THE MOUFANG CONDITION AND ROOT AUTOMORPHISMS FOR SPHERICAL BUILDINGS OF RANK 3

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ABSTRACT. We give direct, geometric constructions for nontrivial root elations for rank 2 residues of higher rank buildings Δ of type B_n, C_n and H_m for $n \in \mathbb{N}$ and $m \in \{3, 4\}$. We show that we can extend these to the ambient building in the case that Δ has type B_n or C_n. With that, we obtain a different proof for the fact that buildings of type B_n and C_n are Moufang. This geometric approach enables us to gain more insight into the root groups associated to these buildings and we obtain new results; Namely, that certain root elations generically fix more points than we previously knew and that every root elation in each point residual can be written as an even self-projectivity. Concerning H_m, we will be able to see in a novel way why thick, spherical buildings of type H_m cannot exist. Altogther, this provides an alternative proof for the fact that all thick, irreducible, spherical buildings Δ of rank 3 have the Moufang property.

CONTENTS

1. INTRODUCTION

In his lecture notes from 1974, Jacques Tits showed that all thick, irreducible, spherical buildings Δ of rank at least three have the Moufang property. This groundbreaking result laid the basis for classifyng all such buildings, which is sometimes described as one of Tits's greatest achievements (see [1, section 0.5]) or one of the great accomplishments of 20th century mathematics in the literature (see [12, preface]). Showing the Moufang property was done by proving a theorem about the extensions of isometries (see [7, Theorem 4.1.2 and Proposition 4.16]), finding an isomorphism φ from an apartment Σ to an apartment Σ', such that Σ and Σ' intersect in a root α and φ acts as the identity on α, and then extending it to an automorphism of the whole building Δ. This proof is general and does not ask for the type of building. However, Jacques

Tits himself noted that his extension theorem is rather technical (see [7, page 7]). Hence, considering the type of the building and using its specific geometry to find direct, geometric constructions for all root elations, can help illuminate why Tits's famous result holds. The author is only aware of the proof introduced by Tits, which relies on the extension theorem, and is unaware of any other proof.

In this article, we give said constructions for rank 2 residues of higher rank buildings in 2.1 and 2.4 for types B_n and C_n and in 3.1 for type H_3. We show in 2.11 and 2.12 that our automorphisms extend to the ambient building for types B_3 and C_3, describe our new results about the fixpoint structure of these root elations in 2.15, and our new results about the connection between root elations and projectivities in 2.14. We give a new geometric proof for the fact that no thick, spherical buildings of types H_3 and H_4 exist in 3.1, which is a direct corollary of 3.1. We note that our constructions for elations also extend to the ambient buildings in the cases B_n and C_n for $n \geq 3$.

The case that Δ has rank 3 is special, because it is the lowest rank for which the Moufang condition is always satisfied and it usually gets easier to show the Moufang condition when the rank gets higher. It is not generally true that rank 2 buildings are Moufang. However, if we consider buildings of type $A_{n \geq 3}$ as an example, it is clearly true that its rank 2 subbuildings are always Moufang: The Lie incidence geometry of a building of type A_n is the geometry of a projective space. Projective planes correspond to the buildings of type A_2 and have been studied extensively by Ruth Moufang, after whom the Moufang condition was named. If we look at buildings of types B_n, their geometries correspond to those of polar spaces. Showing that projective planes – which correspond to the residues of type A_2 – of a polar space of rank 3 are Moufang, requires some work (see [7, Proposition 7.11]). Contrarily, if one considers polar spaces of rank greater than 3, it follows immediately that its projective planes are Moufang, since they are always contained in higher dimensional projective subspaces. Therefore, we will mostly be concerned with the rank 3 case in this article. We will see that buildings of type B_2 and C_2 inside buildings of rank 3 of the same type are always Moufang.

Considering all spherical buildings of rank 3, we observe that we have to examine buildings of type A_3, B_3, C_3, D_3 and H_3. Note that $A_3 = D_3$ and $B_n = C_n$ in our context. As said before, buildings of type A_3 have already been studied extensively and thus will be disregarded in this article.[1] After introducing the concepts that are relevant for this article, we will first cover the case of buildings of type B_n and after that the case of buildings of type H_m ($m \in \{3, 4\}$), which have a much more complicated geometry. In both cases we will first construct root elations for rank 2 residues. In the later case we will find a new proof for the fact that thick, spherical buildings of type H_m do not exist. In the first case we will proceed to show that these constructions extend to the whole space. This also works, if the rank is ≥ 3. Thus, this article lies a basis for developing a new proof for Tits's theorem. As said before, we will be able to state new results about the fixpoint structure of root elations for buildings of type B_3; namely that some of them generically fix more points than we previously knew. We will see that we can write every root elation of a rank 2 residue of a building of type B_3 and H_3 as an even self-projectivity of length 4. In other words: For residues of rank 2 we will show that their little projective group is contained in their special projectivity group.[2]

It is worth mentioning that a simplified version of Tits's extension theorem exists for polar spaces (see [2, Theorem 8.5.5]). However, this does not give us a direct construction for root elations and it assumes that we already know that a certain isomorphism exists ([2, page 400, line 2]).

1.1. **Buildings.** We will define a *building* in the same way as Jacques Tits in [7]. A *building* is a thick, simplicial chamber complex Δ, such that *(i)* Δ is the union of thin chamber subcomplexes

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[1]To see that projective spaces corresponding to thick, irreducible, spherical buildings of type A_n are Moufang, see for example [2, Chapter 6.3].

[2]This result, together with [5, Proposition 2.3], was a motivation for [4, Theorem A].

and we call these *apartments*, *(ii)* for any two simplices, there is an apartment containing both of them and *(iii)* every two apartments are isomorphic through an isomorphism fixing every vertex in their intersection. We call maximal simplices *chambers* and codimension 1 simplices *panels*. Note that, with this definition, every building is automatically *thick*, meaning that every panel is a face of at least three chambers. The apartments of a building are Coxeter complexes ([7, Theorem 3.7]). Given a building Δ, the Coxeter complexes that form the set of apartments of Δ, all stem from the same Coxeter system and thus have the same Coxeter diagram ([7, Proposition 3.15]). This justifies defining the *diagram of a building* Δ as the common Coxeter diagram of the Coxeter complexes. A building is called *irreducible* if its diagram is connected. A Coxeter diagram is called *spherical*, if the corresponding Coxeter group is finite and a building is called *spherical*, if its diagram is spherical. Let I be the vertex set of the diagram of a spherical building Δ. Then we call the cardinality of I the *rank of* Δ. We say that a spherical building Δ of rank n has *type* X_n, if X_n is the type of its Coxeter diagram.

The reader not familiar with buildings can find more information in [1], [7] and [12] among others. It will not be essential for this article to have a deep understanding about general building theory.

For every spherical Coxeter diagram of type X_n, one can define *Lie incidence geometries of type* X_n and get a correspondence between buildings of type X_n and Lie incidence geometries of type X_n. We refer to [2], [6, Section 9.5], [7], [9, Section 9.3] and in particular to [2, Table 4.2] with regards to the following table.

Type of Δ	Lie incidence geometry
A_n	Projective space
B_n, C_n	Polar space
D_n	Top-thin polar space
E_6, E_7, E_8	Parapolar space
F_4	Metasymplectic space
I_j	Generalised polygon
H_3	Generalised icosahedron
H_4	Generalised hypericosahedron

Since, in this article, we want to work with the Lie incidence geometries associated to buildings of type B_n, C_n and H_3, we will, in the following, give some explanations, definitions and state some properties about polar spaces and Lie incidence geometries of type H_3. Note that showing the *Moufang property* for buildings of types B_n and C_n is equivalent to showing the *Moufang property* for the corresponding polar spaces. Hence, we will only define the *Moufang property* properly in this context. For a rough impression and an outlook, we will say: *A spherical building Δ of rank at least two satisfies the Moufang property, if, for each half-apartment α, the group generated by all automorphisms of Δ, which act trivially on every panel of Δ, which contains two chambers in α, acts transitively on the set of all apartments of Δ containing α* (compare [12, Definition 11.1 and 11.2]).

1.2. **Point-Line Geometries.** We start with the most basic and essential definitions.

