is a long root geometry and the proof can be taken over verbatim.

8.4. **Hyperbolic polar spaces.** We first prove some lemmas. When we consider residues of vertices of type 1, that is, the points of the corresponding polar space, we will aim to apply Lemma 8.1. Proposition 8.2 already tells us that we can find a point opposite 4 arbitrarily given points if the underlying field has order at least 4. To handle the case with the field $\mathbb{F}_3$, we recall the following slightly more general results for hyperbolic quadrics, proved in [5].

Lemma 8.6. *If every line of a hyperbolic quadric Q of rank at least 3 contains exactly $s+1$ points, then*

(*i*) *there exists a point non-collinear to each point of an arbitrary set T of $s+1$ (distinct) points, except if these points are contained in a single line, and*

(*ii*) *if Q has even Witt index $2d$, then there exists a maximal singular subspace opposite each member of an arbitrary set T of $s+1$ (distinct) maximal singular subspaces of common type, except if these maximal singular subspaces contain a common singular subspace of codimension 2 in each.*

For a hyperbolic quadric Q of Witt index r, associated to the quadratic form $g\colon V \to \mathbb{K}$ with associated bilinear form $f : V \times V \to \mathbb{K}$, we denote by $\mathsf{PGO}_{2r}(\mathbb{K})$ the group of all elements of $\mathsf{PGL}_{2r}(V)$ preserving f and g. The unique subgroup of index 2 preserving each class of maximal singular subspaces will be denoted by $\mathsf{PGO}^\circ_{2r}(\mathbb{K})$. Note that $\mathsf{PGO}^\circ_6(\mathbb{K})$ is isomorphic to $\mathsf{PSL}_4(\mathbb{K}, 2)$. A *parabolic polarity* of Q is the involution fixing a given parabolic subquadric P of Witt index $r-1$ and interchanging each two maximal singular subspaces of Q containing a common maximal singular subspace of P. Each parabolic polarity belongs to $\mathsf{PGO}_{2r}(\mathbb{K})$, as in V, it is given by $V \to V\colon v \mapsto v - \frac{f(v,w)}{g(w)}w$, for some $w \in V$ with $g(w) \neq 0$.

The following lemma is well-known in the finite case, but we could not find a general reference.

Lemma 8.7. *The group $\mathsf{PGO}_{2r}(\mathbb{K})$ is generated by the parabolic polarities.*

Proof. Let G be the group generated by all parabolic polarities. We first want to prove that all axial elations belong to G. Working in the common perp of two opposite singular subspaces of dimension $r-3$, we may assume for that part that $r=2$. Let $g : V \to \mathbb{K}\colon (x_1, x_2, x_3, x_4) \mapsto x_1x_2 + x_3x_4$. Taking $w = (1, 1, a, 0)$, $a \in \mathbb{K}$, in the above description of a generic parabolic polarity, one calculates that this polarity θ_w is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & a \\ 1 & 0 & 0 & a \\ a & a & -1 & a^2 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Setting $w' = (1, 1, 0, 0)$, we notice that the action of $\theta_w\theta_{w'}$ on the regulus $\mathscr{R}$ containing the lines with equations $X_1 = X_4 = 0$ and $X_2 = X_3 = 0$, coincides with a translation, that is, it maps the line through the point $(1, 0, b, 0)$ to the line through point $(1, 0, b+a, 0)$. Hence G contains a subgroup stabilising $\mathscr{R}$ and acting 2-transitively on it.

Now set $u = (0, a, 1, 1)$ and $u' = (0, 0, 1, 1)$. Then we calculate $\theta := (\theta_w\theta_{w'})(\theta_u\theta_{u'})$ and obtain

$$\begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & a \\ a & a & -1 & a^2 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ -a^2 & 1 & a & a \\ -a & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+a^2 & 0 & 0 & -a \\ 0 & 1 & a & 0 \\ 0 & a & a^2+1 & 0 \\ -a & 0 & 0 & 1 \end{pmatrix}.$$

It can easily be checked that θ stabilises each member of $\mathscr{R}$. Setting $a = \lambda^{-1} - \lambda$, for some $\lambda \notin \{1, -1\}$ (which exists since in the finite case we can refer to the introduction of the Atlas [8], and so we may assume that $\mathbb{K}$ is infinite), we see that θ fixes two lines of the opposite regulus $\mathscr{R}'$, namely the members of the opposite regulus through the points $(1, 0, 0, -\lambda)$ and $(-\lambda, 0, 0, 1)$. By the 2-transitivity mentioned above (applied to $\mathscr{R}'$), we may assume that θ is a collineation pointwise fixing the lines with equations $X_1 = X_3 = 0$ and $X_2 = X_4 = 0$ and acting non-trivially on $\mathscr{R}'$. Let the line of $\mathscr{R}'$ through the point $(1, 0, 0, x)$ briefly be denoted by $[1, x]$. Then the action of θ on $\mathscr{R}'$ is given by $[1, x] \mapsto [1, bx]$, for some $b \in \mathbb{K}$. Note that, interchanging $\mathscr{R}$ with $\mathscr{R}'$, we derived, for each $a \in \mathbb{K}$, the existence of a permutation φ_a of $\mathscr{R}'$ of the form $[1, x] \mapsto [1, x+a]$. Then, for given $k \in \mathbb{K}$, the commutator $\varphi_a^{-1}\theta\varphi_a\theta^{-1}$ acts on $\mathscr{R}'$ as φ_k, and pointwise fixes the line $X_1 = X_3 = 0$, if we choose $a = k(1 - b^{-1})^{-1}$. Hence all axial elations are contained in G, and so the little projective group of Q is contained in G. This implies that we can multiply any member of $\mathsf{PGO}_{2r}(\mathbb{K})$ with an element of G to obtain an element g which pointwise fixes the standard apartment, that is, all basis points of $\mathsf{PG}(V)$. Since it also preserves the form $g\colon V \to \mathbb{K}$, it is represented by a diagonal matrix $\mathsf{diag}(a_1, a_1^{-1}, a_3, a_3^{-1}, \ldots, a_{r-1}, a_{r-1}^{-1})$. Then the product of the parabolic polarities determined by the points $(1, 1, 0, 0, \ldots, 0)$ and $(a_1, 1, 0, 0, \ldots, 0)$ is a collineation represented by $\mathsf{diag}(a_1, a_1^{-1}, 1, 1, \ldots, 1)$. It is now clear how to write g as a product of parabolic polarities. The lemma is proved. □

Theorem 8.8. *Let Δ be the building (of rank $r \geq 4$) associated to a hyperbolic quadric Q of Witt index $r \geq 4$ over the field $\mathbb{K}$. Let F be a simplex of Δ such that $\mathsf{Res}_\Delta(F)$ is irreducible. Then $\Pi(F)$ and $\Pi^+(F)$ are given as in* Table 1. *In Case* (A∗)*, the permutation group $\mathsf{PGL}_r(\mathbb{K}, 2).2$ denotes the extension of $\mathsf{PGL}_r(\mathbb{K}, 2)$ by a symplectic polarity acting on $\mathsf{PG}(r-1, \mathbb{K})$ (and coincides with the group generated by all symplectic polarities). A long hyphen in the table in the column of $\Pi(F)$ means that $\mathsf{typ}(F)$ is not self-opposite and so $\Pi(F)$ is trivially isomorphic to $\Pi^+(F)$ — it must be read as a "bysame" symbol. Grey rows correspond to projectivity groups that are not necessarily the full linear groups.*

Reference	$\mathsf{Res}_\Delta(F)$	$\mathsf{cotyp}(F)$	$\Pi^+(F)$	$\Pi(F)$
(A1)	A_1		$\mathsf{PGL}_2(\mathbb{K})$	$\mathsf{PGL}_2(\mathbb{K})$
(A3)	A_3	$\{r-2, r-1, r\}$	$\mathsf{PGO}_6^\circ(\mathbb{K})$	$\mathsf{PGO}_6(\mathbb{K})$
(A)	A_ℓ, $2 \leq \ell \leq r-2$	$\neq \{r-2, r-1, r\}$	$\mathsf{PGL}_{\ell+1}(\mathbb{K})$	$\mathsf{PGL}_{\ell+1}(\mathbb{K}).2$
(A*)	A_{r-1}, $r \in 2\mathbb{Z}$		$\mathsf{PGL}_r(\mathbb{K}, 2)$	$\mathsf{PGL}_r(\mathbb{K}, 2).2$
(A**)	A_{r-1}, $r \in 2\mathbb{Z}+1$		$\mathsf{PGL}_r(\mathbb{K})$	——
(D)	$\mathsf{D}_{r-2\ell}$, $4 \leq r-2\ell \leq r-1$		$\mathsf{PGO}_{2r-4\ell}^\circ(\mathbb{K})$	$\mathsf{PGO}_{2r-4\ell}^\circ(\mathbb{K})$
(D$'$)	$\mathsf{D}_{r-2\ell+1}$, $4 \leq r-2\ell+1 \leq r-1$		$\mathsf{PGO}_{2r-4\ell+2}^\circ(\mathbb{K})$	$\mathsf{PGO}_{2r-4\ell+2}(\mathbb{K})$

TABLE 1. Projectivity groups in buildings of type D_r over $\mathbb{K}$

Proof. First we notice that, if $\mathbb{K} = \mathbb{F}_2$, then all groups are universal and adjoint (simple) at the same time, so the results follow from Theorem A. Hence we may assume $|\mathbb{K}| \geq 3$. For ease of notation and language, we will speak about plus-type and minus-type of the maximal singular subspaces of Q to distinguish the two different types (arbitrarily).

Also, Case (A1) follows from Theorem D, whereas Case (A) follows from Lemma 7.1 and Theorem 8.4. We now handle the other, less straightforward, cases.

Case (A*) Let M_1, M_2, M_3 be three mutual opposite maximal singular subspaces of plus-type. Let $p_1 \in M_1$ be arbitrary. The maximal singular subspace N through p_1 intersecting M_2 in a submaximal singular subspace (that is, a singular subspace of dimension $r-2$) intersects M_3 in a point p_3, since N is necessarily of minus-type. Hence the maximal singular subspace of minus-type through p_3 intersecting M_1 in a submaximal singular subspace contains p_1. This shows that the projectivity $M_1 \barwedge M_2 \barwedge M_3 \barwedge M_1$ is a duality each point of which is absolute. Lemma 3.2 of [25] implies that it is a symplectic polarity. By conjugation, we can obtain every symplectic polarity of M_1 in this way. Applying Lemma 8.1 together with Lemma 8.6, Case (A*) follows from the fact that the matrix corresponding to a symplectic polarity necessarily has square determinant (and every square can occur).

Case (A3) By Theorem A, every self-projectivity preserves the residual form, hence $\Pi(F) \leq \mathsf{PGO}_6(\mathbb{K})$. Case (A*) for $r = 3$, together with Lemma 7.1 and the fact that $\mathsf{PGO}_6^\circ(\mathbb{K})$ is isomorphic to $\mathsf{PSL}_4(\mathbb{K}, 2)$, concludes this case.

Case (A**) By Theorem A, $\Pi^+(F)$ contains $\mathsf{PSL}_r(\mathbb{K})$. Hence it suffices to show that $\Pi^+(F)$ contains an element of $\mathsf{PGL}_r(\mathbb{K})$ whose corresponding matrix has arbitrary determinant.

Let M_1 and M_3 be two maximal singular subspaces of plus-type intersecting in a subspace U_{13} of dimension $r-3$. Let M_2 be a maximal singular subspace opposite both M_1 and M_3 (then M_2 has plus-type). Let U_{24} be a subspace of M_2 of dimension $r-3$ opposite U_{13}. Let L_1 be the unique line of M_1 collinear to U_{24}. Let L be an arbitrary line in M_1 joining a point $p_{13} \in U_{13}$ with some point $p_1 \in L_1$. Pick $p, p' \in L \setminus \{p_{13}, p_1\}$ and suppose $p \neq p'$ (this is possible as we assume $|\mathbb{K}| \geq 3$).

Let M be the maximal singular subspace of plus-type containing p and intersecting M_2 in a hyperplane. Denote $W = U_{24} \cap M$. Then W has dimension $r-4$ and is collinear to L. The intersection of M and M_3 is a point q, as both have the same type. As both p and p_{13} are collinear to q, also p' is collinear to q. Hence p' is collinear to $\langle q, W\rangle$, and $\langle p', q, W\rangle$ is a singular subspace of dimension $r-2$. Hence there is a unique maximal singular subspace M' of plus-type containing p', q and W. It obviously intersects M_1 in p' and M_3 in q. There is a unique maximal singular subspace M_4 containing U_{24} and intersecting M' in a hyperplane (and hence it is of minus-type). Now with this set-up, one verifies that the projectivity $M_1 \barwedge M_2 \barwedge M_3 \barwedge M_4 \barwedge M_1$ pointwise fixes both U_{13} and L_1, and maps p to p'. Choosing a basis in $U_{13} \cup L_1$, the matrix of such a homology in M_1 is diagonal of the form $\mathsf{diag}(k, k, \ell, \ell, \ldots, \ell)$, and the arbitrariness of p' implies that k and ℓ are also arbitrary. Set $r = 2s+1$. Putting $k = \ell^{-s+1}$, we obtain the determinant $\ell^{-2s+2+2s-1} = \ell$. Since ℓ is arbitrary, the assertion follows.

Case (D$'$) First set $\ell = 1$, that is, $r - 2\ell + 1 = r-1$ and F is just a point of the polar space or hyperbolic quadric Q. Let p_1, p_2, p_3 be three mutual opposite points. Since $p_1^\perp \cap p_2^\perp$ is a hyperbolic quadric of rank $r-1$, we have that $p_1 \cap p_2 \cap p_3$ is either a parabolic subquadric, or a degenerate quadric. In the latter case, $\{p_1, p_2, p_3\}^{\perp\perp}$ is a degenerate plane conic containing p_1, p_2, p_3, and hence p_3 is collinear to either p_1 or p_2, a contradiction. Consequently $p_1 \cap p_2 \cap p_3$ is a parabolic quadric and the projectivity $p_1 \barwedge p_2 \barwedge p_3 \barwedge p_1$ is a parabolic polarity. Clearly, every parabolic polarity of $\mathsf{Res}_\Delta(p_1)$ can be obtained this way. Then Lemma 8.1, Lemma 8.6 and Lemma 8.7 yield $\Pi(p_1) = \mathsf{PGO}_{2r-2}(\mathbb{K})$ and $\Pi^+(p_1) = \mathsf{PGO}^\circ_{2r-2}(\mathbb{K})$.

Now let ℓ be arbitrary (but of course $4 \leq r - 2\ell + 1 \leq r-1$). Since the stabiliser of F in $\mathsf{PGO}_{2r}(\mathbb{K})$ obviously preserves the residual form (in $\mathsf{Res}_\Delta(F)$), we see that $\Pi^+(F)$ is a subgroup of $\mathsf{PGO}^\circ_{2r-4\ell+2}(\mathbb{K})$, and hence coincides with it by Lemma 7.1 and the case $\ell = 1$. In order to show $\Pi(\mathbb{F}) = \mathsf{PGO}_{2r-4\ell+2}(\mathbb{K})$, we only need to exhibit a parabolic polarity as a self-projectivity in $\mathsf{Res}_\Delta(F)$. This is done similarly as in the previous paragraph for the case $\ell = 1$: choose three mutual opposite singular subspaces U_1, U_2, U_3 of dimension $2\ell - 1$ contained in a parabolic subquadric obtained from Q by intersecting Q in its ambient projective space with a subspace of dimension 4ℓ. Suppose also $U_1 \in F$. Then, as before, the projectivity $U_1 \barwedge U_2 \barwedge U_3 \perp U_1$ is a parabolic polarity of $\mathsf{Res}_\Delta(F)$.

Case (D) This is completely similar to the case $\ell > 1$ of Case (D$'$), noting that $\Pi^+(F)$ coincides with $\Pi(F)$ by Theorem B. □

8.5. **Exceptional cases.** Also here, we first prove some lemmas. First we recall the following result from [5] in order to deal with the case of a field of order 3 for simplices of type 7 in E_7.

Lemma 8.9. *If every line of a parapolar space Γ of type $\mathsf{E}_{7,7}$ contains exactly $s+1$ points, then there exists a point at distance 3 from each point of an arbitrary set T of $s+1$ (distinct) points, except if these points are contained in a single line.*

Notation 8.10 (Similitudes—Groups of type D_n)**.** For a hyperbolic quadric Q of Witt index r, associated to the quadratic form $g\colon V \to \mathbb{K}$ with associated bilinear form $f\colon V \times V \to \mathbb{K}$, we denote by $\overline{\mathsf{PGO}}_{2r}(\mathbb{K})$ the group of all elements of $\mathsf{PGL}_{2r}(V)$ preserving f and g up to a scalar multiple. It is the complete linear (algebraic) group of automorphisms of Q, seen as a building of type D_r. The unique subgroup of $\overline{\mathsf{PGO}}_{2r}(\mathbb{K})$ of index 2 preserving each class of maximal singular subspaces will be denoted by $\overline{\mathsf{PGO}}^\circ_{2r}(\mathbb{K})$. It is elementary to see that $\overline{\mathsf{PGO}}_{2r}(\mathbb{K})$ is obtained from $\mathsf{PGO}_{2r}(\mathbb{K})$ by adjoining the appropriate *diagonal automorphisms*, that is, if we assume g in standard form (after introducing coordinates)

$$g\colon \mathbb{K}^{2r} \to \mathbb{K}\colon (x_{-r}, x_{-r+1}, \ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots, x_{r-1}, x_r) \\ \mapsto x_{-r}x_r + x_{-r+1}x_{r-1} + \cdots x_{-2}x_2 + x_{-1}x_1,$$

then we adjoin the linear automorphisms of Q induced by

$$\varphi_k\colon \mathbb{K}^{2r} \to \mathbb{K}^{2r}\colon (x_{-r}, x_{-r+1}, \ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots, x_{r-1}, x_r) \\ \mapsto (x_{-r}, x_{-r+1}, \ldots, x_{-2}, x_{-1}, kx_1, kx_2, \ldots, kx_{r-1}, kx_r),$$

for all $k \in \mathbb{K}^\times$ (and we may assume k is not a square as otherwise the given automorphism is already in $\mathsf{PGO}_{2r}(\mathbb{K})$). We denote the commutator subgroup of $\mathsf{PGO}^\circ_{2r}(\mathbb{K})$ by $\mathsf{P\Omega}_{2r}(\mathbb{K})$. The latter is the simple group $\mathsf{D}_r(\mathbb{K})$ of type D_r over the field $\mathbb{K}$ (see [18]). The group obtained from $\mathsf{P\Omega}_{2r}(\mathbb{K})$ by adjoining the diagonal automorphisms as above is denoted by $\overline{\mathsf{P\Omega}}_{2r}(\mathbb{K})$.

If r is even and $\mathbb{K}$ is not quadratically closed, then $\overline{\mathsf{P\Omega}}_{2r}(\mathbb{K})$ does not coincide with $\overline{\mathsf{PGO}}^\circ_{2r}(\mathbb{K})$ as we will demonstrate later (see Remark 8.20).

Let us call *homology* of a hyperbolic quadric Q as in Notation 8.10 any automorphism of Q pointwise fixing two opposite maximal singular subspaces. The automorphisms φ_k, $k \in \mathbb{K}^\times$, above are homologies. If r is even, then there are two types of such according to which kind of maximal singular subspaces is fixed pointwise (if r is odd, then one always pointwise fixes one maximal singular subspace of each type). We now have the following result, which can be proved using standard arguments similarly to, but simpler than, Lemma 8.7, Lemma 8.13 and Lemma 8.16.

Lemma 8.11. *Let Q be a (non-degenerate) hyperbolic quadric of Witt index r corresponding to the building of type D_r over the field $\mathbb{K}$. Then the following hold.*

(*i*) *The set of all homologies generates $\overline{\mathsf{PGO}}^\circ_{2r}(\mathbb{K})$.*
(*ii*) *If r is even, then the set of homologies pointwise fixing two opposite maximal singular subspaces of only one given type generates $\overline{\mathsf{P\Omega}}_{2r}(\mathbb{K})$.*
(*iii*) *If r is even, then the homologies pointwise fixing two opposite maximal singular subspaces of only one given type, and the elements of $\mathsf{PGO}^\circ_{2r}(\mathbb{K})$ together generate $\overline{\mathsf{PGO}}^\circ_{2r}(\mathbb{K})$.*

We now introduce some notation concerning the exceptional groups of types E_6 and E_7. There does not seem to be standard notation (some people use $\tilde{E}$ and $\hat{E}$, others $\mathrm{SE}_6(\mathbb{K})$ for some of the following groups). The following is partly based on [24].

Notation 8.12 (Groups of type E_6)**.** Let V be a 27-dimensional vector space over the commutative field $\mathbb{K}$, written as the direct sum of three 1-dimensional subspaces and three 8-dimensional subspaces, each of them identified with a split octonion algebra $\mathbb{O}$ over $\mathbb{K}$. We thus write $V = \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O}$. Let $\mathfrak{C} : V \to \mathbb{K}$ be the cubic form defined as

$$\mathfrak{C}(x, y, z; X, Y, Z) = -xyz + xX\overline{X} + yY\overline{Y} + zZ\overline{Z} - (XY)Z - \overline{(XY)Z}.$$

Then we denote by $\mathsf{GE}_6(\mathbb{K})$ the similitudes of $\mathfrak{C}$, that is, the subgroup of $\mathsf{GL}(V)$ preserving $\mathfrak{C}$ up to a multiplicative constant. The subgroup of $\mathsf{GE}_6(\mathbb{K})$ preserving $\mathfrak{C}$ is denoted by $\mathsf{SE}_6(\mathbb{K})$ (and is a subgroup of $\mathsf{SL}(V)$) and the quotients with the respective centres (consisting of scalar matrices) are $\mathsf{PGE}_6(\mathbb{K})$ and $\mathsf{PSE}_6(\mathbb{K})$. The latter is also denoted briefly by $\mathsf{E}_6(\mathbb{K})$ and is simple. The group $\mathsf{PGE}_6(\mathbb{K})$ is the full linear group. The group obtained by adjoining a graph automorphism is denoted by $\mathsf{PGE}_6(\mathbb{K}).2$.