Definition 1.1. A *point-line geometry* is a pair $\Delta = (\mathscr{P}, \mathscr{L})$, where $\mathscr{P}$ is set and $\mathscr{L}$ is a set of subsets of $\mathscr{P}$. The elements of $\mathscr{P}$ are called *points* and the elements of $\mathscr{L}$ are called *lines*. If $p \in \mathscr{P}$ and $L \in \mathscr{L}$ with $p \in L$, we say that the point p *lies on* the line L, and the line L *contains* the point p, or *goes through* p. If two points p and q lie on a common line, they are called *collinear*, denoted $p \perp q$. If they are not contained in a common line, we say that they are *opposite*, denoted $p \equiv q$. For any point p and any subset $P \subseteq \mathscr{P}$, we define

$$p^\perp := \{q \in \mathscr{P} \mid q \perp p\} \text{ and } P^\perp := \bigcap_{p \in P} p^\perp.$$

A *partial linear space* is a point-line geometry in which every line contains at least three points, and where there is a unique line through every pair of distinct collinear points p and q, which is then denoted with pq.

Example 1.2. Let V be a vector space of dimension at least 3. Let $\mathscr{P}$ be the set of 1-dimensional subspaces of V and let $\mathscr{L}$ be the set of 2-dimensional subspaces of V. We can regard $\mathscr{L}$ as $\mathscr{P} \times \mathscr{P}$. Then $(\mathscr{P}, \mathscr{L})$ is called a *projective space (of dimension* $\dim V - 1$*)*.

Definition 1.3. Let $\Delta = (\mathscr{P}, \mathscr{L})$ be a partial linear space.

  (i) A subset S of $\mathscr{P}$ is called a *subspace* of Δ, if every line $L \in \mathscr{L}$ that contains at least two points of S, is contained in S. A subspace that intersects every line in at least a point is called a *hyperplane*. A hyperplane is called *proper*, if it does not consist of the whole point set. We usually regard subspaces of Δ as subgeometries of Δ in the canonical way.

 (ii) A subspace S, in which all points are collinear, or equivalently, for which $S \subseteq S^\perp$, is called a *singular subspace*. If, moreover, S is not contained in any other singular subspace, it is called a *maximal singular subspace*. A singular subspace is called *projective* if, as a subgeometry, it is a projective space.

(iii) For a subset P of $\mathscr{P}$, the *subspace generated by* P is denoted $\langle P \rangle_\Delta = \langle P \rangle$ and is defined to be the intersection of all subspaces containing P. A subspace generated by three mutually collinear points, not on a common line, is called a *plane*. (Note that, in general, this is not necessarily a singular subspace. However, in the cases we will deal with, subspaces generated by pairwise collinear points are singular; in particular planes will be singular subspaces.)

Definition 1.4. Given a point-line geometry, we can construct a graph by drawing a vertex for every point $p \in \mathscr{P}$ and by drawing an edge between two points p and q, if p and q are collinear. This graph is called the *point graph* or *collinearity graph*.

Definition 1.5. Let $n \geq 1$ be a natural number. A *generalised n-gon* is a partial linear space $\Gamma = (\mathscr{P}, \mathscr{L})$, such that the following axioms are satisfied (compare [10, Definition 1.3.1]).

  (i) Γ contains no regular k-gon as a subgeometry, for $2 \leq k < n$.
 (ii) Any two elements of $\mathscr{P} \cup \mathscr{L}$ are contained in some regular n-gon in Γ and these are also called the *apartments of* Γ.
(iii) There exists a regular $(n+1)$-gon as a subgeometry in Γ.

1.3. **Lie Incidence Geometries of Type** $\mathsf{B_n, C_n}$ **and** $\mathsf{D_n}$. As previously mentioned, Lie incidence geometries of types $\mathsf{B_n, C_n}$ and $\mathsf{D_n}$ are polar spaces. Thus, we recall the definition of a polar space and gather some basic properties. We will take the viewpoint of Buekenhout–Shult [3] and note that all results in this section are well-known and were gathered by Hendrik Van Maldeghem in the book *Polar Spaces* [9].

Definition 1.6. A *polar space* is a point-line geometry, in which every line contains at least three points, and for every point x, the set $x^\perp$ is a proper geometric hyperplane.

If two points p and q in a polar space are not collinear, then we call them *opposite*. We will only consider polar spaces of finite rank, that is, such that maximal singular subspaces are generated by a finite number of points. The minimal such number is called the *rank of the polar space*. A *submaximal* singular subspace is a hyperplane of a maximal singular subspace. One can show that all singular subspaces are either empty, points, lines or projective spaces of finite dimension (see [6, Theorem 7.3.6 and Lemma 7.3.8]). Consequently, a polar space is a partial linear space. A polar space is called *top-thin*, if every submaximal singular subspace is contained in exactly two maximal singular subspaces. A polar space is either top-thin or each submaximal singular subspace is contained in at least three maximal singular subspaces (see [9, Theorem 1.7.1]). In the latter case, the polar space is called *thick*.

Remark 1.7. Buildings of type $\mathsf{B_n}$ and $\mathsf{C_n}$ correspond to thick polar spaces. Buildings of type $\mathsf{D_n}$ correspond to top-thin polar spaces (see [9, Section 6.3 & Section 6.4]).

Commonly, when people refer to polar spaces, they mean thick polar spaces. In this article we will do the same. Top-thin polar spaces will not play a role anymore in the following.

Remark 1.8. Polar spaces of rank 2 are also called *generalised quadrangles*.

1.3.1. *Apartments in polar spaces and the Moufang property.*

Lemma 1.9. *Let Δ be a polar space of rank n. Then we can find $2n$ points $p_{-n}, p_{-n+1}, \ldots, p_{-1}$, $p_1, p_2, \ldots, p_n$ such that $p_i \perp p_j$ if, and only if, $i+j \neq 0$, for all $i, j \in \{-n, -n+1, \ldots, -1, 1, \ldots, n\}$.*

Proof. This is [9, Construction 1.5.3]. □

Definition 1.10. The set $\{p_{-n}, p_{-n+1}, \ldots, p_{-1}, p_1, p_2, \ldots, p_n\}$ is called a *polar frame*. An *apartment* of a polar space Δ is the set of all singular subspaces spanned by the points of a polar frame. Hence, the apartments of a generalised quadrangle are regular quadrangles, the apartments of a polar space of rank 3 are regular octahedra and the apartments for polar spaces of higher rank are hyperoctahedra.

Given an apartment $\mathscr{A}$ spanned by a polar frame $\{p_{-n}, p_{-n+1}, \ldots, p_{-1}, p_1, p_2, \ldots, p_n\}$, we can obtain a *half-apartment* α in two different ways:

(i) by removing two collinear points p_i and p_j of the polar frame, all maximal singular subspaces A of $\mathscr{A}$, which contain p_i and p_j and all singular subspaces which are contained in them, which either contain p_i or p_j, or

(ii) by removing one point p_i of the polar frame and all singular subspaces of $\mathscr{A}$, which contain p_i.

Half-apartments are also called *roots*. We will say that a root is *of the first kind*, if we remove two points and *of the second kind*, if we remove one point of the corresponding polar frame. The *inside α^+ of a root α of the first kind* is obtained, when we also remove all maximal singular subspaces which contain either p_i or p_j. The *inside α^+ of a root α of the second kind* is obtained, when we also remove all submaximal singular subspaces, which were contained in one of the removed maximal singular subspaces.

Example 1.11. We will consider apartments of a polar space of rank 3 and mark everything contained in a root in both cases in gray.

The first kind of root.   The second kind of root.
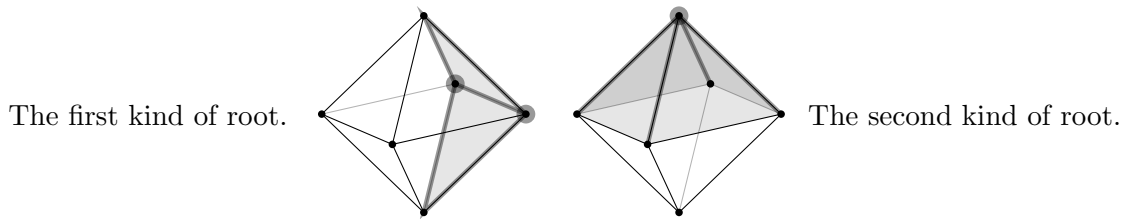
Proposition 1.12. *Each two singular subspaces are contained in a common apartment.*

Proof. This is [9, Proposition 1.6.10]. □

Definition 1.13. Let Δ be a polar space and α a root in Δ. If the group of automorphisms of Δ, which fix α^+ pointwise and stabilise all singular subspaces, which intersect α^+, acts transitively on the set of apartments containing α, then we say that Δ is *Moufang*. The automorphisms of a polar space are also called *collineations* and those, which fix the inside of a root pointwise, are also called *elations*. The group generated by all elations of a polar space is called the *little projective group of Δ.*

1.3.2. *Opposition, Residues and Projections.*

Lemma 1.14. *If p and b are two opposite points in a polar space Δ, then $p^\perp \cap b^\perp$ is a subspace of Δ. If we denote by $\mathscr{L}_{p,b}$ the set of lines completely contained in $p^\perp \cap b^\perp$, then $\Delta_{p,b} = (p^\perp \cap b^\perp, \mathscr{L}_{p,b})$ is a polar space, as soon as $\mathscr{L}_{p,b}$ is non-empty.*

Proof. This is [9, Lemma 2.3.1]. □

Proposition 1.15. *If U is a singular subspace of a polar space Δ, then $p^\perp \cap U$ is either equal to U or a hyperplane of U and $\dim(\langle p, p^\perp \cap U \rangle) = \dim(p^\perp \cap U) + 1$.*

Proof. This is [9, Proposition 1.4.1]. □

The last proposition justifies the following important definition.

Definition 1.16. Let U and V be two singular subspaces. We denote the set of points in U that are collinear to all points of V as $\mathsf{proj}_U(V)$ and call it the *projection of V onto U*. We call U and V opposite if both $\mathsf{proj}_U(V)$ and $\mathsf{proj}_V(U)$ are empty.

Proposition 1.17. *Let Δ be a polar space and let U and V be singular subspaces of Δ. Then the following hold:*

    (i) *$\mathsf{proj}_U(V)$ is a subspace of U.*
    (ii) *$\dim(V) - \dim(\mathsf{proj}_V(U)) = \dim(U) - \dim(\mathsf{proj}_U(V))$.*
    (iii) *Two singular subspaces U and V are opposite if, and only if, $\dim(U) = \dim(V)$ and no point of U is collinear to all points of V.*
    (iv) *Let U be disjoint from a maximal singular subspace M. Then $U' := \langle U, \mathsf{proj}_M U \rangle$ is the unique maximal singular subspace containing U and intersecting M in a singular subspace of dimension $n-2-\dim(U)$. Moreover, U' is the union of all singular subspaces containing U as a hyperplane and intersecting M in at least one point.*
    (v) *Each maximal singular subspace U has an opposite in Δ.*

Proof. This is [9, Note 1.4.4], [9, Proposition 1.4.6], [9, Corollary 1.4.7], [9, Corollary 1.4.8] and [9, Theorem 1.4.9]. □

For our construction, we will want to map from residues to residues and introduce a special way of projecting. For that, we first need the following definition.

Definition 1.18. Let U be a singular subspace of rank at most $n-3$ of a polar space Δ of rank n. Let X_U be the set of all singular subspaces of Δ with dimension $1 + \dim(U)$, which contain U. Let, for each singular subspace V containing U, V/U be the set of elements of X_U contained in V. Let Ω_U be the set of all such V/U, for V ranging through the set of all singular subspaces of Δ containing U. We define a new geometry $\mathsf{Res}_\Delta(U)$ with $\mathscr{P} = X_U$ and $\mathscr{L}$ the set of 1-dimensional subspaces of Ω_U and call it the *residue of U in Δ*.

Proposition 1.19. *The structure $\mathsf{Res}_\Delta(U)$ is a polar space of rank $n-1-\dim(U)$. It is thick if, and only if, Δ is thick, and top-thin if, and only if, Δ is top-thin. If U and U' are two opposite singular subspaces of dimension at most $n-3$, then the polar spaces $\mathsf{Res}_\Delta(U)$ and $\mathsf{Res}_\Delta(U')$ are isomorphic to each other.*

Proof. This is [9, Theorem 1.6.4] and [9, Corollary 1.6.8]. □

Definition 1.20. Let p and b be two opposite points in a polar space Δ of rank 3. Then

$$\mathsf{Res}_\Delta(p) \simeq \mathsf{Res}_\Delta(b) \simeq p^\perp \cap b^\perp =: \Gamma$$

defines a generalised quadrangle in Δ. The apartments of Γ are regular quadrangles. Let u be a point in $p^\perp$. Then we can view u as a point of $\mathsf{Res}_\Delta(p)$, by identifying it with the line pu. We *project pu to b* by projecting b onto pu in the usual way, obtaining a point u' collinear to b and then setting bu' as the image of the *projection of pu onto b*. Like this, we define a *projection from the residue of p to the residue of b* and we also write proj_b^p.

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Definition 1.21. Let p_0, p_1 ..., p_s be points inside a polar space Δ, such that p_i is opposite p_{i-1} and p_{i+1} for $i \in \mathbb{Z}/s\mathbb{Z}$. A composition

$$\mathsf{proj}_{p_s}^{p_{s-1}} \circ \cdots \circ \mathsf{proj}_{p_1}^{p_0}$$

is called a *projectivity*. We call it a *self-projectivity*, if $p_s = p_0$ and *even* if s is even.

All elations that we will construct in this article will be projectivities.

1.4. Lie Incidence Geometries of Type $\mathsf{H_3}$.

Some concepts that we introduced in the previous section, can be defined for all Lie incidence geometries. Since it would be unnecessarily complicated to define them all in full generality, we will now only name a few of them and then talk about how they look for Lie incidence geometries of type $\mathsf{H_3}$. Concepts that we can define for all Lie incidence geometries, but that look different in each, include opposition, residues, apartments and projections and one can figure out how these look for a certain Lie incidence geometry by considering its diagram. How this is done is described in the book *Diagram Geometry* by Francis Buekenhout and Arjeh M. Cohen [2].

The Coxeter diagram for $\mathsf{H_3}$ is the following:
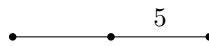
We can deduce from this diagram that a Lie Incidence geometry Δ of type $\mathsf{H_3}$ has planes and generalised pentagons as rank 2 residues and icosahedra as apartments (for details see [2, Section 2.1], [2, Section 2.4]). In the collinearity graph of Δ, vertices can be the same or be at graph-distance 1, 2 or 3.

Definition 1.22. Let Δ be a Lie incidence geometry of type $\mathsf{H_3}$. If two points p and b have distance s in the collinearity graph of Δ, then we will also say that p *and* b *have distance* s. We will call two points p and b *opposite*, if they have distance 3. If two points p and b have distance 2, then we also write $p \perp\!\!\!\perp b$. We denote by $p^{\perp\!\!\!\perp}$ the set of all points of Δ that are at distance 2 to p.

Considering an apartment, we can already see all possible relations that two points can have to each other. We observe the following:

Observation 1.23. Let Δ be a Lie incidence geometry of type $\mathsf{H_3}$ and let p and b be two points in Δ. Then p and b can have one of the following relations:
  (i) p and b are the same and we write $p = b$.
  (ii) p and b are collinear and we write $p \perp b$.
  (iii) p and b are at distance 2 and we write $p \perp\!\!\!\perp b$.
      In that case, $p^\perp \cap b^\perp$ is a unique line in Δ.
  (iv) p and b are at distance 3 and we call them opposite.
      In that case, $p^\perp \cap b^{\perp\!\!\!\perp}$ and $b^\perp \cap p^{\perp\!\!\!\perp}$ are generalised pentagons.

These properties will be all that we need. Later, we want to construct elations for the rank 2 residues that are generalised pentagons. For that, again, we want to define a special way of projecting that we describe in the following.

1.4.1. *The projection map.*

Let Δ be a Lie incidence geometry of type $\mathsf{H_3}$ and let p and b be two opposite points. We define proj_p^b (or proj_p for short, if it is clear from where we are projecting,) as follows.

Projecting a plane through b to the point p: Let π be a plane through b. There exists a unique line in π, for which every point has distance 2 to p. There exists a unique point p' in $p^\perp$ that is collinear to that line. We have $\mathsf{proj}_p(\pi) = pp'$.

Projecting a line through b to the point p: Let L be a line through b. There exists a unique point ℓ on L that has distance 2 to p and a unique line L' in $\ell^\perp \cap p^\perp$. We have $\mathsf{proj}_p(L) = \langle p, L' \rangle$.

If we are given points p_i, such that p_i is opposite both p_{i-1} and p_{i+1} for $i \in \mathbb{Z}/n\mathbb{Z}$ and we project from p_0 to p_1, ..., to p_{n-1}, then we also write $p_0 \overline{\wedge} p_1 \overline{\wedge} \ldots \overline{\wedge} p_{n-1}$. If we want to see how a specific point x of $p_0^\perp$ maps from p_0 to p_1, ..., to p_{n-1} and it is clear that we are considering $\theta \colon p_0 \overline{\wedge} p_1 \overline{\wedge} \ldots \overline{\wedge} p_{n-1}$, then we also write $p_0 x \mapsto \mathsf{proj}_{p_1}(p_0 x) \mapsto \cdots \mapsto x^\theta$.