The cubic form $\mathfrak{C}$ above can also be written without the use of octonions, but using the unique generalised quadrangle $\mathsf{GQ}(2,4)$ of order $(2,4)$, that is, polar space of rank 2 with 3 points on each line and 5 lines through each points. An explicit construction of $\mathsf{GQ}(2,4)$ runs as follows, see Section 6.1 of [22]. Let $\mathscr{P}'$ be the set of all 2-subsets of the 6-set $\{1,2,3,4,5,6,\}$, and define

$$\mathscr{P} = \mathscr{P}' \cup \{1,2,3,4,5,6\} \cup \{1',2',3',4',5',6'\}.$$

Denote briefly the 2-subset $\{i,j\}$ by ij, for all appropriate i,j. Let $\mathscr{L}'$ be the set of partitions of $\{1,2,3,4,5,6\}$ into 2-subsets and define

$$\mathscr{L} = \mathscr{L}' \cup \big\{\{i,j',ij\} \mid i,j \in \{1,2,3,4,5,6\}, i \neq j\big\}.$$

Then $\Gamma = (\mathscr{P},\mathscr{L})$ is a model of $\mathsf{GQ}(2,4)$.

The sets $\{1,2,3,4,5,6\}$ and $\{1',2',3',4',5',6'\}$ have the property that they both do not contain any pair of collinear points, and that non-collinearity is a paring between the two sets. Such a pair of 6-sets is usually called a *double six*.

Define the following set $\mathscr{S}$ of lines of $\mathsf{GQ}(2,4)$.

$$\begin{aligned}\mathscr{S} \;=\; &\{\{14,25,36\},\{15,26,34\},\{16,24,35\},\\ &\{12,2,1'\},\{23,3,2'\},\{13,1,3'\},\\ &\{45,4,5'\},\{56,5,6'\},\{46,6,4'\}\}.\end{aligned}$$

Then $\mathscr{S}$ is a *spread*, that is, a partition of the point set $\mathscr{P}$ into lines.

We now have the following equivalent description of the cubic form $\mathfrak{C}$, see Section 2 of [32]. Let V be the vector space of dimension 27 over $\mathbb{K}$ where the standard basis B is labelled using the elements of $\mathscr{P}$, say $B = \{e_p \mid p \in \mathscr{P}\}$. We denote a generic vector $v \in V$ by $\sum_{p\in\mathscr{P}} x_p e_p$, with $x_p \in \mathbb{K}$. Then

$$\mathfrak{C}(v) = \sum_{\{p,q,r\}\in\mathscr{S}} x_p x_q x_r - \sum_{\{p,q,r\}\in\mathscr{L}\setminus\mathscr{S}} x_p x_q x_r.$$

The projective null set of $\nabla\mathfrak{C}$ is a set of points denoted $\mathscr{E}_6(\mathbb{K})$, and endowed with the lines contained in it, it is a point-line geometry isomorphic to the Lie incidence geometry of type $\mathsf{E}_{6,1}$ over the field $\mathbb{K}$.

With these constructions and notation at hand, we are able to prove the following generation result.

Lemma 8.13. *The products of an even number of symplectic polarities of the building Δ of type E_6 over the field $\mathbb{K}$ generate the Chevalley group $\mathsf{PGE}_6(\mathbb{K})$. The symplectic polarities themselves generate $\mathsf{PGE}_6(\mathbb{K}).2$.*

Proof. We first claim that diagonal automorphisms φ necessarily have a ninth power as determinant. Indeed, φ acts as

$$\varphi\colon V \to V\colon \sum_{p\in\mathscr{P}} x_p e_p \mapsto \sum_{p\in\mathscr{P}} \lambda_p x_p e_p,$$

with $\lambda_p \in \mathbb{K}$. Then φ is a similitude of $\mathfrak{C}$ only if the product $\lambda_p\lambda_q\lambda_r =: \lambda$ is a constant across all lines $\{p,q,r\} \in \mathscr{L}$. Then the determinant of φ is obtained by multiplying this constant over the spread $\mathscr{S}$ and hence the determinant equals λ^9. The claim is proved.

Now a symplectic polarity σ of Δ induces a symplectic polarity in every fixed 5-space of the corresponding Lie incidence geometry Γ of type E_6. Since all symplectic polarities are conjugate, every symplectic polarity of a given 5-space extends to a symplectic polarity of Δ. By the strong transitivity of $\mathsf{Aut}\,\Delta$, we may even assume that two given opposite 5-spaces are fixed (and then, since the fix building has type F_4 and its polar type corresponds to the fixed 5-spaces (as is apparent from [12], all 5-spaces in the span of the two given ones in $\mathsf{PG}(V)$ are fixed). Now, two opposite 5-spaces W and W' are given by the span of the base points corresponding to the two respective 6-sets of a double six. It is easily seen that the product of the symplectic polarities corresponding to the symplectic forms

$x_{-3}y_3+x_{-2}y_2+x_{-1}y_1-x_1y_{-1}x_2y_{-2}-x_3y_{-3}$ and $\lambda x_{-3}y_3+x_{-2}y_2+x_{-1}y_1-x_1y_{-1}x_2y_{-2}-\lambda x_3y_{-3}$,

$\lambda \in \mathbb{K}$, corresponds to diagonal collineation of $\mathsf{PG}(5,\mathbb{K})$ with diagonal $(\lambda,1,1,1,1,\lambda)$. Letting these coordinates correspond naturally to the bases $(e_1,\ldots,e_6)$ and $(e_{1'},\ldots,e_{6'})$ of the subspaces of V corresponding to W and W', respectively, we first derive that the product θ of the corresponding symplectic polarities of Δ acts on $\langle W, W'\rangle$ as

$$(x_1, x_2, \dots, x_6, x_{1'}, x_{2'}, \dots, x_{6'}) \mapsto (\lambda x_1, x_2, x_3, x_4, x_5, \lambda x_6, \lambda x_{1'}, x_{2'}, x_{3'}, x_{4'}, x_{5'}, \lambda x_{6'}).$$

Secondly, since each point $\langle e_{ij}\rangle$ is the unique point of Γ collinear to all $\langle e_\ell\rangle$, except for $\langle e_i\rangle$ and $\langle e_j\rangle$, and to all $\langle e_{\ell'}\rangle$, except for $\langle e_{i'}\rangle$ and $\langle e_{j'}\rangle$ (as follows from Lemma 3.5 in [16]), we see that θ is a diagonal automorphism. Now one easily calculates that θ is uniquely determined by its restriction to $\langle W, W'\rangle$ and maps e_{ij} to e_{ij} if $|\{i,j\} \cap \{1,6\}| = 1$, to λe_{ij} if $\{i,j\} \cap \{1,6\} = \varnothing$, and to $\lambda^{-1} e_{ij}$ if $\{i,j\} = \{1,6\}$. Correspondingly, the determinant of θ is λ^9. Now clearly the diagonal automorphisms generate non-trivial elements of $\mathsf{PSE}_6(\mathbb{K})$. Since the latter is simple, and since the subgroup of $\mathsf{E}_6(\mathbb{K}).2$ generated by all symplectic polarities of Δ is normal, the group generated by arbitrary products of an even number of symplectic polarities contains $\mathsf{PSE}_6(\mathbb{K})$. Since it also contains all diagonal automorphisms by the above, the assertions follow. □

Notation 8.14 (Groups of type E_7)**.** Let V be a 56-dimensional vector space over the commutative field $\mathbb{K}$, written as the direct sum of two 1-dimensional subspaces and two 27-dimensional subspaces. We now briefly recall the construction given in Section 10 of [15], and we refer the reader there for details. Let $\Gamma_1 = (V_1, E_1)$ be the Schläfli graph and let $\Gamma_2 = (V_2, E_2)$ be the Gosset graph. Note that the Schläfli graph is the graph with vertices the points of $\mathsf{GQ}(2,4)$, adjacent if not collinear in $\mathsf{GQ}(2,4)$. One can describe Γ_2 in terms of Γ_1 as follows. Let $\Gamma_1' = (V_1', E_1')$ and $\Gamma_1'' = (V_1'', E_1'')$ be two disjoint copies of Γ_1 and consider two symbols ∞' and ∞''. Then the vertices of Γ_2 are the vertices of Γ_1' and Γ_1'' together with ∞' and ∞''. The vertex ∞' (resp. ∞'') is adjacent to all vertices of Γ_1' (resp. Γ_1''). Adjacency inside Γ_1' and Γ_1'' is as in Γ_1, and a vertex of Γ_1' is adjacent to the vertex of Γ_1'' if the corresponding vertices of Γ_1 are at distance 2 from one another. We now fix a Hermitian spread $\mathscr{S}$ of Γ_1, and denote by $\mathscr{S}'$ and $\mathscr{S}''$ the copies of $\mathscr{S}$ in Γ_1' and Γ_1'', respectively.

Label the basis vectors of V by the vertices of the Gosset graph Γ_2. We define for each cross-polytope, which we will call a hexacross of Γ_2, and for each pair of opposite hexacrosses, a quadratic form, determined up to a non-zero scalar. Later on, we will use precisely these quadratic forms to describe a variety denoted by $\mathscr{E}_7(\mathbb{K})$, which will define the geometry $\mathsf{E}_{7,7}(\mathbb{K})$ (see Theorem 8.15 below).

We use coordinates relative to the standard basis of V, denoting the variable related to the basis vector corresponding to the vertex v of Γ_2 by X_v. The set of all quadratic forms will (only) depend on Γ_2, the vertex ∞' of Γ_2 and the spread $\mathscr{S}'$ of V_1'. We will refer to the first two classes of quadratic forms below as the *short quadratic forms belonging to* $(\Gamma_2, \infty', \mathscr{S}')$, and to those of the last two classes as the *long quadratic forms belonging to* $(\Gamma_2, \infty', \mathscr{S}')$. Hence there are four classes in total.

- *Let Q be a hexacross defined by a vertex $v'' \in \Gamma_1''$, that is, $Q = (\Gamma_2(v'') \cap V_1') \cup \{\infty', v''\}$.* There are exactly two vertices i, j of $\Gamma_2(v'') \cap V_1'$ belonging to a common member of $\mathscr{S}'$. Let P be the partition of $(\Gamma_2(v'') \cap V_1') \setminus \{i,j\}$ in pairs of non-adjacent vertices. We define the quadratic form

$$\beta_Q : V \to \mathbb{K} : (X_v)_{v \in V_2} \mapsto -X_i X_j + X_{\infty'} X_{v''} + \sum_{\{k,\ell\} \in P} X_k X_\ell.$$

  Similarly, one defines 27 quadratic forms using a hexacross defined by a vertex of Γ_1' and ∞''.
- *Let Q be a hexacross consisting of the union of a* 6*-clique W' of Γ_1' and a* 6*-clique W'' of Γ_1''.*
  There are unique 3-cliques $C_1, C_2 \in \mathscr{C}$ with $C_1 \cup C_2 = W'$. For each $w' \in W'$, let $w'' \in W''$ denote the unique vertex of W'' not adjacent to w'. Then we define the quadratic form

$$\beta_Q : V \to \mathbb{K} : (X_v)_{v \in V_2} \mapsto \sum_{w' \in C_1} X_{w'} X_{w''} - \sum_{w' \in C_2} X_{w'} X_{w''}.$$

- *Let (Q', Q'') be a pair of opposite hexacrosses with $\infty' \in Q'$ and $\infty'' \in Q''$.*
  Then Q' and Q'' have a unique vertex v' and v'' in Γ_1'' and Γ_1', respectively. For each $w' \in Q'$, let $w'' \in Q''$ denote the unique vertex of Γ_2 opposite w'. Then we define the quadratic form

$$\beta_{Q',Q''} : V \to \mathbb{K} : (X_v)_{v \in V_2} \mapsto -X_{\infty'} X_{\infty''} - X_{v'} X_{v''} + \sum_{w' \in Q' \setminus \{\infty', v'\}} X_{w'} X_{w''}.$$

- *Let (Q', Q'') be a pair of opposite hexacrosses with $\infty' \notin Q'$ and $\infty'' \notin Q''$.*
  Set $W' = Q' \cap V_1'$ and $W'' = Q'' \cap V_1'$. For each $w \in W' \cup W''$, let w_* be the vertex of Γ_2 opposite w. Then we define the quadratic form

$$\beta_{Q',Q''} : V \to \mathbb{K} : (X_v)_{v \in V_2} \mapsto \sum_{w' \in W'} X_{w'} X_{w'_*} - \sum_{w'' \in W''} X_{w''} X_{w''_*}.$$

We now recall Theorem 10.6 of [15].