2. The Moufang Property for Polar Spaces.

2.1. Generalised Quadrangles in Polar Spaces are Moufang.
Let $\tilde{\Delta}$ be a polar space of rank $n \geq 3$. Let Γ be an arbitary generalised quadrangle in Δ arising as the intersection of $U^\perp$ with $V^\perp$, for two opposite singular subspaces of dimension $n-3$. Then Γ is contained in some polar space Δ of rank 3 inside $\tilde{\Delta}$ and there exist two opposite points p and b in Δ, such that $\Gamma = p^\perp \cap b^\perp$. Furthermore, Γ is isomorphic to $\mathsf{Res}_\Delta(p)$ and $\mathsf{Res}_\Delta(b)$. In the following we prove the Moufang condition for Γ. For that we have to show that the group of automorphisms of Γ that pointwise fix the inside of an arbitrary half-apartment α^+ of an arbitrary apartment α of Γ and stabilise all elements incident with some element on the inside of the half-apartment α^+, acts transitively on the set of apartments containing α^+. We will give direct constructions for the desired automorphisms for both kinds of roots.

2.1.1. *First kind of root.*

Lemma 2.1. *Let Δ be a polar space of rank 3 and let Γ be the generalised quadrangle obtained from two opposite points p, b of Δ by considering $p^\perp \cap b^\perp$. Then all roots of the first kind of Γ are Moufang.*

Proof. Let α be an arbitrary apartment of Γ. Let q and d be two collinear points in α. Let u be the point in α collinear to q but not to d and let n be the point in α collinear to d but not to q. Our goal is to construct an automorphism that fixes qd pointwise, stabilises the lines that intersect q or d and moves the line nu to a line $n'u'$, such that $n' \in nd \setminus \{n\}$ and $u' \in uq \setminus \{u\}$. Let j be a point on pq not equal to p or q. The lines ju and pu' intersect in a point that we will denote by i. Set $\ell := \mathsf{proj}_{bd}(i)$.
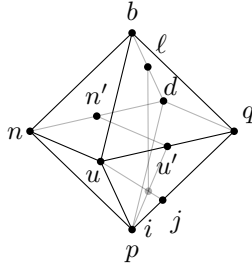
The points p and b, b and j, j and ℓ and ℓ and p are opposite. We define $\theta \colon \mathsf{Res}(p) \to \mathsf{Res}(p)$ as follows:

$$\theta := \mathsf{proj}_p^\ell \circ \mathsf{proj}_\ell^j \circ \mathsf{proj}_j^b \circ \mathsf{proj}_b^p.$$

The map in Γ defined by $x \mapsto \theta(px) \cap b^\perp$ is a collineation of Γ that we also will denote with θ (and there is no danger of confusion).

Since qd is in $p^\perp \cap b^\perp \cap j^\perp \cap \ell^\perp$, every point of qd is fixed by θ.

Let π be an arbitrary plane through pq. Let π' be the unique plane through b intersecting π in a line,. Then $\pi' = \mathsf{proj}_b^p(\pi)$. Since $j \in \pi$, the unique plane through j intersecting π is π again. This means that $\mathsf{proj}_j^b \circ \mathsf{proj}_b^p(\pi) = \pi$. The same token shows $\mathsf{proj}_p^\ell \circ \mathsf{proj}_\ell^j(\pi) = \pi$ and so $\theta(\pi) = \pi$.

Now let π be an arbitrary plane through pd. Set $\pi' = \mathsf{proj}_b^p(\pi$. As in the previous paragraph, since $\ell \in \pi'$, we find $\mathsf{proj}_\ell^j \circ \mathsf{proj}_j^b(\pi') = \pi'$. Projecting π' onto $\mathsf{Res}(p)$ yields π again. Hence $\theta(\pi) = \pi$ again.

Intersecting with Γ we already see that θ is a root elation fixing all points on dq and all lines through d and q. There remains to show that θ maps un to $u'n'$

The line pu maps to bu under the first projection and then to ju under the second projection. Since i is the unique point of ju that ℓ is collinear to, the line ju maps to the line $i\ell$ under the third projection and then to ip under the fourth projection. Since i is on pu', we have $pu' = pi$ and with that $\theta(u) = u'$. Projecting preserves incidence and therefore the unique point on nd that u' is collinear to has to be the point $\theta(n) = n'$. Hence the line nu moves to $n'u'$ under θ. $\qquad\square$

Observation 2.2. If we change the position of l in our construction and define it to be the projection of the intersection point of pv' and jv onto bn, then

$$\mathsf{proj}_p^l \circ \mathsf{proj}_l^j \circ \mathsf{proj}_j^b \circ \mathsf{proj}_b^p$$

is an automorphism that stabilises all planes through pq or pn and moves the point $v \neq u$ on uq to the point $v' \neq u$ on uq.
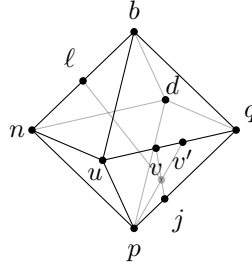
Observation 2.3. If we change the position of l in our construction and define it to be on the line bq, but not equal to q, then

$$\mathsf{proj}_p^l \circ \mathsf{proj}_l^j \circ \mathsf{proj}_j^b \circ \mathsf{proj}_b^p$$

is an automorphism that fixes every plane containing pq and every line that is contained in either $\langle p, u, q \rangle$ or $\langle p, d, q \rangle$.
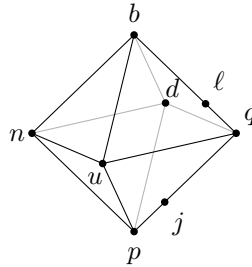
If $[\mathsf{proj}_{[\mathsf{proj}_{bn}(j),j]}(l), l]$ intersects pn, then θ is the identity. In that case the lines pn, $[\mathsf{proj}_{bn}(j), j]$, bq, pq, $[\mathsf{proj}_{[\mathsf{proj}_{bn}(j),j]}(l), l]$ and bn form a grid.

2.1.2. *Second kind of root.*

Lemma 2.4. *Let Δ be a polar space of rank 3 and let Γ be the generalised quadrangle obtained from two opposite points p, b of Δ by considering $p^\perp \cap b^\perp$. Then all roots of the second kind of Γ are Moufang.*

Proof. As in the previous proof, let α be an apartment of Γ defined by a polar frame $\{d, q, u, n\}$, where d is collinear to q and n, and opposite u. Let α' be another apartment of Γ spanned by a polar frame $\{d, q, u', n\}$, where u' is a point collinear to n and q, and opposite d and u. In the

following we will construct an automorphism of Γ that fixes dq and dn pointwise, moves u to u' and fixes all lines through d.

Pick any point on bu not equal to b or u and denote it by j'. Since b and p, and u and u' are opposite, and $u \perp p \perp u' \perp b \perp u$, the lines bu and pu' are opposite. Let j'' be the projection of j' onto pu'. The line bd is opposite $j'j''$, because j' is not collinear to d, and if b were collinear to j'', then b would be collinear to all points on $u'p$, a contradiction. Let ℓ be the projection of d onto $j'j''$. Arguing as above, we see that bu and pd are opposite. Let j be the projection of j' onto pd. Since $j' \perp n$ and $j'' \perp n$, we conclude $\ell \perp n$. In the same way $\ell \perp q$.

Next we will show that ℓ is opposite both p and j. The point p is collinear to j'' but not to j', since pd and bu are opposite and p is already collinear to u. In particular, p is not collinear to all points of $j'j''$ and not collinear to l. Suppose that j were collinear to j''. Then j would be collinear to all of pu' and u' would be collinear to p and j, so to all of pd. That contradicts the fact that u' and d are opposite. That means j' is the unique point on $j'j''$ that j is collinear to. In particular j is not collinear to ℓ.
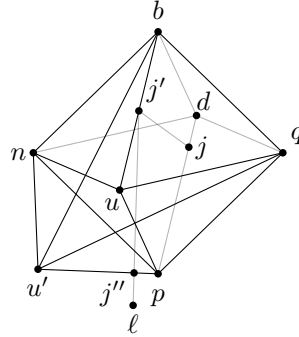
Since $p \equiv b \equiv j \equiv \ell \equiv p$, we can project in each case from one residue into the other.

We define $\theta \colon \mathsf{Res}(p) \to \mathsf{Res}(p)$ as follows:

$$\theta := \mathsf{proj}_p^\ell \circ \mathsf{proj}_\ell^j \circ \mathsf{proj}_j^b \circ \mathsf{proj}_b^p$$

From the discussion above follows that both j and ℓ are collinear to all points of the lines dq and dn. This readily implies that θ fixes every point on $dq \cup dn$. Also, since j is contained in each plane through the line pd, we find that each plane through pd is stabilised by θ. Hence θ induces a root elation in Γ. We now check that $\theta(u) = u'$.