Theorem 8.15. *The variety $\mathscr{E}_7(\mathbb{K})$, obtained by intersecting the respective null sets in $\mathbb{P}(V)$ of the bundle $\mathscr{B}$ of 126 quadratic forms β_Q, for Q ranging over the set of hexacrosses of Γ_2, and the 63 quadratic forms $\beta_{Q',Q''}$, with $\{Q', Q''\}$ ranging over the set of pairs of opposite hexacrosses of Γ_2, naturally defines a point-line geometry isomorphic to the Lie incidence geometry of type $\mathsf{E}_{7,7}$ over the field $\mathbb{K}$.*

Then we denote by $\mathsf{GE}_7(\mathbb{K})$ the group of linear automorphisms of V that stabilise the bundle $\mathscr{B}$, that is, the subgroup of $\mathsf{GL}(V)$ that maps every quadratic form defined by $\mathscr{B}$ to a linear combination of quadratic forms defined by $\mathscr{B}$. The subgroup $\mathsf{GE}_7(\mathbb{K}) \cap \mathsf{SL}_{56}(V)$ is denoted by $\mathsf{SE}_7(\mathbb{K})$ and the quotients with the respective centres (consisting of scalar matrices) are $\mathsf{PGE}_7(\mathbb{K})$ and $\mathsf{PSE}_7(\mathbb{K})$. The latter is also denoted briefly by $\mathsf{E}_7(\mathbb{K})$ and is simple (as follows with similar methods as in [24]; however we will not need this fact and therefore we do not include a proof).

Lemma 8.16. *The Chevalley group $\mathsf{PGE}_7(\mathbb{K})$ is generated by the automorphisms of the building Δ of type E_7 over the field $\mathbb{K}$ pointwise fixing two opposite residues of type 7.*

Proof. We denote the base vector of V corresponding to a vertex $v \in V_2$ by e_v, and the corresponding coordinate of a generic vector by x_v. Suppose first that we have some diagonal automorphism φ of Δ which maps the coordinate x_v to $\lambda_v x_v$, for all $v \in V_2$. We may assume $\lambda_{\infty'} = 1$. The projective points $\langle e_{v'} \rangle$, for $v' \in V_1'$ generate a 26-dimensional space intersecting $\mathscr{E}_7(\mathbb{K})$ in a set of points isomorphic to $\mathscr{E}_6(\mathbb{K})$. It follows from the proof of Lemma 8.13 that there is a constant $\lambda \in \mathbb{K}^\times$ such that the product $\lambda_{v_1'} \lambda_{v_2'} \lambda_{v_3'}$ is equal to λ as soon as v_1', v_2', v_3' correspond to a line of the $\mathsf{GQ}(2,4)$ underlying Γ_1'. Let $v. \in V_1'$ arbitrary, and let $v'' \in V_1''$ be the corresponding vertex (these are at distance 3 in the Gosset graph). Then, since φ is a similitude of each short quadratic form, we infer that $\lambda_{v''} \lambda_{v'} = \lambda$ (consider the hexacross defined by ∞' and v''). Hence, with obvious notation, we deduce that $\lambda_{v_1''} \lambda_{v_2''} \lambda_{v_3''} = \lambda^2$. Since φ is a similitude of any long quadratic form "containing" ∞' and ∞'', we deduce $\lambda_{\infty''} = \lambda$. Now the determinant of the matrix of φ is equal to $\lambda^9 \cdot \lambda^{18} \cdot \lambda = \lambda^{28}$. Hence diagonal automorphisms have determinant a 28th power.

Now let θ be an automorphism of Δ pointwise fixing the residues of two opposite points. We may take for the latter $\langle e_{\infty'} \rangle$ and $\langle e_{\infty''} \rangle$. Since θ fixes all lines through these points, and since the points $\langle e_{v'}$, $v' \in V_1'$, are the unique points on lines through $\langle e_{\infty'} \rangle$ at distance 2 from $\langle e_{\infty''} \rangle$, we see that θ is a diagonal automorphism. We assume again that θ maps the coordinate x_v to $\lambda_v x_v$, for all $v \in V_2$, with $\lambda_{\infty'} = 1$. Since θ fixes the residue of $\langle e_{\infty'} \rangle$ linewise, we see that $\lambda_{v'} = \lambda$ is a constant for all $v' \in V_1'$. As above, it now easily follows that $\lambda_{v''} = \lambda^2$, for all $v'' \in V_1''$, and $\lambda_{\infty''} = \lambda^3$. Hence the determinant of the matrix of θ is equal to $\lambda^{27} \cdot \lambda^{54} \cdot \lambda^3 = \lambda^{84}$, which is a 28th power modulo a 56th power. Moreover, it is easy to see that θ induces similitudes on all short and long quadratic forms, for arbitrary $\lambda \in \mathbb{K}$. Hence all 28th powers occur. Now the assertion follows similarly as the end of the proof of Lemma 8.13. $\square$

Lemma 8.1 and Lemma 8.9 can be used to determine $\Pi^+(F)$ and $\Pi(F)$ for simplices of type 7 in buildings of type E_7. However, in general, the number of possibilities for triangles of mutually opposite simplices is too large to be practical or useful. The following result provides an alternative to Lemma 8.1. The condition that J is a self-opposite type is not essential, but convenient, and we will only need it in that case.

Lemma 8.17. *Let Δ be a spherical building over the type set I and let $J \subseteq I$ be a self-opposite type. Suppose that for each pair of simplices F, F' of type J, the subgraph $\Gamma_J^{\{F,F'\}}$ of Γ_J induced by the vertices opposite both F and F' is connected. Suppose also that there is a simplex of type J opposite any given set of three simplices of type J. Let F be a given simplex of type J. Denote by $\Pi_4(F)$ the set of all self-projectivities $F \barwedge F_2 \barwedge F_3 \barwedge F_4 \barwedge F$ of F of length 4 with $F \sim F_3$, $F_2 \sim F_4$. Suppose that $\Pi_4(F)$ is geometric. Then $\Pi^+(F) = \langle \Pi_4(F) \rangle$.*

Proof. We first prove the following property for four simplices F_1, F_2, F_3, F_4, where $\mathsf{typ}(F_1) = \mathsf{typ}(F_3) = J$ and both F_2 and F_4 are opposite both F_1 and F_3.

(*) *The projectivity $\rho\colon F_1 \barwedge F_2 \barwedge F_3 \barwedge F_4$ can be written as a product of a perspectivity $F_1 \barwedge F_4$ and conjugates of members of $\Pi_4(F_1)$.*

Indeed, let $F_1 = F_1' \sim F_2' \sim \cdots F_n' = F_3$ be a path in $\Gamma_J^{\{F_2,F_4\}}$. Define $\rho_i\colon F_4 \barwedge F_i' \barwedge F_2 \barwedge F_{i+1}' \barwedge F_4$, $i \in \{1, 2, \ldots, n-1\}$. Denote by ρ_0 the perspectivity $F_1 \barwedge F_4$. Then it is elementary to see that $\rho = \rho_0\rho_1\rho_2 \cdots \rho_{n-1}$. So, since $\Pi_4(F)$ is geometric, it suffices to show that each ρ_i can be written as the product of conjugates of members of $\Pi_4(F_1)$. It follows from letting $(F_4, F_i', F_2, F_{i+1}')$ play the role of (F_1, F_2, F_3, F_4) in the previous argument that ρ_i is a product of conjugates of members of $\Pi_4(F_{i+1}')$. Hence (*) is proved.

Now let $\rho\colon F \barwedge F_2 \barwedge F_3 \barwedge \cdots \barwedge F_{2\ell-1} \barwedge F_{2\ell} \barwedge F$ be an arbitrary even projectivity. We prove by induction on $\ell \in \{1, 2, \ldots\}$ that ρ is the product of conjugates of members of $\Pi_4(F)$. This is trivial for $\ell = 1$ and it is equivalent to (*) for $\ell = 2$. So let $\ell \geq 3$. Select a simplex F_2' opposite each of F, F_3 and F_5 (since these all have the same type, this is still possible if J is not self-opposite). Setting

$$\begin{cases} \rho_1^*\colon F \barwedge F_2 \barwedge F_3 \barwedge F_2' \barwedge F, \\ \rho_2^*\colon F_2' \barwedge F_3 \barwedge F_4 \barwedge F_5 \barwedge F_2', \\ \rho'\colon F \barwedge F_2' \barwedge F_5 \barwedge F_6 \cdots \barwedge F_{2\ell} \barwedge F, \end{cases}$$

we see that, if $\rho_0\colon F \barwedge F_2'$, we have

$$\rho = \rho_1^* \cdot (\rho_0\rho_2^*\rho_0^{-1}) \cdot \rho',$$

where we know by the induction hypothesis that all factors are products of members of $\Pi_4(F)$, using the fact that $\Pi_4(F)$ is geometric (and hence closed under conjugation). □

In order to be able to apply Lemma 8.17, we have to check the conditions in the various cases. It turns out we will use Lemma 8.17 in exactly three different cases, for which we now prove the condition on the corresponding graph Γ_J.

Lemma 8.18. *Let Δ be the spherical building over the field $\mathbb{K}$, $|\mathbb{K}| > 2$, of type either E_6 or E_7. Let $J = \{2\}$ if $\mathsf{typ}\,\Delta = \mathsf{E}_6$, and $J \in \{\{1\}, \{3\}\}$ if $\mathsf{typ}\,\Delta = \mathsf{E}_7$. Let v and v' be vertices of type J. Then the subgraph Γ'_J of Γ_J induced on the vertices opposite both v and v' is connected. If $J \neq \{3\}$, then, more generally, the complement of the union of two geometric hyperplanes of the corresponding geometries of type $\mathsf{E}_{n,J}$, $n = 6, 7$, is connected.*

Proof. The claims we refer to in this proof are those of Section 8.3.