We have $bu = \mathsf{proj}_b^p(pu)$. Since $j \perp j' \in bu$, we have $jj' = \mathsf{proj}_j^b(bu)$. Since $j' \perp \ell$, we have $j'\ell = \mathsf{proj}_\ell^j(jj')$. Finally, Since $p \perp j'' \in j'\ell$, we have $pj'' = \mathsf{proj}_p^\ell(j'\ell)$. But $j'' \in pu'$, hence $pj' = u'$. It follows that $\theta(pu) = pu'$ and so $\theta(u) = u'$. This proved the lemma. $\quad\square$

Corollary 2.5. *Let Δ be a polar space of rank at least 3 and let Γ be the generalised quadrangle obtained from two opposite singular subspaces U, V of Δ of dimension $n-3$ by considering $U^\perp \cap V^\perp$. Then Γ is a Moufang quadrangle.*

Proof. The claim follows from Lemma 2.1 and 2.4. $\quad\square$

2.2. **Polar Spaces of Rank 3 are Moufang.** In the following we show that the elations we found for generalised quadrangles in 2.1 extend to elations of the ambient polar space. We will only show this in detail for the case that the latter has rank 3. However, it can be done for arbitrary rank $n \geq 3$ and we will comment on that at the end of this section.

2.2.1. *First kind of half-apartment.* Let Δ be a polar space of rank 3. We want to show that for every line dq in Δ through two arbitrary, collinear points d and q, we can construct an automorphism $\phi\colon \Delta \to \Delta$ that fixes all planes through dq pointwise, all lines through either d or q linewise and acts transitively on the points of the lines through either d or q, which are not contained in a plane through dq.

Lemma 2.6. *Let d, q be two collinear points in the polar space Δ. Let m, m' be two collinear points of Δ not collinear to q, but such that $d \in mm'$. Then there exists a unique permutation η of the set of points $d^{\perp} \cup q^{\perp}$ with the following properties.*

(i) *All points of $d^{\perp} \cap q^{\perp}$ are fixed.*
(ii) *The collineation induced by η in each plane π containing d or q, but not both, is a translation of π with axis $\mathsf{proj}_{\pi}(dq)$ and centre d or q, respectively.*
(iii) *η preserves collinearity.*

Proof. Let π be a given plane containing dq. Let β be the plane spanned by m and $L := \mathsf{proj}_{\pi}(m) \ni d$. Then π contains also m'. Since projective planes inside polar spaces are Moufang, there exists an elation η of β with axis L and center d, such that $\eta(m) = m'$.

Now let M be a line through q not collinear to d. Then M is opposite every line of β except for L. For every line $K \subseteq \beta$, with $d \in K \neq L$, we define an action $\eta_K : M \to M : x \mapsto \mathsf{proj}_M^K(\eta(\mathsf{proj}_K^M(x)))$. We claim

(*) The map η_K is independent of K.

Indeed, let K_1 and K_2 be two lines in β through d distinct from L. Let $x_1 \in K_1 \setminus \{d\}$ be arbitrary. Set $y_1 := \mathsf{proj}_M(x_1)$. If $z_1 = \mathsf{proj}_L(y_1)$, then $x_1 z_1 = \mathsf{proj}_\beta(y_1)$. It follows that $\mathsf{proj}_{K_2}(y_1) = x_1 z_1 \cap K_2 =: x_2$, and hence $z_1 \in \eta(x_1)\eta(x_2)$ as $\eta(z_1) = z_1$. Now $z_1 = L \cap \mathsf{proj}_\pi(M)$ and as such, z_1 is also equal to $\mathsf{proj}_L(\eta_{K_1}(x_1) = \mathsf{proj}_L(\eta_{K_2}(x_2)$. It follows that $\mathsf{proj}_\beta(\eta_{K_1}(x_1) = \eta(x_1)z_1$, which by the above coincides with $\eta(x_2)z_1$. Hence $\eta_{K_1}(y_1)$ is collinear to $\eta(x_2)$ and with that we conclude that $\eta_{K_2}(y_1) = \eta_{K_1}(y_1)$. The claim is proved.

From now one, we will abbreviate the definition of η_K above by the sentence "we copy the action of η on K to M".

Thanks to (*), we can denote η_K simply by η. Now let α be any plane through M. We claim that

(**) There exists a unique elation in α with axis $A := \mathsf{proj}_\alpha(dq)$ and centre q extending the action of η.

Indeed, We can copy the action of η on the line mm' to any line of α through q, and we denote the corresponding maps by η. Then the arguments of the proof of (*) imply that, if x and y are two points of α with $q \notin xy$, then xy and $\eta(x)\eta(y)$ intersect A in the same point. But this is exactly equivalent to showing that η preserves collinearity, when extending η to A by stating that η fixes each point of A. The claim now follows.

Let β' be any plane intersecting β in a line through d. Then (**) with the roles of d and q interchanged implies that there is a unique elation in β with axis $\mathsf{proj}_{\beta'}(dq)$ agreeing with θ on the line $\beta \cap \beta'$. By (*), the action of that elation on any line of β' through d (not collinear to q) copied to M agrees with η. If we keep doing this procedure, then, since in $\mathsf{Res}(d)$, the geometry opposite dq is connected (see hvmbook), the action of η on M, copied on every line through d opposite M, with additionally the identity on $d^{\perp} \cap q^{\perp}$, defines a permutation η preserving collinearity and inducing a nontrivial elation in every plane through d not containing q.

Note that η, acting on $d^{\perp}$ as defined in the previous paragraph, is independent of the line M through q in α, as follows from (*) with d and q interchanged. Likewise we can now extend η from α to $q^{\perp}$, preserving collinearity, and the action on every line through q not collinear to d is copied from the action of η on any line through d not collinear to q.

Since "copying the action" just means that collinear pairs of points from $d^{\perp} \times q^{\perp}$ are mapped to collinear pairs of points, the lemma is proved. $\qquad\square$

We now make the connection between η as defined in Lemma 2.6 and the axial elation defined in Lemma 2.1.

Lemma 2.7. *Let d, q be two collinear points in the polar space Δ. Let m, m' be two collinear points of Δ not collinear to q, but such that $d \in mm'$. Let η be the unique permutation of $d^\perp \cup p^\perp$ with the properties mentioned in* Lemma 2.6. *Let p and b be two opposite points both collinear to d, q and m. Let θ be the root elation of $p^\perp \cap b^\perp$ with root (q, qd, d) mapping m to m'. Then the action of θ on every line through q or d in $p^\perp \cap d^\perp$ coincides with the action of η on that line.*

Proof. Let j be an arbitrary point on $pq \setminus \{p, q\}$. Let β be the plane spanned by b, m and d. Set $K := \mathsf{proj}_\beta(j)$ and note $d \in K$. Define ℓ as the intersection of the line bd with the line joining m with $bm' \cap K$. With the proof of Lemma 2.1, we find that $p \,\overline{\wedge}\, b \,\overline{\wedge}\, j \,\overline{\wedge}\, d \,\overline{\wedge}\, p$ moves pm to pm'. Hence this defines θ in $p^\perp \cap b^\perp$. The action of θ on the line mm' is given by $x \mapsto lx' \cap mm'$, where $x' = bx \cap K$. It is well known that this is the restriction to mm' of the elation of β with axis bd, centre d mapping m to mm'. Hence $\theta(x) = \eta(x)$, for all $x \in mm'$. Since the actions on the lines of $p^\perp \cap d^\perp$ through q are obtained by copying, for both η and θ, these action coincide. By copying these to the lines through d, the lemma follows. $\qquad\square$

The following is an immediate consequence of the foregoing.

Corollary 2.8. *Let d, q be two collinear points in the polar space Δ. Let m, m' be two collinear points of Δ not collinear to q, but such that $d \in mm'$. Let η be the unique permutation of $d^\perp \cup p^\perp$ with the properties mentioned in* Lemma 2.6. *Let p and b be two opposite points both collinear to d, q and m. Let θ be the root elation of $\mathsf{Res}(p)$ with root $(pq, \langle p, q, d \rangle, pd)$ mapping pm to pm'. Let θ_b be the collineation of $\mathsf{Res}(b)$ defined by $L \mapsto \mathsf{proj}_b(\theta(\mathsf{proj}_p(L)))$. Then θ_b is the unique root elation of $\mathsf{Res}(b)$ for which the action on the lines through b collinear to q or d coincides with the action of η.*

The definition of the mapping $L \mapsto \mathsf{proj}_b(\theta(\mathsf{proj}_p(L)))$ will be abbreviated by θ *is copied from p to d.*

The next lemma allows us to extend η to the whole polar space Δ in an unambiguous way.

Lemma 2.9. *Let, with previous notation, the point x be collinear to a unique point y of dq. Let p and b be two opposite points collinear to all of q, b and x. Then there exists a unique root elation $\eta_{p,b}$ in $p^\perp \cap b^\perp$ corresponding to the root (d, dq, q) having the same action as η on the points collinear to q or d. Moreover, $\eta_{p,b}(x)$ does not depend on p and b.*

Proof. The existence and uniques of $\eta_{p,b}$ follows from Lemma 2.7. Now let o be another point collinear to all of q, d, x and assume first that o is opposite both p and b. Let θ_p, θ_b and θ_o be the root elation of $\mathsf{Res}(p), \mathsf{Res}(b)$ and $\mathsf{Res}(o)$, respectively, for which the action on the lines through p, b, o, respectively, collinear to q (or d) coincides with the action of η. By Corollary 2.8, θ_p copied from p to b is θ_b; θ_b copied from b to o is θ_0 and θ_p copied from p to o is again θ_o. It follows that $\theta_o(ox)$ intersects both $\theta_p(px)$ and $\theta_b(bx)$ nontrivially. But $\theta_p(px) = p\eta_{p,b}(x)$ and $\theta_b(bx) = b\eta_{p,b}(x)$. Since these two lines are not contained in a plane, the line $\theta_0(0x)$, which intersects both, contains the intersection point $\eta_{p,b}(x)$. Hence $\eta_{p,o}(x) = \eta_{p,b}(x) = \eta_{o,b}(x)$.

Now assume that o is collinear to b. Note that o is then opposite p. Note also that $o \in by$, as, denoting the plane spanned by p, d, b by β, the line by coincides with $\mathsf{proj}_\beta(x)$. Since $\eta_{p,b}(x)$ is also collinear to y, we find that $o \perp \eta_{p,b}(x)$. Hence $\eta_{p,b}(x) = \eta_{p,o}(x)$.

Now let p', b' be two other opposite points collinear to p, b, x. At most one of p, b is collinear to b', say p is opposite b'. By the foregoing $\eta_{p,b}(x) = \eta_{p,b'}(x) = \eta_{p',b'}(x)$, and the lemma is proved. $\qquad\square$

Proposition 2.10. *Let d, q be two collinear points in the polar space Δ. Let m, m' be two collinear points of Δ not collinear to q, but such that $d \in mm'$. Using the notation of the previous lemmas, the automorphism $\phi : \Delta \to \Delta$ defined by*

$$\phi(x) := \begin{cases} x, & \text{if } x \text{ is in a plane with } dq. \\ \eta(x), & \text{if } x \text{ collinear to either } q \text{ or } d, \text{ not in a plane with } dq. \\ \eta_{p,b}(x) & \text{if } x \text{ collinear to neither } d \text{ nor } q \text{ ; } p \equiv b, x, d, q \in p^\perp \cap b^\perp, \end{cases}$$

is a well-defined collineation on the whole polar space Δ that fixes all planes through dq pointwise, stabilises all lines through either d or q and maps m to m'.

Proof. We have already proved that ϕ is well-defined. Since interchanging the roles of m and m' clearly results in a two-sided inverse map, we deduce that ϕ is bijective. It suffices to prove that ϕ maps lines into lines.

This is certainly obviously true for lines intersecting the line bq, and for lines collinear to either q or d. There are two possibilities left.

   (i) Let L be a line collinear to a unique point $e \in dq \setminus \{d, q\}$. The projection p of q onto L is collinear to dq. Then p is fixed. For each point $x \in L \setminus \{p\}$, Lemma 2.9 ensures that $\phi(x)$ is contained in the line $\theta_p(L)$ (with the notation of the proof of Lemma 2.9).

   (ii) Let L be opposite dq. We can then pick two opposite points p, b collinear to both L and qd. Then, by Lemma 2.9 again, $\phi(L) = \eta_{p,b}(L)$ is a line.

The proposition is proved. $\hfill\square$

The arbitrariness of m and m' now yields the following consequence.

Corollary 2.11. *Let Δ be a polar space of rank 3. Then all roots of the first kind are Moufang*

2.2.2. *Second kind of half-apartment.*

Lemma 2.12. *Let Δ be a polar space of rank 3. Then all roots of the second kind are Moufang.*

Proof. Let α and β be two planes intersecting in a point o. Let L and M be two opposite lines in α and β respectively and let p and p' be two points opposite o in $L^\perp \cap M^\perp$. We aim to construct a collneation ϕ of Δ stabilising all lines through o, fixing $\alpha \cup \beta$ pointwise, and mapping p to p'. Let x be a point on L. Let η_x be the root elation of $\mathsf{Res}(x)$ fixing all lines of α through x, fixing al lines through x that intersect β, stabilizing all planes through ox and mapping xp to xp'. We can copy that action onto every point $y \in \beta$ opposite x, meaning, we define η_y as the unique collineation of $\mathsf{Res}(y)$ mapping any line K through y to $\mathsf{proj}_y(\eta_x(\mathsf{proj}_x K))$. Now let $z\alpha$ be such that $z \neq x$ and the line xz does not contain o. Let $\eta_{z,y}$ be the action of η_y copied onto z, for any $y \in \beta$ opposite z. Clearly $\eta_{z,y}$ is always a root elation in $\mathsf{Res}(z)$ with axes α and $\langle z, \mathsf{proj}_\beta(z) \rangle$ and centre oz. But also, each plane through the line xz has the same image under η_x and under $\eta_{z,y}$. Since a root elation in $\mathsf{Res}(z)$ with centre oz is determined by the image of any plane not through oz, we conclude that $\eta_z := \eta_{z,y}$ is independent of y. Interchanging the roles of x and z, we conclude that there exists a unique root elation η_a of $\mathsf{Res}(a)$, for each point of α such that the set of all such root elations, together with all η_b for $b \in \beta$, is closed under copying from points to opposite points.

Now let w be an arbitrary point of Δ opposite o. We define $\phi(w) = w$, for $w \in \alpha \cup \beta$. If $w \notin \alpha \cup \beta$, then set $A := \mathsf{proj}_\alpha(w)$ and $B := \mathsf{proj}_\beta(w)$. Select $a \in A$ and let π be the plane spanned by w and A. Set $\pi' = \eta_a(\pi)$. Select $b \in B$ end let σ be the plane spanned by B and w. By copying, $\sigma' := \eta_b(\sigma)$ intersects π' in a unique point, which we define to be $\phi(w)$. Note that, by copying, $\eta_a(aw) = a\phi(w)$, for all $a \in A$ and $\eta_b(bw) = b\phi(w)$, for all $b \in B$.

Now let $w \perp o$. There are two possibilities. First suppose $A := \mathsf{proj}_\alpha(w)$ and $B := \mathsf{proj}_\beta(w)$ are not coplanar. For an arbitrary point $a \in A \setminus \{o\}$, we define $\phi(w) = \eta_a(aw) \cap \langle w, B \rangle$. Similarly as in the previous paragraph we find that this is independent of $a \in A \setminus \{o\}$, and also $\phi(w) = \eta_b(bw) \cap \langle w, A \rangle$, for all $b \in B \setminus \{o\}$.

Secondly, suppose A and B as defined above are coplanar. Then for each $a \in A \setminus \{o\}$, we have $\eta_a(aw) = aw$ and we define $\phi(w) = w$.

Hence in all cases we have that, if $c \in \alpha \cup \beta \setminus \{o\}$ is collinear to w, then $\eta_c(cw) = s\phi(c)$.

It remains to show that lines are mapped to lines. If a line K is not collinear to o, then it is collinear to unique distinct points $a \in \alpha$ ad $b \in \beta$, and the images under ϕ of the points of K are all contained in te line $\eta_a(\langle a, K \rangle) \cap \eta_b(\langle b, K \rangle)$. Applying the inverse mapping (obtained by interchanging the roles of p and p'), we see that $\phi(K)$ is a line.

Finally, let K be collinear to o. Each point $w \in K$ is mapped onto a point $\phi(w) \in ow$. Consider a plane $\pi \not\ni o$ through K. By the foregoing, the image $\phi(\pi)$ is contained in a plane which intersects $\langle o, K \rangle$ in the set $\{\phi(w) \mid w \in K\}$. Since distinct planes sharing at least two points intersect in lines, we see that $\phi(K)$ is a line.