We start by proving the cases $J = \{2\}$ if $\mathsf{typ}\,\Delta = \mathsf{E}_6$ and for $J = \{1\}$ if $\mathsf{typ}\,\Delta = \mathsf{E}_7$. Indeed, let C be the complement of two proper geometric hyperplanes H_1, H_2 of the long root geometry Γ of type E_6 or E_7 over the field $\mathbb{K}$, with $|\mathbb{K}| > 2$. Let p and q be two points of C. A moment's thought reveals that we may assume that p and q are opposite in Γ. The equator geometry $E(p, q)$ is not the union of two geometric hyperplanes, because lines contain at least 3 points and hence, if $E(p,q)$ is contained in neither H_1 nor H_2, then there is some point $r \in E(p,q) \cap C$. Claim 3 yields paths in C connecting p with r and r with q. So, we may assume that $E(p,q) \subseteq H_1$. Since $E(p,q)$ contains planes, we find a point $x \in E(p,q) \cap H_1 \cap H_2$. Let ξ be the symp through p and x, and let ζ be a symp opposite ξ through q. Set $H_{\xi,1} := H_1 \cap \xi$, $H_{\xi,2} := H_2 \cap \xi$, and let $H_{\xi,3}$ be the (pointwise) projection of $\zeta \cap H_1$ onto ξ; likewise let $H_{\xi,4}$ be the pointwise projection of $\zeta \cap H_2$ onto ξ (projecting here means mapping to the symplectic points). Since the projection of q is x, we see that $H_{\xi,1} \cap H_{\xi,2} \ni x \notin H_{\xi,3} \cap H_{\xi,4}$. Hence Claim 2 yields a point $z \in C \cap \xi$ such that the unique point z' of ζ symplectic to z also belongs to C. Now Claim 3 yields paths $p \perp\!\!\!\perp z \perp\!\!\!\perp z' \perp\!\!\!\perp q$ and the proof for these two cases is finished.

Now assume $J = \{3\}$ and Δ has type E_7. Consider the parapolar space of type $\mathsf{E}_{7,7}$ corresponding to Δ. Then v and v' correspond to two 5-spaces V and V'. Adjacency in $\Gamma_{\{3\}}$ coincides with the $\sim$ relation of the proof of Claim 6 in any symp containing the two 5-spaces.

Let W_1 and W_2 be two arbitrary 5-spaces both opposite both V and V'. We show that they belong to the same connected component of the graph $\Gamma'_{\{3\}}$. As before, we may assume that they are opposite in Δ. Consider opposite symps $\xi_1 \supseteq W_1$ and $\xi_2 \supseteq W_2$. Let U_i and U'_i be the projection of V and V', respectively, onto ξ_i, $i = 1, 2$. Then, by Proposition 2.1, a 5-space in ξ_i is opposite V or V' if, and only if, it is opposite W_i or W'_i, respectively, $i = 1, 2$. Let Z_1 and Z'_1 be the projections of U_2 and U'_2, respectively, onto ξ_1. Let L_1 be a line of ξ_1 disjoint from $U_1 \cup U'_1 \cup Z_1 \cup Z'_1$, guaranteed to exist by Claim 4. Let L_2 be its projection onto ξ_2. Then L_2 is disjoint from $U_2 \cup U'_2$. Claim 5 now yields a 5-space W^*_i through L_i disjoint from $U_i \cup U'_i$, $i = 1, 2$, and Claim 6 for $d = 6$ implies that it now suffices to prove that W^*_1 and W^*_2 are in the same connected component of $\Gamma'_{\{3\}}$.

Let ζ be the symp through L_1 and L_2. Let U and U' be the projections of V and V', respectively, onto ζ. Since no point of L_i, $i = 1, 2$, is collinear to a point of $V \cup V'$, the 5-spaces U and U' are disjoint from $L_1 \cup L_2$. Claim 5 yields 5-spaces Y_1 and Y_2 containing L_1 and L_2, respectively, disjoint from $U \cup U'$. Claim 6 for $d = 6$ implies that Y_1 and Y_2 are in the same connected component of $\Gamma'_{\{3\}}$. Hence, in order to prove the lemma, it suffices to show that W^*_i and Y_i are in the same connected component, for $i = 1, 2$. But this now follows from Claim 7 for $d = 5$ applied to the residue of L_i.

The lemma is proved. □

Theorem 8.19. *Let Δ be a building of type E_6, E_7 or E_8 over the field $\mathbb{K}$. Let F be a simplex of Δ such that $\mathsf{Res}_\Delta(F)$ is irreducible. Then $\Pi(F)$ and $\Pi^+(F)$ are given as in* Table 2*, where the last column contains a checkmark if $\mathsf{typ}(F)$ is polar closed. Again, a long hyphen in the table in the column of $\Pi(F)$ means that $\mathsf{typ}(F)$ is not self-opposite and so $\Pi(F)$ is trivially isomorphic to $\Pi^+(F)$ — it must again be read as a "bysame" symbol. Grey rows correspond to projectivity groups which are not necessarily full linear groups.*

Proof. The case (A1) was handled in Theorem D. We now handle the other cases. Note that we may again assume that $|\mathbb{K}| \geq 3$ as otherwise the linear groups are unique.

Cases (A2) and (A3) Every subdiagram of type A_2 or A_3 of E_r, $r = 6, 7, 8$, is contained in a subdiagram of type A_3 or A_4, respectively. Then the assertions all follow from Corollary 7.2 and Theorem 8.4.

Case (A4) If $2 \notin \mathsf{cotyp}(F)$ for E_6, or if $\mathsf{cotyp}(F) \neq \{1, 2, 3, 4\}$ for $\mathsf{E}_7, \mathsf{E}_8$, then we can again embed the diagram of $\mathsf{Res}_\Delta(F)$ in diagram of type A_5 and use Corollary 7.2 and Theorem 8.4. Now suppose $2 \in \mathsf{cotyp}(F)$ for E_6 and $\mathsf{cotyp}(F) = \{1, 2, 3, 4\}$ for E_7 and E_8. Then the assertion follows from Corollary 7.2 and Case (A**) for $r = 5$ of Theorem 8.8.

Case (A5) In the Coxeter diagram of type E_8, every subdiagram of type A_5 is contained in one of type A_6 and hence the assertion for E_8 follows from Corollary 7.2. The same thing holds

Reference	$\mathsf{typ}(\Delta)$	$\mathsf{Res}_\Delta(F)$	$\mathsf{cotyp}(F)$	$\Pi^+(F)$	$\Pi(F)$	
(A1)		A_1		$\mathsf{PGL}_2(\mathbb{K})$	$\mathsf{PGL}_2(\mathbb{K})$	
(A2)	E_6	A_2	$\{2,4\}$	$\mathsf{PGL}_3(\mathbb{K})$	$\mathsf{PGL}_3(\mathbb{K}).2$	
	E_6	A_2	$\neq\{2,4\}$	$\mathsf{PGL}_3(\mathbb{K})$	——	
	$\mathsf{E}_7,\mathsf{E}_8$	A_2		$\mathsf{PGL}_3(\mathbb{K})$	$\mathsf{PGL}_3(\mathbb{K}).2$	
(A3)	E_6	A_3	$\{3,4,5\}$	$\mathsf{PGL}_4(\mathbb{K})$	$\mathsf{PGL}_4(\mathbb{K})$	✓
	E_6	A_3	$\neq\{3,4,5\}$	$\mathsf{PGL}_4(\mathbb{K})$	——	
	$\mathsf{E}_7,\mathsf{E}_8$	A_3		$\mathsf{PGL}_4(\mathbb{K})$	$\mathsf{PGL}_4(\mathbb{K}).2$	
(A4)	E_6	A_4		$\mathsf{PGL}_5(\mathbb{K})$	——	
	$\mathsf{E}_7,\mathsf{E}_8$	A_4		$\mathsf{PGL}_5(\mathbb{K})$	$\mathsf{PGL}_5(\mathbb{K}).2$	
(A5)	E_6	A_5		$\mathsf{PSL}_6(\mathbb{K},3)$	$\mathsf{PSL}_6(\mathbb{K},3)$	✓
	E_7	A_5	$\{2,4,5,6,7\}$	$\mathsf{PSL}_6(\mathbb{K},2)$	$\mathsf{PSL}_6(\mathbb{K},2).2$	
	E_7	A_5	$2\notin\mathsf{cotyp}(F)$	$\mathsf{PGL}_6(\mathbb{K})$	$\mathsf{PGL}_6(\mathbb{K}).2$	
	E_8	A_5		$\mathsf{PGL}_6(\mathbb{K})$	$\mathsf{PGL}_6(\mathbb{K}).2$	
(A6)	$\mathsf{E}_7,\mathsf{E}_8$	A_6		$\mathsf{PGL}_7(\mathbb{K})$	$\mathsf{PGL}_7(\mathbb{K}).2$	
(A7)	E_8	A_7		$\mathsf{PGL}_8(\mathbb{K})$	$\mathsf{PGL}_8(\mathbb{K}).2$	
(D4)	E_6	D_4		$\overline{\mathsf{PGO}}_8^\circ(\mathbb{K})$	$\overline{\mathsf{PGO}}_8(\mathbb{K})$	
	$\mathsf{E}_7,\mathsf{E}_8$	D_4		$\overline{\mathsf{PGO}}_8^\circ(\mathbb{K})$	$\overline{\mathsf{PGO}}_8^\circ(\mathbb{K})$	✓
(D5)	E_6	D_5		$\overline{\mathsf{PGO}}_{10}^\circ(\mathbb{K})$	——	
	$\mathsf{E}_7,\mathsf{E}_8$	D_5		$\overline{\mathsf{PGO}}_{10}^\circ(\mathbb{K})$	$\overline{\mathsf{PGO}}_{10}(\mathbb{K})$	
(D6)	E_7	D_6		$\overline{\mathsf{P\Omega}}_{12}(\mathbb{K})$	$\overline{\mathsf{P\Omega}}_{12}(\mathbb{K})$	✓
	E_8	D_6		$\overline{\mathsf{PGO}}_{12}^\circ(\mathbb{K})$	$\overline{\mathsf{PGO}}_{12}^\circ(\mathbb{K})$	✓
(D7)	E_8	D_7		$\overline{\mathsf{PGO}}_{14}^\circ(\mathbb{K})$	$\overline{\mathsf{PGO}}_{14}(\mathbb{K})$	
(E6)	$\mathsf{E}_7,\mathsf{E}_8$	E_6		$\mathsf{PGE}_6(\mathbb{K})$	$\mathsf{PGE}_6(\mathbb{K}).2$	
(E7)	E_8	E_7		$\mathsf{PGE}_7(\mathbb{K})$	$\mathsf{PGE}_7(\mathbb{K})$	✓

TABLE 2. Projectivity groups in the exceptional cases $\mathsf{E}_6,\mathsf{E}_7,\mathsf{E}_8$

in the Coxeter diagram of type E_7 if the subdiagram of type A_5 does not contain the vertex of type 2.

Now suppose Δ is the building of type E_6 over the field $\mathbb{K}$, and F is a vertex of type 2.

We argue in the corresponding Lie incidence geometry of type $\mathsf{E}_{6,1}$. There, F is a 5-space. Let F_1,F_2,F_3 be three 5-spaces, with both F_1 and F_3 opposite both F and F_2, and with F adjacent to F_2, and F_1 adjacent to F_3, that is, $\pi_0 := F\cap F_2$ and $\pi_1 := F_1\cap F_3$ are planes. We also initially assume that π_0 and π_1 are opposite. Consider the projectivity $\rho\colon F\barwedge F_1\barwedge F_2\barwedge F_3\barwedge F$. We claim that ρ fixes each point of π_0. Indeed, let $p_0\in\pi_0$ be such a point. Then clearly, since $F\cap F_2$ contains p_0, the projectivities $F\perp F_1\barwedge F_2$ and $F_2\perp F_3\barwedge F$ fix p_0, hence $p_0^\rho=p_0$ and the claim is proved. Likewise, ρ fixes each point of F collinear with a point of π_1. The set of such points forms a plane π_0' of F, disjoint from π_0. Choosing a basis of F in $\pi_0\cup\pi_0'$, a matrix of ρ is a diagonal matrix with diagonal elements three times 1 and three times some scalar $k\in\mathbb{K}$. We now show that k can be arbitrary. This is equivalent to showing that,

(*) given F_1,F_2 and F_3 as above, given a line L_0 in F containing points $x_0\in\pi_0$ and $x_0'\in\pi_0'$, and given two points $p,q\in L_0\setminus\{x_0,x_0'\}$, we can re-choose F_3 through π_1 such that ρ maps p to q.

We now prove (*). Let p_1 be the projection of p onto F_1 and let p_2 be the projection of p_1 onto F_2. If p and p_2 were not collinear, then the symp $\xi(p,p_2)$ would contain p_1 and π_0, leading to additional points in π_0 collinear to p_1 inside $\xi(p,p_2)$, contradicting the fact that F and F_1 are opposite and hence p_1 is far from F. Hence there is some singular 4-space U containing π_0, L and p_2. (Note that, since U intersects F in a 3-space, Lemma 2.4(ii) implies that U is really a 4-space and not a 4′-space.) Set $\xi := \xi(x_0,p_1)$. Then ξ contains p,q,p_1,p_2 and the unique point $x_1 \in \pi_1$ collinear to x_0'. It is clear that π_1 intersects ξ in only x_1, as otherwise there would be a point of π_1 collinear to x_0, contradicting the fact that π_0 and π_1 are opposite. So, Lemma 2.5 yields a plane $\alpha \subseteq \xi$ collinear to π_1. Lemma 2.4 implies that α and π_1 are contained in a unique 4′-space U_2, which is itself contained in a unique 5-space F_3. Now both q and p_2 are (inside ξ) collinear to all points of respective lines of α, implying that they are collinear to a common point $p_3 \in F_3$. Now (*) follows.

It now also follows that the set of such projectivities ρ (as above with π_0 and π_1 opposite) is geometric (they are the homologies with two disjoint planes as centres). Now we drop the assumption of π_0 being opposite π_1. We claim that in this more general case, the projectivity ρ, as defined above, is the product of homologies with disjoint planes as centres. Indeed, set $\pi_0' := \mathsf{proj}_F^{F_1}(\pi_1)$ as above. If π_0' is disjoint from π_0, then by Proposition 2.1, π_0 and π_1 are opposite. Now we treat the other cases. Set $d = \dim(\pi_0 \cap \pi_0')$ and note that $d=-1$ is precisely the case we already proved.

- $d=0$ Let π_2 be a plane in F sharing a line with π_0 but disjoint from π_0'. Then it is easy to check that the unique 4-space U containing the 3-space generated by π_2 and π_0 is disjoint from $\mathsf{proj}_{F_2}^{F_1}(\pi_1)$. Hence there exists a 5-space $F_2' \neq F$ containing π_2 and opposite both F_1 and F_3, and we have that π_1 is opposite both $F \cap F_2'$ and $F_2 \cap F_2'$. We can now write ρ as the product of $F \barwedge F_1 \barwedge F_2' \barwedge F_3 \barwedge F$ and the conjugate of $F_3 \barwedge F_2' \barwedge F_1 \barwedge F_2 \barwedge F_3$ by $F \barwedge F_3$, reducing this case to the case $d=-1$, which we already proved.
- $d=1$ Let π_2 be a plane in F sharing a line with π_0 and exactly one point (necessarily in π_0) with π_0'. Then, similarly as in the case $d=0$, we can choose a 5-space $F_2' \neq F$ through π_2 opposite both F_1 and F_3 and such that π_1 has a unique point collinear to some point of $F \cap F_2'$, and that point is also the unique point of π_1 collinear to some point of $F_2 \cap F_2'$. We have hence reduced this case to two times the case $d=0$, which we proved above.
- $d=2$ This case is similarly reduced to the case $d=1$. We leave the (straightforward) details to the reader.

The claim is proved. Hence, thanks to Lemma 8.18, we can apply Lemma 8.17 and obtain that $\Pi^+(F)$ is generated by all homologies with disjoint planes as centres. This group contains $\mathsf{PSL}_6(\mathbb{K})$ and then clearly corresponds to all 6×6 matrices with a determinant equal to some nonzero 3th power. Also, $\Pi(F) = \Pi^+(F)$ by virtue of Theorem B.

Now suppose $\mathsf{cotyp}(F) = \{2,4,5,6,7\}$ in case of E_7. Here we can take for F a pair consisting of a 5-space W and a symp ξ containing W in the Lie incidence geometry of type $\mathsf{E}_{7,7}$ over the field $\mathbb{K}$. We employ the same method as in the previous case (Case A_5 in E_6), noting that a projectivity $\{W,\xi\} \barwedge \{W',\xi'\} \barwedge \{W'',\xi''\}$, where the simplices $\{W,\xi\}$ and $\{W'',\xi''\}$ are adjacent, is trivial as soon as $W = W''$, and so we may always assume that in such a (sub)sequence $W \neq W''$ and $\xi = \xi''$. However, since the action of the projectivity is apparently independent of the symps ξ and ξ', we may only consider projections from 5-spaces onto 5-spaces. Hence let W_1, W_2, W_3 be three 5-spaces with both W_1 and W_3 opposite both W and W_2, and $\Sigma_1 := W_1 \cap W_3$ and $\Sigma_0 := W \cap W_2$ 3-spaces such that the symps ξ_0 and ξ_1 containing W, W_2, and W_1, W_3, respectively, are also opposite. Similarly as in the previous case (type A_5 inside E_6), we may from the beginning assume that Σ_0 and Σ_1 are opposite 3-spaces. Set $L_0 := \mathsf{proj}_W^{W_1}(\Sigma_1)$. Then, by Proposition 2.1 L_0 and Σ_0 are disjoint. Set $L_2 := \mathsf{proj}_{W_2}^{W_1}(\Sigma_1)$, then likewise L_2 and Σ_0 are disjoint. Let x_0 be an arbitrary point on L_0. Then inside ξ_0 one sees that there is a unique point x_2 on L_2 collinear to x_0. We claim that the projectivity $\rho_1 \colon W \perp W_1 \perp W_2$ maps x_0 to x_2. Indeed, set $W_1' := \mathsf{proj}_{\xi_0}^{\xi_1}(W_1)$. Then, again by Proposition 2.1, W_1' is disjoint from both W

and W_2. Set $U_1 := \mathsf{proj}^{W}_{W_1}(x_0)$ and $U_1' := \mathsf{proj}^{\xi_1}_{\xi_0}(U_1)$ and note that $\Sigma_1 \subseteq U_1$. Then $U_1' \subseteq W_1'$. Since x_0 is at distance 2 from each point of U_1, it follows by Lemma 2.6 that x_0 is collinear to all points of U_1'. Hence x_0 is contained in the unique $5'$-space V_0 of ξ_0 containing U_1'. Likewise, if $x_2' = \mathsf{proj}^{W_1}_{W_2}(U_1)$, then $x_2' \in V_0$. Hence x_0 and x_2', which is contained in L_2 as U_1 contains Σ_1, are collinear. Consequently, $x_2' = x_2$ and the claim is proved.

It now also follows that $\rho_3\colon W_2 \barwedge W_3 \barwedge W$ maps x_2 back to x_0, since x_0 is the unique point on L_0 collinear to x_2. Consequently, the projectivity $\rho\colon W \barwedge W_1 \barwedge W_2 \barwedge W_3 \barwedge W$ fixes each point of L_0. It is easy to see that it also fixes every point of Σ_0. Hence it is a homology corresponding to a diagonal matrix with the diagonal consisting of four times a 1 and two times a scalar $k \in \mathbb{K}^\times$. If we can now show that every nonzero scalar k can occur, then, similarly to the case A_5 in E_6, using Lemma 8.17 and Lemma 8.18, we are done.

But it follows from the arguments in the previous paragraphs that the projectivity ρ_1 coincides with the projectivity $W \barwedge W_1' \barwedge W_2$ inside ξ_0. Likewise the projectivity ρ_3 coincides with the projectivity $W_2 \barwedge W_3' \barwedge W$ inside ξ_0, with $W_3' := \mathsf{proj}^{\xi_1}_{\xi_0}(W_3)$. Now the assertion follows with exactly the same arguments as Case (A**) in the proof of Theorem 8.8.

This concludes Case (A5).

Case (A6) In a Coxeter diagram of type E_8 a subdiagram of type A_6 is either contained in a subdiagram of type A_7, in which case we are done by Corollary 7.2 and Theorem 8.4, or it has type $\{2,4,5,6,7,8\}$, in which case it is contained in a subdiagram of type D_7 and we are done by Corollary 7.2 and Theorem 8.8(A**).

In a Coxeter diagram of type E_7 a subdiagram of type A_6 necessarily has type $\{1,3,4,5,6,7\}$. Although a proof using the Lie incidence geometry of type $\mathsf{E}_{7,7}$, where F is a 6-space, is feasible, we prefer to consider the Lie incidence geometry of type $\mathsf{E}_{7,1}$, where the proof is entirely similar to the case (A7) of A_7 in E_8 below. So we refer to that case for the details.

Case (A7) In a Coxeter diagram of type E_8 a subdiagram of type A_7 has cotype 2, which corresponds to a 7-space. Let Δ be the building of type E_8 over the field $\mathbb{K}$.

Let W_0 and W_2 be two 7-spaces in Δ intersecting in a 4-space that we denote by U. Let W_1 be a 7-space opposite to both W_0 and W_2. Then U projects to a plane α in W_1. Projection here means that U and α are special; for every pair of points (u,a) with $u \in U$ and $a \in \alpha$, there exists a point $p_{u,a}$ that is collinear to u and a. Let U' be a 4-space in W_1 that has no intersection with α. Let W_3 be a 7-space that intersects W_1 in U' and is opposite both W_0 and W_3. Set

$$\rho\colon W_0 \barwedge W_1 \barwedge W_2 \barwedge W_3 \barwedge W_0.$$

We claim that points of U are fixed under ρ. Indeed, a point of $W_0 \cap W_2$ first maps to a hyperplane of W_1, then back to itself, then to a hyperplane of W_3 and again back to itself. That means U is fixed pointwise under ρ. The claim is proved.

The projection of U' onto W_0 is a plane that we will denote by β. A point p of β maps to a hyperplane of W_1 that contains U' and intersects α in a line. We will call this hyperplane H. The projection of H onto W_2 is a point that we will denote by p'. We claim that the point p is collinear to p'.

Indeed, suppose for a contradiction that p were not collinear to p'. The points p and p' cannot have distance 3 or be a special pair, since they are both collinear to each point of U. That means p and p' are symplectic. The symp $\xi(p,p')$ intersects W_0 and W_2 each in a $6'$ space that contains U. The projection of the $6'$-space $W_0 \cap \xi(p,p')$ onto W_1 is a point w. The point w is in α, since α was the projection of W_0 onto W_1. Let F be a $6'$ space in W_1 not through w. Then F is opposite $W_0 \cap \xi(p,p')$. Let ξ be the unique symp through F. Then ξ is opposite $\xi(p,p')$.