This completes the proof of the proposition. $\square$

Corollary 2.13. *Let Δ be a polar space of rank 3. Then Δ is Moufang.*

Proof. The claim follows from Corollary 2.11 and Lemma 2.12 $\square$

2.3. Corollaries.

Corollary 2.14. *For polar spaces of rank 3, every root elation in each point residual can be written as an even self-projectivity of length 4.*

Proof. This follows from the fact that, in our proofs, we constructed every elation by projecting from the residue of a point b, to that of a point p, to that of a point l, to that of a point j, and back to the residue of b for an appropriate choice of points b, p, l and j in each case. $\square$

Corollary 2.15. *Root elations associated to roots α of the first kind of polar spaces of rank 3 not only stabilise all planes through the pointwise fixed line that is the intersection L of the two planes in α^+, but fix them pointwise. They also stabilise all lines through either of the two points of α^+ on L. Root elation associated to roots β of the second kind stabilise all lines through the central point of β^+.*

Proof. This follows from the proofs of Proposition 2.10 and Lemma 2.12. $\square$

2.3.1. *Rank $n \geq 3$.*

For the induction, suppose we showed that the elations θ extend in the case that the polar space Δ has rank $n-1$. Suppose $\tilde{\Delta}$ is a polar space of rank n containing Δ. Let $\mathscr{A}$ be some apartment of $\tilde{\Delta}$ spanned by a polar frame $\{p_{-n}, p_{-n+1}, \ldots, p_{-1}, p_1, p_2, \ldots, p_n\}$. Since we know that the extension works for Δ, we know that θ is determined for all singular subspaces spanned by the points $\{p_{-n+1}, p_{-n+2}, \ldots, p_{-1}, p_1, p_2, \ldots, p_{n-1}\}$. Now, with the exact same techniques as before, we can copy the action of θ to the surrounding singular subspaces.

Let Δ be a Lie incidence geometry of type H_3. As before, we would like to construct elations for rank 2 residues of Δ first. We will focus on the rank 2 residues that are generalised pentagons. For every generalised pentagon Γ in Δ, we can find two opposite points b and p, such that $b^\perp \cap p^{\perp\perp} = \Gamma$. An elation θ of Γ will fix the inside α^+ of a half-apartment α of Γ pointwise and stabilise all lines through two points on the inside of that half-apartment. Since we are generally interested in the Moufang property, we would like to construct θ in a way, such that α gets mapped to a different apartment α'. Note that this is only possible, if Δ correspond to a thick, spherical building of type H_3.

Proposition 3.1. *Suppose thick, spherical buildings of type H_3 existed. Let Δ be a Lie incidence geometry corresponding to such a building and let p and b be two opposite points of Δ. Let Σ be an apartment of the generalised pentagon $b^\perp \cap p^{\perp\perp}$ with points $\{b_0, b_1, b_2, b_3, b_4\}$, such that $b_i \perp b_{i+1}$ for $i \in \mathbb{Z}/5\mathbb{Z}$. Then we can construct an elation that fixes the lines $b_0 b_1$ and $b_1 b_2$ pointwise, stabilises all lines through b_0 and b_1 in $b^\perp \cap p^{\perp\perp}$ and maps Σ to an apartment Σ' with points $\{b_0, b_1, b_2, b_3', b_4'\}$, such that $b_2 \perp b_3' \perp b_4' \perp b_0$, $b_3 \neq b_3'$ and $b_4 \neq b_4'$. Since p, b, Σ and Σ' were arbitrary, this would show, that rank 2 residues of Δ are Moufang.*

Proof. Let $p_0, \ldots, p_4$ be points that span a pentagon in $p^\perp \cap b^{\perp\perp}$, such that p_i is collinear to $b_{i-1} b_i$ for $i \in \mathbb{Z}/5\mathbb{Z}$. Let b_4' be a point on $b_0 b_4$ not equal to b_0 or b_4 and let b_3' be a point in $b^\perp \cap p^{\perp\perp}$ that is collinear to b_2 and b_4'. We want to define a map $\theta : \mathsf{Res}(b) \to \mathsf{Res}(b)$, such that θ fixes $b_0 b_1$ and $b_1 b_2$ pointwise, stabilises all lines through b_0 and b_1 and moves b_3 to b_3' and b_4 to b_4'. Let d be an arbitrary point on bb_1 not equal to b or b_1.

The point d has distance 2 to all points of a unique line through p_0 in $\langle p, p_0, p_4 \rangle$ that intersects pp_4 in some point p_4'. Every point on bb_1 has distance 2 to p_3. With that, d has distance 2 to all points of the line $p_3 p_4'$ in $\langle p, p_3, p_4 \rangle$. We define $d_4 := d^\perp \cap (p_0 p_4')^\perp$ and $d_3 := d^\perp \cap (p_3 p_4')^\perp$.

We consider the projectivity $\varphi : b \overline{\wedge} p \overline{\wedge} d$ and observe the following, using the notation introduced in 1.4.1:

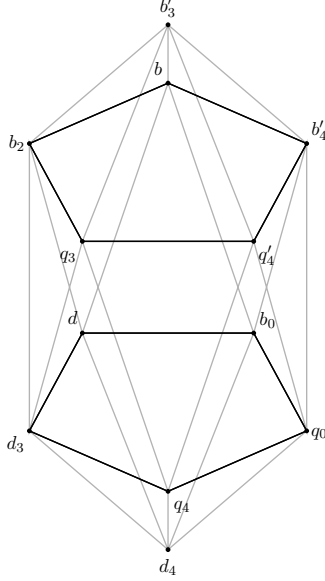$$bb_2 \mapsto \langle p, p_2, p_3 \rangle \mapsto db_2$$
$$\langle b, b_1, b_2 \rangle \mapsto pp_2 \mapsto \langle d, b_1, b_2 \rangle = \langle b, b_1, b_2 \rangle$$
$$bb_1 \mapsto \langle p, p_1, p_2 \rangle \mapsto db_1$$
$$\langle b, b_0, b_1 \rangle \mapsto pp_1 \mapsto \langle d, b_0, b_1 \rangle = \langle b, b_0, b_1 \rangle$$
$$bb_0 \mapsto \langle p, p_0, p_1 \rangle \mapsto db_0$$
$$\langle b, b_0, b_4 \rangle \mapsto pp_0 \mapsto \langle d, b_0, d_4 \rangle$$
$$bb_4 \mapsto \langle p, p_0, p_4 \rangle \mapsto dd_4$$
$$\langle b, b_3, b_4 \rangle \mapsto pp_4 \mapsto \langle d, d_3, d_4 \rangle$$
$$bb_3 \mapsto \langle p, p_3, p_4 \rangle \mapsto dd_3$$
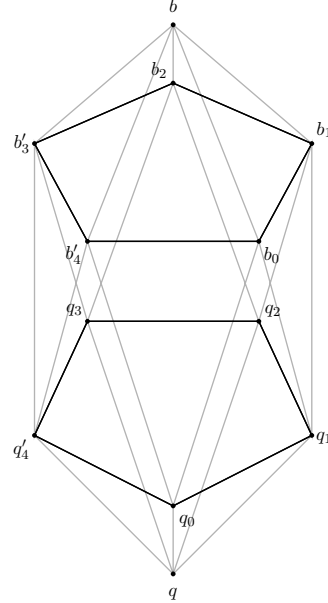$$\langle b, b_2, b_3 \rangle \mapsto pp_3 \mapsto \langle d, b_2, d_3 \rangle$$
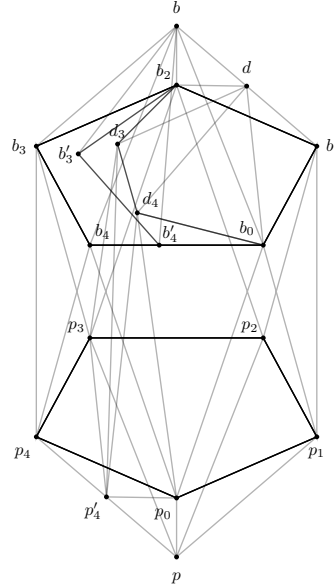
Let x be some arbitrary point on $b_0 b_1$. Then bx maps to a plane through pp_1 and then to dx. Let y be some arbitrary point on $b_1 b_2$. Then by maps to a plane through pp_2 and then to dy.

We observe that the points b_3' and d_4 are opposite and consider an apartment Σ containing b_3', b, b_2, b_4', b_0, d, d_3 and d_4. The points b_4' and d_4 have distance 2 and are both collinear to b_0. Let q_0 be another point on the line $(b_4')^\perp \cap d_4^\perp$ through b_0. We label the leftover point of the pentagon $d_4^\perp \cap (b_3')^{\perp\perp}$ in Σ by q_4. Let q_4' be the point in Σ collinear to q_0, q_4, b_3' and b_4' and q_3 be the point collinear to q_4', q_4, b_2, b_3' and d_3'. With that, all points of Σ are labeled. The point b_1 is on the line bd. Let q be the unique point on q_4q_4' with distance 2 to b_1.