The projections of p and q onto W_1 define the two $6'$-spaces of W_1, for which it is not the case, that all points are opposite p and q and the intersection of them is a 5-space that we will denote by V_1. Let V_0 be the 5-space that is the projection of V_1 onto $\xi(p,p')$. The 5-space spanned by U and p in $\xi(p,p')$ is disjoint from V_0, but p and p' are both collinear to V_0. Let x be an arbitrary point in V_1 and x' be a point at distance 2 from x in V_0. The point x' is collinear to both p and p'. There exists a unique symp ξ' that intersects $\xi(p,p')$ exactly in x' and ξ exactly in x.

The $6'$-spaces $\langle V_0, p\rangle$ and $\langle V_0, p'\rangle$ in $\xi(p,p')$ are of the same type as the $6'$ spaces $W_0 \cap \xi(p,p')$ and $W_2 \cap \xi(p,p')$. But the intersections $\langle V_0, p\rangle \cap (W_0 \cap \xi(p,p'))$ and $\langle V_0, p'\rangle \cap (W_2 \cap \xi(p,p'))$ only contain a unique point. That contradicts the fact that subspaces of the same type in polar spaces of type D_7 have to intersect in even codimension. We conclude that p and p' have to be collinear and the claim is proved.

Now we claim that all points of β are fixed under ρ. Indeed, the point p projects to the hyperplane H in W_1. This hyperplane projects to p' in W_2. This already implied that p and p' are collinear. If we project a point of W_2 to W_3 and to W_0, we also get, that this point and the point in W_0 have to be collinear. The point p is the only point of β that p' is collinear to. Hence p' projects to a hyperplane in W_3 and then back to p.

Let xy be a line in W_0 between a point $x \in U$ and $y \in \beta$. Let a and b be two distinct points on xy not equal to either x or y. We claim that we can re-define W_3 such that ρ maps a to b. Let $a' := W_0 \barwedge W_1 \barwedge W_2(a)$, $b' := W_0 \barwedge W_1 \barwedge W_2(b)$. Since a, b, a', y, y' and U form a 6-space, $\langle a, b, a'\rangle$ forms a plane, ba' and yy' intersect in a point s.

Let W' be a 7-space through U and s. The projection of W' onto U' is a 7-space that is not opposite W' and that we will denote by W_3. Since W_3 and W' are not opposite, there exists, by Lemma 2.14, a plane γ_3 in W_3 such that no point of γ_3 is opposite any point of W' and there exists a plane γ' in W' such that no point of γ' is opposite any point of W_3.

Let y' be the point in γ' on a line with the point y of β. The line yy' is the unique line between γ' and β. The points y and y' are not opposite all points of U'. Hence the point s is not opposite all points of U'. The point s is also not opposite any of the points of γ_3 since $s \in W'$. That means s is not opposite any point of W_3 and hence has to be contained in γ'.

The line $a'b$ contains s. That means the points a' and b project to the same hyperplane of W_3. Then the point a maps to a hyperplane of W_1, then to a' in W_2 and the point a' projects to a hyperplane of W_3 and then to b. Our claim is proved.

By Theorem A, $\Pi^+(W_0)$ contains $\mathsf{PSL}_8(\mathbb{K})$. By the above, it also contains all diagonal matrices with diagonal $(1,1,1,1,1,k,k,k)$, $k \in \mathbb{K}^\times$ arbitrarily, and the entries k can be anywhere. This readily implies that $\Pi^+(W_0)$ contains the diagonal matrices with diagonal $(1,1,1,1,1,1,1,k)$, hence all homologies, hence $\Pi^+(W_0) = \mathsf{PGL}_8(\mathbb{K})$ and $\Pi(W_0) = \mathsf{PGL}_8(\mathbb{K}).2$.

Case (D5) We first consider the case of a Coxeter diagram of type E_6. Without loss of generality, we may assume that F has type 6. Hence we consider F as a symp in a geometry of type $\mathsf{E}_{6,1}$ over the field $\mathbb{K}$.

Let p_1 be a point in Δ and ξ_0 a symp opposite p_1 in Δ. Let U be a maximal singular subspace in ξ_0. Then U is a 4-space. Let ξ_2 be another symp through U opposite p_1. Opposite a 4-space are lines. Let L be a line through p_1 opposite U and $V := \mathsf{proj}_{\xi_0}(L)$. Let p_3 be any point on L opposite both ξ_0 and ξ_2, so that we have a projectivity $\rho : \xi_0 \barwedge p_1 \barwedge \xi_2 \barwedge p_3 \barwedge \xi_0$. We will show that ρ fixes U and V pointwise.

First let x be a point in U. Then x projects to a symp $\xi(x,p_1)$, then back to x, since $x \in \xi_0 \cap \xi_2$, then to a symp $\xi(x,p_3)$ and then again back to x.

Now let y be a point in V. The point $y = y_0$ projects to a symp $\xi(y_0,p_1) = \xi_y$ and then to a point $y_2 \in \xi_2$. Suppose y_0 and y_2 were not collinear. The symp ξ_y has to contain the closure of y_0 and y_2. Both y_0 and y_2 are collinear to a 3-space of U. The intersection of these 3-spaces contains a plane. That means that the closure has to contain a plane of U that then had to be contained in ξ_y. But that contradicts the fact that U and p_1 are opposite, because p_1 would have to be collinear to elements of that plane. It follows that $y_0 \perp y_2$. Now, since $V = \mathsf{proj}_{\xi_0}(L)$, we see that $L \subseteq \xi_y$. So y_2 continues mapping to ξ_y and then back to y_0. Hence points of V are fixed.

Next we want to show that we can always define p_3 on L in a way, such that the projectivity ρ defined above maps an arbitrary point p on a line xy, with $x \in U$ and $y \in V$, to another arbitrary point q on xy for $p \notin U, V$ and $q \notin U, V$. Given U, V, L and p_1 as before and a line xy as described above, let p be an arbitrary point on xy not in U or V. Then projecting p to p_1 yields a symp $\xi(p, p_1)$ that projects to a point p_2 onto ξ_2. Let $y_2 := \mathsf{proj}_{\xi_2}(\mathsf{proj}_{p_1}(y))$. By the previous paragraph, the points x, y, y_2 generate a singular plane, which contains p, q and p_2. Let $a := p_2q \cap yy_2$. Suppose a were collinear to p_1. Then a would be in $\xi(p, p_1)$ and $\xi(p, p_1)$ would contain the plane $\langle x, y, y_2 \rangle$ and in particular the line xy. But that contradicts the fact that $\xi(p, p_1)$ intersects ξ_0 only in p. It follows that a is not collinear to p_1. That means a is collinear to a different point of L that we will define as p_3. This point p_3 is not collinear to p_2 as otherwise $\xi(p, p_1$ would contain L, forcing $p \in V$, a contradiction. Since a and p_2 are in $\xi(p_3, p_2)$, $\xi(p_3, p_2)$ contains the whole line ap_2 and hence the point q. With that it follows that p maps to $\xi(p, p_1)$ to p_2 to $\xi(p_3, p_2) = \xi(p_3, q)$, and finally to q.

Now Lemma 8.11(i) proves the assertion.

In a Coxeter diagram of type E_7 or E_8, a subdiagram of type D_5 is always contained in a subdiagram of type E_6, and so we can apply Corollary 7.2, the previous paragraphs, and Theorem B.

Case (D4) Each subdiagram of type D_4 in a diagram of type E_n, $n = 6, 7, 8$, is contained in a subdiagram of type D_5. It follows that, if F is a simplex of cotype D_4 in a building Δ of type E_n, $n = 6, 7, 8$, then there is a subsimplex $F' \subseteq F$ of cotype D_5. By the previous case and Theorem A, the stabiliser of F' in the little projective group $\mathsf{Aut}^\dagger(\Delta)$ of Δ acts on $\mathsf{Res}_\Delta(F')$ as the complete linear type preserving group of automorphisms. Hence the stabiliser of F in $\mathsf{Aut}^\dagger(\Delta)$, acting on $\mathsf{Res}_\Delta(F)$ contains the stabiliser in the full linear type preserving group of $\mathsf{Res}_\Delta(F')$ of the vertex $F \setminus F'$. This is clearly also the full linear type preserving group of $\mathsf{Res}_\Delta(F)$.

Now, in case of $\mathsf{typ}(\Delta) = \mathsf{E}_6$, it follows from Theorem B that $\Pi^+(F)$ has index 2 in $\Pi(F)$, and so $\Pi(F)$ is the full linear group of the corresponding polar space of $\mathsf{Res}_\Delta(F)$. In case of E_7 or E_8, Theorem B implies that $\Pi(F) = \Pi^+(F)$.

Case (D7) In a Coxeter diagram of type E_8 a subdiagram of type D_7 is missing vertex 1. So F is a symp in the geometry Γ of type $\mathsf{E}_{8,8}$ over the field $\mathbb{K}$.

Let ξ_0 and ξ_2 be two symps in Γ intersecting in a 6-space that we will denote by U. Let ξ_1 be a symp opposite ξ_0 and ξ_2, and U' the $6'$-space that is the projection of U onto ξ_1. Opposite U' in ξ_1 is a 6-space V. Let V' be the $6'$-space that is the projection of V onto ξ_0. Suppose there is a symp $\xi_3 \supseteq V$ opposite both ξ_0 and ξ_2, such that we have a projectivity $\rho\colon \xi_0 \barwedge \xi_1 \barwedge \xi_2 \barwedge \xi_3 \barwedge \xi_0$. We first show that ρ fixes every point of U and V'. For points of U the assertion follows immediately. A point v_0 of V' first projects to a point v_1 in V, then to some point in ξ_2 and then back to v_1. The point v_1 maps back to v_0.

Next we want to show that there always exists a ξ_3 through V opposite both ξ_0 and ξ_2 and that we can define it in a way, such that we can map an arbitrary point p on a line xy, with $x \in U$ and $y \in V'$, to a point q on xy for $p \notin U, V'$ and $q \notin U, V'$.

We define:

$$y =: y_0,\ \mathsf{proj}_{\xi_1}(y_0) =: y_1,\ \mathsf{proj}_{\xi_2}(y_1) =: y_2,$$
$$p =: p_0,\ \mathsf{proj}_{\xi_1}(p_0) =: p_1,\ \mathsf{proj}_{\xi_2}(p_1) =: p_2.$$

First we want to show that $p_2 \perp q$. Suppose p_2 and q were not collinear. Then there exists a symp $\xi(p_2, q) = \xi(p_2, p)$ and a point $u \in U \cap \xi(p_2, p)$ such that p_1 is opposite u. But by Lemma 2.9(iii) the point p_1 is supposed to be symplectic to only one point of $\xi_{p_2,p}$, and it is already symplectic to p and p_2. It follows that $p_2 \perp q$ and there exists a plane $\langle p, q, p_2 \rangle$. Define $a := y_0y_2 \cap p_2q$.

Let $\xi(U,a)$ be the unique symp through U containing a, according to Lemma 2.12. Define $\xi_3 := \mathsf{proj}_V(\xi(U,a))$. Then $\xi(U,a)$ is the unique symp through U not opposite ξ_3

In $\xi(U,a)$ let, according to Lemma 2.13, W be the maximal singular subspace, such that no point of W is opposite (at distance 3 from) any point of ξ_3. Let W_3 be the maximal singular subspace of ξ_3, such that no point of W_3 is opposite any point of $\xi(U,a)$. The points y and y_2 are not opposite any point of V. Hence a is not opposite any point of V. But the point a is also not opposite any point of W_3 by the definition of W_3. Hence $a \in W$.