We see that q has distance 3 to b and d, distance 2 to b_0, b_1, b_2, b_3', b_4', d_3 and d_4 and distance 1 to q_0, q_4, q_4', q_3, q_2 and q_1. We consider an apartment containing $\langle q, q_0, q_4'\rangle$ and $\langle b, b_1, b_2\rangle$. We denote the point in that apartment that is collinear to q, q_0, b_0 and b_1 by q_1 and the point that is collinear to q, q_3, b_1 and b_2 by q_2.

The apartment containing $\langle b_3', b, b_2\rangle$ and $\langle d_4, q_0, q_4\rangle$ The apartment containing $\langle b, b_1, b_2\rangle$ and $\langle q, q_0, q_4'\rangle$

We set $\theta := b \barwedge p \barwedge d \barwedge q \barwedge b$ and claim that θ has the desired properties. Indeed, we observe the following.

$$bb_2 \mapsto \langle p, p_2, p_3\rangle \mapsto db_2 \mapsto \langle q, q_2, q_3\rangle \mapsto bb_2$$
$$\langle b, b_1, b_2\rangle \mapsto pp_2 \mapsto \langle d, b_1, b_2\rangle = \langle b, b_1, b_2\rangle \mapsto qq_2 \mapsto \langle b, b_1, b_2\rangle$$
$$bb_1 \mapsto \langle p, p_1, p_2\rangle \mapsto db_1 \mapsto \langle q, q_1, q_2\rangle \mapsto bb_1$$
$$\langle b, b_0, b_1\rangle \mapsto pp_1 \mapsto \langle d, b_0, b_1\rangle = \langle b, b_0, b_1\rangle \mapsto qq_1 \mapsto \langle b, b_0, b_1\rangle$$
$$bb_0 \mapsto \langle p, p_0, p_1\rangle \mapsto db_0 \mapsto \langle q, q_0, q_1\rangle \mapsto bb_0$$
$$\langle b, b_0, b_4\rangle \mapsto pp_0 \mapsto \langle d, b_0, d_4\rangle \mapsto qp_0 \mapsto \langle b, b_0, b_4\rangle$$
$$bb_4 \mapsto \langle p, p_0, p_4\rangle \mapsto dd_4 \mapsto \langle q, p_0, q_4\rangle \mapsto bb_4'$$
$$\langle b, b_3, b_4\rangle \mapsto pp_4 \mapsto \langle d, d_3, d_4\rangle \mapsto qq_4 \mapsto \langle b, b_3', b_4'\rangle$$
$$bb_3 \mapsto \langle p, p_3, p_4\rangle \mapsto dd_3 \mapsto \langle q, q_3, q_4\rangle \mapsto bb_3'$$
$$\langle b, b_2, b_3\rangle \mapsto pp_3 \mapsto \langle d, b_2, d_3\rangle \mapsto qq_3 \mapsto \langle b, b_2, b_3'\rangle$$

Let x be some arbitrary point on b_0b_1. Then bx maps to a plane through pp_1, to dx, to a plane through qp_1 and back to bx.

Let y be some arbitrary point on b_1b_2. Then by maps to a plane through pp_2, to dy, to a plane through qq_2 and back to by.
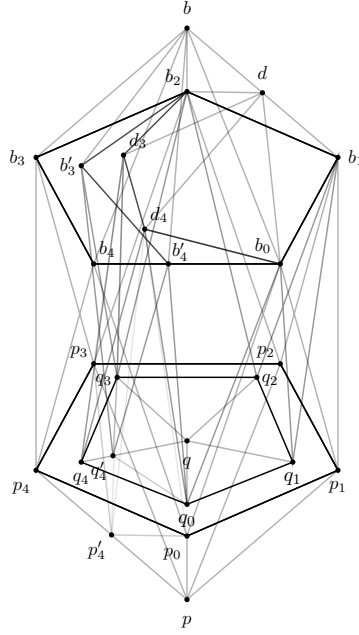
The established points and lines relevant to determining the images under θ.

The last things we have to show, are that all lines through b_0 and b_1 in $b^\perp \cap p^{\perp\perp}$ are stabilised under θ. For that, we will first consider an arbitrary plane β through bb_1. Since bb_1 first maps to $\langle p, p_1, p_2 \rangle$, β will first map to a line in $\langle p, p_1, p_2 \rangle$. Since $\mathsf{proj}_d(\langle p, p_2, p_2 \rangle) = db_1$ and $d \in \beta$, this line will map back to β. Similarly, $\mathsf{proj}_q(\beta)$ is a line in $\langle q, q_1, q_2 \rangle$ that will map back to β, when we project to b. This shows that all lines through b_1 in $b^\perp \cap p^{\perp\perp}$ are stabilised under θ.

For the lines through b_0, we first consider the projectivity $\psi : p \,\overline{\wedge}\, b \,\overline{\wedge}\, q \,\overline{\wedge}\, d \,\overline{\wedge}\, p$. We see that p_1p_2 is fixed pointwise under ψ. Furthermore, a plane γ through pp_1 maps to a line through b in $\langle b, b_0, b_1 \rangle$ that intersect b_0b_1 in some point g, to a plane through qq_1, to a line through d in $\langle b, b_0, b_1 \rangle$ that intersect b_0b_1 in the same point g and back to γ. Thus, planes through pp_1 are stabilised under ψ and in the same way we can see that planes through pp_2 are stabilised.

We can define a projectivity $\theta' := p \,\overline{\wedge}\, b \,\overline{\wedge}\, q' \,\overline{\wedge}\, d' \,\overline{\wedge}\, p$, where q' is a point on pp_1 and d' is definied analogously to q previously, opposite p and d' and such that $(\mathsf{proj}_p^b)^{-1} \circ \theta' \circ \mathsf{proj}_p^b$ moves b_3' to b_3. Because of the given symmetry, θ' will fix p_0p_1 and p_1p_2 pointwise and stabilise all planes through pp_1.

Now we can consider $\psi' : b \,\overline{\wedge}\, p \,\overline{\wedge}\, d' \,\overline{\wedge}\, q' \,\overline{\wedge}\, b$ and see that ψ' fixes b_0b_1 pointwise and stabilises all planes through bb_0 and bb_1.

We can see that

$$\psi' \circ \theta^{-1} = b \,\overline{\wedge}\, q \,\overline{\wedge}\, d \,\overline{\wedge}\, p \,\overline{\wedge}\, b \,\overline{\wedge}\, p \,\overline{\wedge}\, d' \,\overline{\wedge}\, q' \,\overline{\wedge}\, b$$

fixes all planes through bb_1 and all points on b_0b_1, since both ψ' and θ^{-1} do that. But then $\psi' \circ \theta^{-1}$ fixes an apartment in $b^\perp$ and therefore has to be the identity. Thus, θ is a root elation.

$\square$

3.1. Non-existence of Thick, Sphercial Buildings of Types H_3 and H_4.

In 1976, an article by Jacques Tits got published, showing that Moufang m-gons only exist for $m \in \{2, 3, 4, 6, 8\}$ (see [8, Théorème 1]). Another result by Tits – contained in ([7, Addenda, page 275]) – shows that thick, spherical buildings of type H_3 and H_4 do not exist. To summarise the proof of Tits briefly: Suppose thick, spherical buildings of type H_3 (or H_4) existed. According to [7, Addenda, page 274], these would have to be Moufang. With that, it would follow that the rank 2 residues of a Moufang spherical building of type H_3, which form generalised pentagons (and which are also contained in H_4), also satisfy the Moufang condition [7, Addenda, page 274].

But that is impossible, because no such generalised pentagons exist according to [8, Théorème 1]. Since the proof of [7, Addenda, page 274] relies on the extension theorem [7, Theorem 4.1.2], which is, as mentioned in the introduction, rather technical, there have been attempts to find different ways to prove the non-existence.

One different way of proving that there are no thick, spherical buildings of type H_3 and H_4, without using the extension theorem, was found by Hendrik Van Maldeghem in 1995 (see [11, Section 5]). Van Maldeghem's proof works by first assuming that thick, spherical buildings of type H_3 exist, showing that the rank 2 residues that are generalised pentagons are regular and showing that regular generalised pentagons do not exist.

In this article, we also did not need the extension theorem, only assumed that thick, spherical buildings of type H_3 exist and then gave geometric constructions for nontrivial root elations of generalised pentagons; implying the Moufang condition for them, which then, again, leads to a contradiction using [8, Théorème 1]. This has not been done before and is a more direct and less intricate way of proving the non-existence of thick, spherical buildings of type H_3 and H_4, since we do not need the concept of regularity.

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