The same points of ξ_3 are not opposite q and p_2, since, if we take an arbitrary point s in ξ_3 which is not opposite q, then s is not opposite a and q, so also not opposite p_2 and the same if we switch the roles of q and p_2. Let q_3 be the unique point of ξ_3 symplectic to q. Then q is opposite $\xi_3 \setminus q_3^\perp$. But then $\xi_3 \setminus q_3^\perp = \xi_3 \setminus p_3^\perp$. The perps of points s and t in a polar space are the same, if $s = t$. Hence it follows that $q_3 = p_3$. That means q is the unique point of ξ_0 that p_3 is symplectic to. With that we get that $\rho(p) = q$.

Now, as before, Lemma 8.11(i) proves the assertion.

Case (D6) We first treat the case of type D_6 inside type E_7. Let ξ be a symp of the geometry of type $\mathsf{E}_{7,7}$ over the field $\mathbb{K}$. We first claim that $\Pi(\xi)$, which is equal to $\Pi^+(\xi)$ by Theorem B, contains all homologies pointwise fixing two ξ-opposite maximal singular 5-spaces. Let M_{13} and M be two such subspaces of ξ. Let ξ_3 be an arbitrary symp distinct from ξ and containing M_{13}. Let ξ_2 be a symp opposite both ξ and ξ_3 (and note that this implies that each point of ξ_2 is opposite some point of ξ). There is a unique maximal singular subspace M_{24} contained in ξ_2 each point of which is collinear to some point of M, that is, $M_{24} = \mathsf{proj}^{\xi}_{\xi_2}(M)$. Let L be any given line in ξ joining a point $p_{13} \in M_{13}$ and $p \in M$. Choose two points $q, q' \in L \setminus \{p_{13}, p\}$. Set $q_2 = \mathsf{proj}_{\xi_2}(q)$ and $q_3 = \mathsf{proj}_{\xi_3}(q_2)$.

If q were not collinear to q_3, then the symp containing them would contain a 3-dimensional subspace of M_{13} and q_2; this would imply that q_2 is close to ξ, contradicting Lemma 2.6 in view of our remark in the previous paragraph that says that q_2 is opposite some point of ξ. Hence $\langle q, q_3, q' \rangle$ is a plane π, contained in the symp ζ containing p_{13} and q_2. Let ξ' be any symp containing, M_{24}, but distinct from ξ_2. Let p_{24} be the unique point of ξ_2 collinear to p, and note $p_{24} \in M_{24}$, and that p_{24} and q_2 are collinear. Hence $p_{24} \in \zeta$. This implies that $\zeta \cap \xi'$ is either a line or a 5-space through p_{24}. In the latter case p_{13}, being collinear with more than one point of that intersection, is close to ξ', contradicting Lemma 2.6 and the fact that M_{24} is opposite M_{13}, and hence p_{13} is opposite points of ξ'. Hence $\zeta \cap \xi'$ is a line $K \ni p_{24}$. If q_2 were not collinear to K, then ζ would contain a 3-space of M_{24}, again a similar contradiction (since ζ contains p_{13}). The planes π and $\langle q_2, K \rangle$ are easily seen to be opposite in ζ, hence there is a unique point $q_4 \in \langle q_2, K \rangle$ collinear to both q_3 and q'. Now let ξ_4 be the symp containing M_{24} and q_4, whose existence follows from Lemma 2.7. Then one checks that ξ_4 is opposite both ξ and ξ_3, and the projectivity $\xi \barwedge \xi_2 \barwedge \xi_3 \barwedge \xi_4 \barwedge \xi$ pointwise fixed both M_{13} and M, and maps q to q'. This proves the claim.

Now, if we want to apply Lemma 8.17, then we have to show that every projectivity

$$\rho\colon \xi_0 \barwedge \xi_1 \barwedge \xi_2 \barwedge \xi_3 \barwedge \xi_0,$$

with $M_0 := \xi_0 \cap \xi_2$ and $M_1 = \xi_1 \cap \xi_3$ singular 5-spaces, is the product of similar projectivities, but with M_0 opposite M_1. So suppose M_0 and M_1 are not opposite. As for the case of type A_5 in type E_6, there are 3 cases to consider, and they are again all quite similar to each other, so we consider for example the case where the set of points of M_0 collinear to a point of M_1 is a line L (the other possibilities are a 3-space and the whole space M_1). Then we consider an appropriate 5-space M_2 in ξ_0 intersecting M_0 in a 3-space contained in M_0, and disjoint from L. Then we find a symp ξ_2' containing M_2, opposite both ξ_1 and ξ_3, and intersecting ξ_2 in a 5-space opposite M_1. As in the case of type A_5 in type E_6, we can now write ρ as the product of $\xi_0 \barwedge \xi_1 \barwedge \xi_2' \barwedge \xi_3 \barwedge \xi_0$ and the conjugate of $\xi_3 \barwedge \xi_2' \barwedge \xi_1 \barwedge \xi_2 \barwedge \xi_3$ by $\xi_0 \barwedge \xi_3$.

Now we can use Lemma 8.11 and, thanks to Lemma 8.18, also Lemma 8.17 to conclude that $\Pi^+(F) = \overline{\mathsf{P\Omega}}_{12}(\mathbb{K})$.

Now consider the case of type D_6 inside type E_8 and let F be a simplex of cotype D_6. On the one hand, since the D_6 subdiagram is contained in a E_7 subdiagram, $\Pi^+(F)$ contains $\overline{\mathsf{P\Omega}}_{12}(\mathbb{K})$. On the other hand, since the D_6 subdiagram is contained in a D_7 subdiagram, $\Pi^+(F)$ contains $\mathsf{PGO}^\circ_{12}(\mathbb{K})$, as follows from Theorem 8.8(D′). Now Theorem B and Lemma 8.11(iii) yield $\Pi(F) = \Pi^+(F) = \overline{\mathsf{PGO}}^\circ_{12}(\mathbb{K})$.

Case (E6) Let Γ be the parapolar space of type $\mathsf{E}_{7,7}$ over the field $\mathbb{K}$. Let p_1, p_2, p_3 be three mutually opposite points of Γ. If we show that the self-projectivity $\rho\colon p_1 \barwedge p_2 \barwedge p_3 \perp p_1$ is always a symplectic polarity, then Lemma 8.1 and Lemma 8.9 implies that $\Pi(p)$ is generated by all the symplectic polarities. By Proposition 6.8(i) of [13], ρ pointwise fixes a subbuilding of type F_4. More exactly, if $\mathsf{Res}_\Delta(p_1)$ is viewed as a parapolar space Γ_{p_1} of type $\mathsf{E}_{6,1}$ with the lines through p_1 as points, then ρ pointwise fixes a geometric hyperplane inducing in Γ_{p_1} a geometry of type $\mathsf{F}_{4,4}$ over the field $\mathbb{K}$. It follows from [12] that ρ is a symplectic polarity. Now Lemma 8.13 shows that $\Pi^+(p)$ is $\mathsf{PGE}_6(\mathbb{K})$ and $\Pi(p)$ is $\mathsf{PGE}_6(\mathbb{K}).2$. Since these groups are the respective full linear type preserving and full not necessarily type preserving groups, Corollary 7.2 implies the assertion for type E_6 in type E_8.

Case (E7) In a Coxeter diagram of type E_8 a subdiagram of type E_7 is missing vertex 8. Hence, given a building Δ of type E_8 over the field $\mathbb{K}$, a simplex $F = \{p\}$ with residue of type E_7 can be thought of as being a point in the Lie incidence geometry Γ of type $\mathsf{E}_{8,8}$ over the field $\mathbb{K}$. Let p_1 and p_3 be collinear points of Γ opposite p and let p_2 be a point of Γ collinear to p and opposite both p_1 and p_3. Note that the lines through p form the point set of a Lie incidence geometry of type $\mathsf{E}_{7,7}$, where a line consists of all lines of Γ through p contained in a given plane of Γ. Set $\rho\colon p \barwedge p_1 \barwedge p_2 \barwedge p_3 \barwedge p$. Obviously, if π is a plane through the line pp_2, then $\mathsf{proj}^{p_1}_{p_2}\mathsf{proj}^{p}_{p_1}(\pi) = \pi$ and $\mathsf{proj}^{p_3}_{p}\mathsf{proj}^{p_2}_{p_3}(\pi) = \pi$. Hence ρ fixes π. Likewise, if L is the projection onto p of the line $L_1 := p_1p_3$, all planes through L are fixed by ρ. By the basic properties of Lie incidence geometries of type $\mathsf{E}_{7,7}$, there is a unique line M through p no point of which is special with any point of L (it is the projection of L on π in the residue $\mathsf{Res}_\Delta(p)$). Let K_1 and K_2 be two distinct lines in π containing p and distinct from both pp_2 and M. Let M_1 be the line of π containing all points of π not opposite p_1. Set $\{x_1\} = K_1 \cap M_1$. Then there exists a unique point x_1' with $x_1 \perp x_1' \perp p_1$. So the line p_1x_1' is the projection onto p_1 of the line K_1. Since $p_2 \perp x_1$, we see that the line p_2x_1 is the projection of the line p_1x_1' onto p_2.

Set $\{x_2\} = p_2x_1 \cap K_2$ and redefine p_3 as the projection of x_2 onto the line L_1. If $x_2 \perp x_2' \perp p_3$, then it is clear that p_2x_1 is projected onto p_3x_2' from p_2 onto p_3, and p_3p_2' is projected back onto $px_2 = K_2$ from p_3 onto p. Hence ρ maps K_1 to K_2 and the result now follows from Lemma 8.16. □

This concludes the proofs of all our main results. We conclude the paper with some remarks.

Remark 8.20. It now follows from Theorem 8.19 that $\overline{\mathsf{P\Omega}}_{12}(\mathbb{K})$ does not always coincide with $\overline{\mathsf{PGO}}_{12}(\mathbb{K})$. Indeed, if it did, then the special projectivity groups in the buildings of type E_7 of all irreducible residues of types contained in D_6 would be the full linear groups. This contradicts the second grey row of Table 2 for fields containing non-square elements.

Remark 8.21. The argument for case E_7 in E_8 of the proof of Theorem 8.19 could also be used for the cases of D_6 in E_7 and A_5 in E_6, if we would use the corresponding long root geometries. We chose to use the simpler and more accessible Lie incidence geometries of types $\mathsf{E}_{7,7}$ and $\mathsf{E}_{6,1}$, respectively, instead, also as a warm-up for the more complicated cases such as A_5 in E_7 and D_7 in E_8.

Remark 8.22. In the course of the proof of Theorem 8.19 we do not really need the full strength of Lemmas 8.11(i), 8.13 and 8.16, since we know by Theorem A that also the little projective group is already contained in the group we want to generate. This knowledge would simplify the proof, since we would only have to prove that the little projective group together with the said homologies generate the full linear group.

Remark 8.23. One could ask what to expect of the case where the diagram is not simply laced. For starters, the description of all spherical buildings is more complicated. Secondly, Theorem D will not hold anymore in full generality. Indeed, there are polar spaces of rank n where $\Pi^+(F)$ is not permutation equivalent to $\mathsf{PGL}_2(\mathbb{K})$, for F of cotype n, even if the set of maximal singular subspaces through a submaximal singular subspace carries in a natural way the structure of a projective line over $\mathbb{K}$ (like a symplectic polar space). However, analogues, appropriately phrased, of Theorems B and C should still hold. Also, Theorem A remains through across all types.

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